BLOCH-TO-BMOA COMPOSITIONS ON COMPLEX BALLS

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ABSTRACT. Let φ be a holomorphic map between complex unit balls. We characterize those φ for which the composition operator $f \mapsto f \circ \varphi$ maps the Bloch space into BMOA.

1. INTRODUCTION

Let $H(B_m)$ denote the space of holomorphic functions in the unit ball B_m of $\mathbb{C}^m$, $m \geq 1$.

1.1. Spaces under consideration. The Bloch space $\mathcal{B}(B_m)$ consists of those functions $f \in H(B_m)$ for which

$$\|f\|_{\mathcal{B}(B_m)} = |f(0)| + \sup_{w \in B_m} |\mathcal{R}f(w)|(1 - |w|^2) < \infty,$$

where

$$\mathcal{R}f(w) = \sum_{j=1}^m w_j \frac{\partial f}{\partial w_j}(w), \quad w \in B_m,$$

is the radial derivative of f . The Hardy space $H^p(B_n)$, $p > 0$, $n \geq 1$, consists of functions $f \in H(B_n)$ such that

$$\|f\|_{H^p(B_n)}^p = \sup_{0 < r < 1} \int_{\partial B_n} |f(r\zeta)|^p d\sigma_n(\zeta) < \infty,$$

where σ_n is the normalized Lebesgue measure on the sphere ∂B_n . Also, we consider $\text{BMOA}(B_n)$, the space of holomorphic functions that have bounded mean oscillation on ∂B_n . Equivalent definitions of $\text{BMOA}(B_n)$ are given in Section 2.

1.2. Composition operators. Given a holomorphic map $\varphi : B_n \rightarrow B_m$, the composition operator $C_\varphi : H(B_m) \rightarrow H(B_n)$ is defined by the following identity:

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(B_m), \quad z \in B_n.$$

Various properties of C_φ are presented in the monographs [5, 12]. In this paper, we describe those φ for which C_φ maps $\mathcal{B}(B_m)$ into $\text{BMOA}(B_n)$.

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1.2.1. *Bloch-to-BMOA composition operators: $m = 1$.* There is a series of results about the operators under consideration. In particular, characterizations of the bounded operators $C_\varphi : \mathcal{B}(B_1) \rightarrow \text{BMOA}(B_n)$ were obtained in [10]; see also [7, 9, 13]. The cases $n = 1$ and $n \geq 2$ are rather different. Indeed, let $\varphi : B_n \rightarrow B_1$ be a holomorphic Lipschitz function of order 1. Then C_φ does not map $\mathcal{B}(B_1)$ into $\text{BMOA}(B_n)$ when $\|\varphi\|_\infty = 1$ and $n = 1$, but C_φ maps $\mathcal{B}(B_1)$ into $\text{BMOA}(B_n)$ when $n \geq 2$ (see [4] and [10], respectively). Motivations and further references are given in [3, 10].

1.2.2. *Bloch-to-BMOA composition operators: $m \geq 2$.* To the best knowledge of the author, for arbitrary $n, m \in \mathbb{N}$, the problem in question was considered only in [3], where the bounded and compact composition operators $C_\varphi : \mathcal{B}(B_m) \rightarrow \text{BMOA}(B_n)$ are characterized under an additional regularity assumption about φ . Namely, the operator C_φ is bounded if and only if

$$(1.1) \quad \frac{(1 - |z|^2)|\mathcal{R}\varphi(z)|^2}{(1 - |\varphi(z)|^2)^2} d\nu_n(z) \quad \text{is a Carleson measure,}$$

where ν_n is Lebesgue measure on B_n and $\nu_n(B_n) = 1$.

In the present paper, we use the Möbius-invariance of the spaces $\text{BMOA}(B_n)$ and $\mathcal{B}(B_m)$. So, for $z \in B_n$, let ϕ_z denote the involution of B_n such that $\phi_z(0) = z$. Let β_m denote the Bergman metric on the ball B_m . The main result of this paper is the following theorem:

Theorem 1.1. *Let $\varphi : B_n \rightarrow B_m$ be a holomorphic map. Then the following properties are equivalent:*

$$(1.2) \quad C_\varphi : \mathcal{B}(B_m) \rightarrow \text{BMOA}(B_n) \text{ is a bounded operator;}$$

$$(1.3) \quad \sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} \beta_m(\varphi(\phi_z(r\zeta)), \varphi(z)) d\sigma_n(\zeta) < \infty.$$

For $m = 1$, Theorem 1.1 was proved in [7]; see also [10].

1.3. Comments.

1.3.1. *Property (1.3) and hyperbolic BMOA.* Recall that the Garsia seminorm on $\text{BMOA}(B_n)$ is defined by the identity

$$\|f\|_{G^1(B_n)} = \sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} |f(\phi_z(r\zeta)) - f(z)| d\sigma_n(\zeta).$$

Therefore, (1.3) reduces to the property $\|f\|_{G^1(B_n)} < \infty$ when φ is replaced by $f \in H(B_n)$ and β_m is replaced by the Euclidean metric. So, as in [9, 14] for $n = m = 1$, we say that (1.3) defines the hyperbolic BMOA class. However, other names have been used for this class; see [7].

1.3.2. *Carleson measures.* As observed in [3], the implication (1.1) $\Rightarrow$ (1.2) holds for all holomorphic maps $\varphi : B_n \rightarrow B_m$. Hence, Theorem 1.1 guarantees that (1.1) implies (1.3) for arbitrary φ . So, one could expect that (1.3) implies (1.1) for all φ . If this is the case, then it would be interesting to find a direct proof of the implication in question.

1.3.3. *Bounded mean oscillation.* The classical seminorm on $\text{BMOA}(B_n)$ is defined by the identity

$$\|f\|_{\text{BMOA}(B_n)} = \sup \frac{1}{\sigma_n(Q)} \int_Q |f^* - f_Q^*| d\sigma_n,$$

where f^* is the boundary function of f , f_Q^* is the average of f^* over Q , and the supremum is taken over all quasi-balls $Q = Q_r(\eta) = \{\xi \in \partial B_n : |1 - \langle \eta, \xi \rangle| < r\}$, $\eta \in \partial B_n$; see Section 2.4 for details.

The hyperbolic analog of the property $\|f\|_{\text{BMOA}(B_n)} < \infty$ is the following one:

$$(1.4) \quad \sup_{r>0, \eta \in \partial B_n} \frac{1}{Q_r(\eta)} \int_{Q_r(\eta)} \beta_m(\varphi^*(\zeta), \varphi(\eta\sqrt{1-r^2})) d\sigma_n(\zeta) < \infty.$$

The relations between (1.3), (1.4) and similar properties will be considered elsewhere.

2. PRELIMINARIES

Basic properties of $\mathcal{B}(B_m)$ and $\text{BMOA}(B_n)$ are collected in this section. Further details are given in [1, 15]; see also [6] for $n = m = 1$.

2.1. **Automorphisms of the unit ball.** The automorphism group of B_n , denoted by $\text{Aut}(B_n)$, consists of all biholomorphic mappings from B_n onto B_n . Given $z \in B_n$, the involution (or the Möbius transform) $\phi_z \in \text{Aut}(B_n)$ is defined for $\lambda \in B_n$ as follows:

$$\begin{aligned} \phi_z(\lambda) &= -\lambda \quad \text{when } z = 0, \\ \phi_z(\lambda) &= \frac{z - P_z\lambda - \sqrt{1-|z|^2}Q_z\lambda}{1 - \langle \lambda, z \rangle} \quad \text{when } z \in B_n \setminus \{0\}, \end{aligned}$$

where $P_z\lambda = |z|^{-2}\langle \lambda, z \rangle z$, $Q_z\lambda = \lambda - P_z\lambda$. To distinguish the involutions of B_m , we write ψ_w , $w \in B_m$ in the place of ϕ_z , $z \in B_n$.

2.2. **Hyperbolic BMOA.** The hyperbolic BMOA is defined by (1.3) in terms of the Bergman metric β_m on B_m . Note that

$$\beta_m(w_1, w_2) = C \log \frac{1 + |\psi_{w_1}(w_2)|}{1 - |\psi_{w_1}(w_2)|}, \quad w_1, w_2 \in B_m.$$

So, a holomorphic map $\varphi : B_n \rightarrow B_m$ is in the hyperbolic BMOA if and only if

$$(2.1) \quad \sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} \log \frac{1}{1 - |\psi_{\varphi(z)}(\varphi(\phi_z(r\zeta)))|^2} d\sigma_n(\zeta) < \infty.$$

2.3. **Seminorms on the Bloch space.** For $f \in H(B_m)$, put

$$|\tilde{\nabla} f(w)|^2 = (1 - |w|^2) (|\nabla f(w)|^2 - |\mathcal{R}f(w)|^2), \quad w \in B_m,$$

where

$$\nabla f(w) = \left(\frac{\partial f}{\partial w_1}(w), \dots, \frac{\partial f}{\partial w_m}(w) \right)$$

is the complex gradient of f .

Let $\widetilde{\mathcal{B}}(B_m)$ denote the quotient of $\mathcal{B}(B_m)$ by the space of constant functions. Then $\widetilde{\mathcal{B}}(B_m)$ is a Banach space with respect to the following norms:

$$\begin{aligned} & \sup_{w \in B_m} |\mathcal{R}f(w)|(1 - |w|^2); \\ & \sup_{w \in B_m} |\nabla f(w)|(1 - |w|^2); \\ & \|f\|_{\widetilde{\mathcal{B}}(B_m)} = \sup_{w \in B_m} |\widetilde{\nabla} f(w)|. \end{aligned}$$

Clearly, the above expressions are seminorms on $\mathcal{B}(B_m)$; these seminorms degenerate exactly on the constant functions. The main advantage of $\|\cdot\|_{\widetilde{\mathcal{B}}(B_m)}$ is its Möbius-invariance. Namely,

$$\|f \circ \psi\|_{\widetilde{\mathcal{B}}(B_m)} = \|f\|_{\widetilde{\mathcal{B}}(B_m)}$$

for all $\psi \in \text{Aut}(B_m)$, $f \in \mathcal{B}(B_m)$.

Also, a function $f \in H(B_m)$ belongs to $\mathcal{B}(B_m)$ if and only if there exists a constant $C > 0$ such that

$$(2.2) \quad |f(w_1) - f(w_2)| \leq C\beta_m(w_1, w_2) \quad \text{for all } w_1, w_2 \in B_m.$$

2.4. Seminorms on BMOA. For $\zeta \in \partial B_n$ and $r > 0$, put

$$Q_r(\zeta) = \{\xi \in \partial B_n : |1 - \langle \zeta, \xi \rangle| < r\}.$$

Recall that the radial limits $|f^*(\zeta)| = \lim_{r \rightarrow 1-} |f(r\zeta)|$ are defined for σ_n -almost all $\zeta \in \partial B_n$ for every $f \in H^1(B_n)$. Let $\text{BMOA}(B_n)$ denote the space of functions $f \in H^1(B_n)$ such that

$$(2.3) \quad |f(0)|^p + \sup_{\zeta \in \partial B_n, r > 0} \frac{1}{\sigma_n(Q)} \int_Q |f^* - f_Q^*|^p d\sigma_n < \infty,$$

where $p = 1$, $Q = Q_r(\zeta)$ and

$$f_Q^* = \frac{1}{\sigma_n(Q)} \int_Q f^* d\sigma_n.$$

The norm $\|f\|_{\text{BMOA}(B_n)}$ is defined as the left-hand side of (2.3) with $p = 1$.

By the John-Nirenberg theorem, there exist constants $A(n) > 0$ and $C(n) > 0$ such that

$$(2.4) \quad \int_{\partial B_n} \exp(|f^*(\zeta)|) d\sigma_n(\zeta) \leq C(n)$$

for all $f \in \text{BMOA}(B_n)$ with $\|f\|_{\text{BMOA}(B_n)} \leq A(n)$. The John-Nirenberg inequality guarantees that $f \in \text{BMOA}(B_n)$ if and only if $f \in H^2(B_n)$ and (2.3) holds with $p = 2$.

The proof of Theorem 1.1 will be based on the following fact: (2.3) holds for $p = 1$ or for $p = 2$ if and only if

$$(2.5) \quad \|f\|_{G^p(B_n)} = \sup_{z \in B_n} \|f \circ \phi_z - f(z)\|_{H^p(B_n)} < \infty$$

for $p = 1$ or for $p = 2$. The above seminorms degenerate exactly on the constant functions. Let $\widetilde{\text{BMOA}}(B_n)$ denote the quotient of $\text{BMOA}(B_n)$ by the space of constant functions. Then $\widetilde{\text{BMOA}}(B_n)$ is a Banach space with respect to the Garsia norm $\|\cdot\|_{G^p(B_n)}$, $p = 1$ or $p = 2$.

3. AUXILIARY RESULTS

3.1. Aleksandrov–Ryll–Wojtaszczyk polynomials. Ryll and Wojtaszczyk [11] constructed holomorphic polynomials which proved to be very useful for many problems of function theory in the unit ball. We need the following improvement of the Ryll–Wojtaszczyk theorem.

Theorem 3.1 ([2, Theorem 4]). *Let $m \in \mathbb{N}$. Then there exists $\delta = \delta(m) \in (0, 1)$ and $J = J(m) \in \mathbb{N}$ with the following property: For every $d \in \mathbb{N}$, there exist holomorphic homogeneous polynomials $W_j[d]$ of degree d , $1 \leq j \leq J$, such that*

$$(3.1) \quad \|W_j[d]\|_{L^\infty(\partial B_m)} \leq 1 \quad \text{and}$$

$$(3.2) \quad \max_{1 \leq j \leq J} |W_j[d](\xi)| \geq \delta \quad \text{for all } \xi \in \partial B_m.$$

3.2. A reverse estimate for Bloch functions. For $m = 1$, it is known that the following lemma holds with $J(1) = 1$; see [8, Lemma 2.1].

Lemma 3.2. *Let $m \in \mathbb{N}$ and let $0 < p < \infty$. Then there exist constants $J = J(m) \in \mathbb{N}$, $\tau_{m,p} > 0$ and there exist functions $F_{j,x} \in \mathcal{B}(B_m)$, $1 \leq j \leq J$, $0 \leq x \leq 1$, such that $\|F_{j,x}\|_{\mathcal{B}(B_m)} \leq 1$, $F_{j,x}(0) = 0$, and*

$$\sum_{j=1}^J \int_0^1 |F_{j,x}(w)|^p dx \geq \tau_{m,p} \left(\log \frac{1}{1 - |w|^2} \right)^{\frac{p}{2}}$$

for all $w \in B_m$.

Proof. Let the constant $\delta \in (0, 1)$ and the polynomials $W_j[d]$, $1 \leq j \leq J$, $d \in \mathbb{N}$, be those provided by Theorem 3.1. For $k \in \mathbb{Z}_+$, let R_k denote the Rademacher function:

$$R_k(x) = \text{sign} \sin(2^{k+1}\pi x), \quad x \in [0, 1].$$

For each non-dyadic $x \in [0, 1]$, consider the functions

$$F_{j,x}(w) = \frac{1}{4} \sum_{k=0}^{\infty} R_k(x) W_j[2^k](w), \quad w \in B_m, \quad 1 \leq j \leq J.$$

Estimate (3.1) guarantees that

$$(1 - |w|^2) |(\mathcal{R}F_{j,x})(w)| \leq \frac{1 - |w|^2}{4} \sum_{k=0}^{\infty} 2^k |w|^{2^k} \leq \frac{1 - |w|^2}{2} \sum_{n=1}^{\infty} |w|^n \leq 1$$

for all $w \in B_m$. Observe that $F_{j,x}(0) = 0$; hence, $\|F_{j,x}\|_{\mathcal{B}(B_m)} \leq 1$.

Next,

$$C_p \int_0^1 |F_{j,x}(w)|^p dx \geq \left(\sum_{k=0}^{\infty} |W_j[2^k](w)|^2 \right)^{\frac{p}{2}}$$

by [16, Chapter V, Theorem 8.4]. Given positive numbers a_j , $1 \leq j \leq J = J(m)$, we have

$$\left(\sum_{j=1}^J a_j \right)^{\frac{p}{2}} \leq C_{p,m} \sum_{j=1}^J a_j^{p/2}.$$

Hence,

$$C_{p,m} \sum_{j=1}^J \int_0^1 |F_{j,x}(w)|^p dx \geq \left(\sum_{k=0}^{\infty} \sum_{j=1}^J |W_j[2^k](w)|^2 \right)^{\frac{p}{2}}.$$

Recall that $W_j[2^k]$ is a homogeneous polynomial of degree 2^k ; thus,

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{j=1}^J |W_j[2^k](w)|^2 &\geq \delta^2 \sum_{k=0}^{\infty} |w|^{2^{k+1}} \\ &\geq \delta^2 \sum_{n=1}^{\infty} \frac{|w|^{2n}}{n} = \delta^2 \log \frac{1}{1 - |w|^2}, \quad w \in B_m, \end{aligned}$$

by (3.2). So,

$$\sum_{j=1}^J \int_0^1 |F_{j,x}(w)|^p dx \geq \left(\frac{\delta^2}{C_{m,p}} \log \frac{1}{1 - |w|^2} \right)^{\frac{p}{2}},$$

as required. $\square$

4. BLOCH-TO-BMOA COMPOSITION OPERATORS

4.1. Proof of Theorem 1.1. Assume that (1.2) holds. Note that $C_\varphi 1 = 1$; hence, $C_\varphi : \widetilde{\mathcal{B}}(B_m) \rightarrow \text{BMOA}(B_n)$ is a bounded operator. Using (2.5) with $p = 2$, we obtain

$$(4.1) \quad \sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} |f \circ \varphi \circ \phi_z(r\zeta) - f \circ \varphi(z)|^2 d\sigma_n(\zeta) \leq C \|f\|_{\widetilde{\mathcal{B}}(B_m)}^2.$$

Let the constant $\tau = \tau_{m,2} > 0$ and the functions $F_{j,x}$, $1 \leq j \leq J$, $0 \leq x \leq 1$, be those provided by Lemma 3.2 for $p = 2$. Note that $\|F_{j,x}\|_{\widetilde{\mathcal{B}}(B_m)} \leq C \|F_{j,x}\|_{\mathcal{B}(B_m)} \leq C$. Recall that $\|\cdot\|_{\widetilde{\mathcal{B}}(B_m)}$ is Möbius-invariant; hence, $\|F_{j,x} \circ \psi_{\varphi(z)}\|_{\widetilde{\mathcal{B}}(B_m)} \leq C$, where the constant $C > 0$ does not depend on $z \in B_n$. Also, we have $F_{j,x} \circ \psi_{\varphi(z)}(\varphi(z)) = F_{j,x}(0) = 0$. Thus, by (4.1) with $f = F_{j,x} \circ \psi_{\varphi(z)}$,

$$\int_{\partial B_n} |F_{j,x} \circ \psi_{\varphi(z)}(\varphi(\phi_z(r\zeta)))|^2 d\sigma_n(\zeta) \leq C$$

for all $z \in B_n$, $0 < r < 1$, $0 \leq x \leq 1$. Hence,

$$\sum_{j=1}^J \int_0^1 \int_{\partial B_n} |F_{j,x} \circ \psi_{\varphi(z)}(\varphi(\phi_z(r\zeta)))|^2 d\sigma_n(\zeta) dx \leq C, \quad z \in B_n, \quad 0 < r < 1.$$

Therefore, Fubini's theorem and Lemma 3.2 guarantee that

$$\tau \int_{\partial B_n} \log \frac{1}{1 - |\psi_{\varphi(z)}(\varphi(\phi_z(r\zeta)))|^2} d\sigma_n(\zeta) \leq C, \quad z \in B_n, \quad 0 < r < 1.$$

So, we obtain (2.1) or, equivalently, (1.3).

To prove the converse implication, assume that (1.3) holds, that is,

$$\sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} \beta_m(\varphi(\phi_z(r\zeta)), \varphi(z)) d\sigma_n(\zeta) < \infty.$$

Let $f \in \mathcal{B}(B_m)$. Then

$$|f(\varphi(\phi_z(r\zeta))) - f(\varphi(z))| \leq C\beta_m(\varphi(\phi_z(r\zeta)), \varphi(z))$$

by (2.2). Hence,

$$\begin{aligned} & \sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} |f(\varphi(\phi_z(r\zeta))) - f(\varphi(z))| d\sigma_n(\zeta) \\ & \leq C \sup_{z \in B_n} \sup_{0 < r < 1} \int_{\partial B_n} \beta_m(\varphi(\phi_z(r\zeta)), \varphi(z)) d\sigma_n(\zeta) < \infty. \end{aligned}$$

Using (2.5) with $p = 1$, we have $f \circ \varphi \in \text{BMOA}(B_n)$. So, (1.2) holds by the closed graph theorem. The proof of Theorem 1.1 is complete.

4.2. A necessary condition. The space $\text{BMOA}(B_n)$ is not a lattice, so it is not expected that (1.2) is equivalent to a restriction on $|\varphi^*(\zeta)|$, $\zeta \in \partial B_n$. However, applying Lemma 3.2, we obtain a related explicit condition, which is necessary for (1.2).

Proposition 4.1. *Let $\varphi : B_n \rightarrow B_m$ be a holomorphic map. Assume that $C_\varphi : \mathcal{B}(B_m) \rightarrow \text{BMOA}(B_n)$ is a bounded operator. Then there exist constants $\varepsilon = \varepsilon(n, m, \|C_\varphi\|_{\mathcal{B} \rightarrow \text{BMOA}}) > 0$ and $C = C(n) > 0$ such that*

$$\int_{\partial B_n} \exp\left(\varepsilon \log \frac{1}{1 - |\varphi^*(\zeta)|^2}\right)^{\frac{1}{2}} d\sigma_n(\zeta) \leq C.$$

Proof. The operator C_φ maps $\mathcal{B}(B_m)$ to the Hardy space $H^2(B_n)$. So, arguing as in the proof of the implication (1.2) $\Rightarrow$ (1.3), we obtain

$$\sup_{0 < r < 1} \int_{\partial B_n} \log \frac{1}{1 - |\varphi(r\zeta)|^2} d\sigma_n(\zeta) < \infty.$$

Hence,

$$\int_{\partial B_n} \log \frac{1}{1 - |\varphi^*(\zeta)|^2} d\sigma_n(\zeta) < \infty$$

by Fatou's lemma. In particular,

$$(4.2) \quad |\varphi^*(\zeta)| < 1$$

for σ_n -almost every $\zeta \in \partial B_n$. If (4.2) holds, then

$$(4.3) \quad f(\varphi^*(\zeta)) = \lim_{r \rightarrow 1^-} f(\varphi(r\zeta)) = (f \circ \varphi)^*(\zeta)$$

for any $f \in H(B_m)$.

Now, let the constant $\tau = \tau_{m,1} > 0$ and the functions $F_{j,x} \in \mathcal{B}(B_m)$, $1 \leq j \leq J$, $0 \leq x \leq 1$, be those provided by Lemma 3.2 with $p = 1$. We have $\|F_{j,x} \circ \varphi\|_{\text{BMOA}(B_n)} \leq \|C_\varphi\|_{\mathcal{B} \rightarrow \text{BMOA}}$. Thus, for $\delta = A(n)\|C_\varphi\|_{\mathcal{B} \rightarrow \text{BMOA}}^{-1}$, (4.3) and (2.4) guarantee that

$$\begin{aligned} \int_{\partial B_n} \exp(\delta |F_{j,x}(\varphi^*(\zeta))|) d\sigma_n(\zeta) &= \int_{\partial B_n} \exp(\delta |(F_{j,x} \circ \varphi)^*(\zeta)|) d\sigma_n(\zeta) \\ &\leq C(n), \quad 1 \leq j \leq J, \quad 0 \leq x \leq 1, \end{aligned}$$

where the constants $A(n) > 0$ and $C(n) > 0$ are those provided by the John–Nirenberg theorem for $\text{BMOA}(B_n)$. Therefore,

$$\begin{aligned} C(n) &\geq \int_{\partial B_n} \frac{1}{J} \sum_{j=1}^J \int_0^1 \exp(\delta |F_{j,x}(\varphi^*(\zeta))|) dx d\sigma_n(\zeta) \\ &\geq \int_{\partial B_n} \exp\left(\frac{\delta}{J} \sum_{j=1}^J \int_0^1 |F_{j,x}(\varphi^*(\zeta))| dx\right) d\sigma_n(\zeta) \\ &\geq \int_{\partial B_n} \exp\left(\frac{\tau\delta}{J} \sqrt{\log \frac{1}{1 - |\varphi^*(\zeta)|^2}}\right) d\sigma_n(\zeta) \end{aligned}$$

by Fubini's theorem, Jensen's inequality and Lemma 3.2. $\square$

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