A TREE CHARACTERIZATION OF THE POINT OF CONTINUITY PROPERTY IN GENERAL BANACH SPACES

GINÉS LÓPEZ PÉREZ AND JOSÉ ANTONIO SOLER ARIAS

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ABSTRACT. We obtain a characterization of the point of continuity property for general bounded subsets in Banach spaces in terms of trees. For this we introduce the notion of a topologically weakly null tree and, as a consequence, we get that a general Banach space satisfies the point of continuity property if, and only if, every seminormalized topologically weakly null tree has a boundedly complete branch.

1. INTRODUCTION

A bounded subset K of a Banach space has the point of continuity property (PCP) if every nonempty closed subset admits a point of continuity for the identity map from the weak to norm topologies. From [1] it is known that K fails PCP iff there exists a countable set $C \subset K$ and $\delta > 0$ so that $\text{diam}(U) > \delta$ for all nonempty relatively open $U \subset C$.

A Banach space X has the PCP if its closed unit ball has the PCP. Furthermore, from [3] and [5], a separable X has the PCP iff it admits a boundedly complete skipped blocking finite dimensional decomposition (BCSBFDD) $(F_j)_{j=1}^\infty$. This means that $X = [F_j : j \in \mathbb{N}]$, $F_j \cap [F_k : k \neq j] = \{0\}$ for all j and if (x_n) is any seminormalized skipped block sequence w.r.t. (F_j) , then (x_j) is a boundedly complete basic sequence. (x_j) is skipped w.r.t. (F_j) means that for some $p_1 < p_2 < \dots$, $x_j \in [F_i : p_j < i < p_{j+1}]$. For $S \subseteq X$, $[S]$ denotes the closed linear span of S .

Much has been written about the PCP, and we refer the reader to [1], [3] and [5] for further background.

In this paper we prove

Theorem A. *Let K be a bounded subset of a Banach space X . The following are equivalent:*

- 1) K fails the PCP.
- 2) *There exists a seminormalized topologically weakly null tree $\{x_A\}_{A \in [\mathbb{N}^{<\omega}]}$ in X such that*

$$\left\{ \sum_{B \leq A} x_B : A \in [\mathbb{N}]^{<\omega} \right\} \subseteq K.$$

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Theorem B. *Let X be a Banach space. The following are equivalent:*

- i) X has the PCP.
- ii) Every seminormalized topologically weakly null tree in X admits a boundedly complete branch.

$[\mathbb{N}]^{<\omega}$ denotes the set of all finite sequences in $\mathbb{N}$ ordered by extension. A branch of a tree $\{x_A\}$ is a sequence $(x_\emptyset, x_{(n_1)}, x_{(n_1, n_2)}, \dots)$ for some sequence $(n_i)_{i=1}^\infty$ of positive integers. A tree $\{x_A\}$ is weakly null if $x_{A, n} \xrightarrow{\omega} 0$ as $n \rightarrow \infty$ for all $A \in [\mathbb{N}]^{<\omega}$. It is topologically weakly null if $0 \in \overline{\{x_{A, n}\}_{n=1}^\infty}^\omega$.

Theorem B was known in the case that X^* is separable [4] or more generally if X contains no isomorph of ℓ_1 [6], where in ii) the word “topologically” is deleted. If X does not contain ℓ_1 , then any sequence with 0 in its weak closure contains a subsequence that is weakly null (see [2] and [8]). It follows that any topologically weakly null tree in a space not containing ℓ_1 admits a full subtree that is weakly null. Hence Theorem B yields these earlier results and extends them to a general characterization.

2. THE PROOFS

Proof of Theorem A. 1) $\Rightarrow$ 2). If K fails the PCP there exist $\delta > 0$ and a countable set $C \subseteq K$ with $\text{diam } U > 2\delta$ for all relatively open nonempty $U \subseteq C$. Let $y_\emptyset \in C$. Then $y_\emptyset \in \overline{C \setminus B(y_\emptyset, \delta)}^\omega$, where $B(y_\emptyset, \delta)$ is the closed δ -ball centered at $y_\emptyset$. Order $C \setminus B(y_\emptyset, \delta)$ as $(y_{(n)})_{n=1}^\infty$. Note that $y_\emptyset \in \overline{\{y_{(n)}\}}^\omega$. We then repeat this step for each $y_{(n)}$ to obtain $y_{(n, m)}$'s and so on for each y_A once it is defined.

Set $x_\emptyset = y_\emptyset$ and for $A \in [\mathbb{N}]^{<\omega}$, set $x_{(A, n)} = y_{(A, n)} - y_A$. It follows that $\|x_A\| > \delta$ for all A (we could have chosen $\|y_\emptyset\| > \delta$ to start) and $\{x_A\}$ is a topologically weakly null tree. Also for all $A \in [\mathbb{N}]^{<\omega}$,

$$\sum_{B \leq A} x_B = y_A \in K.$$

2) $\Rightarrow$ 1). Let $C = \{\sum_{B \leq A} x_B : A \in [\mathbb{N}]^{<\omega}\}$. Then C satisfies the criterion mentioned earlier for failing the PCP.

We next prove Theorem B. First we add two equivalent conditions:

- iii) Every seminormalized topologically weakly null tree $\{x_A\}$ in X satisfies

$$\sup \left\{ \sum_{j=1}^k x_{(n_1, \dots, n_j)} : k \in \mathbb{N}, n_1, n_2, \dots \in \mathbb{N} \right\} = \infty.$$

- iv) For every seminormalized topologically weakly null tree $\{x_A\}$ in X there exist integers $n_1, n_2, \dots \in \mathbb{N}$ with

$$\sup_k \left\| \sum_{j=1}^k x_{(n_1, \dots, n_j)} \right\| = \infty. \quad \square$$

Proof of Theorem B. ii) $\Rightarrow$ iv) $\Rightarrow$ iii) are trivial. iii) $\Rightarrow$ i) follows from Theorem A. i) $\Rightarrow$ ii) follows the fact that $[x_A : A \in [\mathbb{N}]^{<\omega}]$ is a separable space with the PCP and thus has a BCSBFDD (F_j) . Indeed it is easy to see that if $\{x_A\}$ is a seminormalized topologically weakly null tree, then some branch is a perturbation of a skipped block sequence w.r.t. (F_j) and hence is boundedly complete. $\square$

Remarks. **1.** It seems unlikely that given a seminormalized topologically weakly null tree one can select a full subtree which is topologically weakly null and basic. This is easily done if the original tree is weakly null. It remains open if X failing the PCP implies X contains a subspace Y , failing the PCP, with a basis.

2. Tree characterizations are often equivalent to certain games. In the infinite asymptotic game played in a separable space X , (I) chooses X_1 a finite co-dimensional subspace of X and (II) chooses x_1 in the unit sphere of X_1 . Plays alternate thusly forever.

Problem. Is X having the PCP equivalent to (I) having a winning strategy to force (II) to select a boundedly complete basic sequence?

This is the case if X^* is separable (see [7]).

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071-GRANADA, SPAIN

E-mail address: glopezp@ugr.es

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071-GRANADA, SPAIN

E-mail address: jasoler@ugr.es