QUASI-WANDERING SUBSPACES IN A CLASS OF REPRODUCING ANALYTIC HILBERT SPACES

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ABSTRACT. We prove that a class of reproducing analytic Hilbert spaces with $\mathcal{U}$ -invariant kernels on the unit ball has the quasi-wandering property for the d -shift operator tuple.

1. INTRODUCTION

For a given operator T on a separable Hilbert space H , a closed subspace M of H is called an *invariant* subspace for T if $TM \subset M$. For an invariant subspace M of H for T , the space $M \ominus TM$ is called the *wandering subspace* and $P_M TM^\perp$ the *quasi-wandering subspace* respectively for M , where and in what follows P_M denotes the projection onto a closed subspace M and $M^\perp = H \ominus M$, $TM^\perp = \{Tx : x \in M^\perp\}$.

For a subset E of H , we shall denote by $[E]_T$ the smallest invariant subspace of H for T containing E . In other words, $[E]_T$ is the norm-closed linear span of functions of the form $T^k \psi$ for $\psi \in E$ and $k = 0, 1, \dots$.

An operator T on a Hilbert space H has the *wandering property* (briefly, the W-property) if for each nontrivial invariant subspace M of H for T , $M = [M \ominus TM]_T$ and has the *quasi-wandering property* (briefly, the QW-property) if $M = [P_M TM^\perp]_T$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and H denote a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. There are many examples where the shift operator S , defined by

$$(Sf)(z) = zf(z), \quad f \in H, z \in \mathbb{D},$$

has the W-property. In [Beu], Beurling proved that if $M \neq 0$ is an invariant subspace of the Hardy space $H^2(\mathbb{D})$, then $M \ominus SM$ is a one-dimensional subspace spanned by an inner function η and

$$M = [\eta]_S = [M \ominus SM]_S.$$

Beurling's theorem has played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. However, despite the great development in these fields over the past fifty years, it is only recently that progress has been made in proving analogues for the other classical Hilbert spaces,

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the Dirichlet space and the Bergman space. In [Ric], Richter proved that the analogue of Beurling's theorem is true in the Dirichlet space $\mathcal{D}$. It is well known that the invariant subspace lattice of the Bergman space $L_a^2(\mathbb{D})$ is very complicated. In fact, the dimension of the wandering subspace $M \ominus SM$ can be an arbitrary positive integer or ∞ ([Hed]). However, a big breakthrough in the study of the analogue of Beurling's theorem on the Bergman space was made by A. Aleman, S. Richter and C. Sundberg ([ARS]). They proved that any invariant subspace M of the Bergman space $L^2(\mathbb{D})$ also has the W-property.

The quasi-wandering subspace was studied recently by Izuchi in [Izu]. Let M be a nontrivial invariant subspace of $H^2(\mathbb{D})$ for S . Since in this case S is an isometry, i.e., $S^*S = 1$, then one easily sees that $P_M SM^\perp \subset M \ominus SM$. On the other hand, it is easy to check (see [Izu]) that

$$(1.1) \quad \dim M \ominus SM \leq \dim P_M SM^\perp.$$

Thus we have

$$P_M SM^\perp = M \ominus SM.$$

So in the one-variable Hardy space case, a quasi-wandering subspace coincides with a wandering subspace for M . It is not the case in the Bergman space although (1.1) still holds. However, in [Izu] it is shown that the shift operator on the Bergman space $L_a^2(\mathbb{D})$ has the QW-property. This result is seen as a counterpart to the theorem of Aleman-Richter-Sundberg.

We say that a closed subspace M of Hilbert space H is invariant for a d -tuple of operators $\bar{T} = (T_1, \dots, T_d)$, if $T_i M \subset M$ for all $i = 1, \dots, d$. Set

$$P_M \bar{T} M^\perp = P_M T_1 M^\perp + \dots + P_M T_d M^\perp.$$

We say that an operator tuple $\bar{T} = (T_1, \dots, T_d)$ has the QW-property if for each nontrivial invariant subspace M of H for $\bar{T}$, $M = [P_M \bar{T} M^\perp]_{\bar{T}}$.

Motivated by the work of [Izu], in this paper we will consider the QW-property for a d -tuple of operators $\bar{S} = (S_1, \dots, S_d)$, called d -shift, on a reproducing analytic Hilbert space H_d^k on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ defined by a $\mathcal{U}$ -invariant kernel k . Here

$$(S_i f)(z) = z_i f(z), \quad f \in H_d^k, z = (z_1, \dots, z_d) \in \mathbb{B}_d,$$

and a reproducing kernel $k : \mathbb{B}_d \times \mathbb{B}_d \rightarrow \mathbb{C}$ with $k(\cdot, 0) \equiv 1$ is $\mathcal{U}$ -invariant if for each unitary map $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$, $k(Uz, Uw) = k(z, w)$ for all $z, w \in \mathbb{B}_d$. We find a class of reproducing kernel Hilbert spaces of analytic functions on $\mathbb{B}_d$, which includes many of the classic Hilbert spaces, such as the Dirichlet space and the weighted Bergman spaces, whose shift operators have the QW-property.

In order to state our main result, let us introduce two kinds of reproducing kernels. Fix a constant v with $v > 0$. Let $H_d^{k_v}$ be the Hilbert space on the unit ball defined by the following $\mathcal{U}$ -invariant reproducing kernel:

$$k_v(z, w) = \frac{1}{(1 - \langle z, w \rangle)^v}.$$

Note that when $v = 1$, $H_d^{k_v}$ is the Arveson space H_d^2 , which is studied by Arveson in [Arv]. When $v = d$, $H_d^{k_v}$ is the usual Hardy space $H^2(\mathbb{B}_d)$, and when $v > d$, $H_d^{k_v}$ is the weighted Bergman space $L_a^2(\mathbb{B}_d, (1 - |z|^2)^{v-d-1} dV)$. Here dV denotes the normalized volume measure on $\mathbb{B}_d$.

Recall that a reproducing kernel k is called a complete Nevanlinna-Pick kernel if $k(\cdot, 0) \equiv 1$ and if $1 - 1/k(z, w)$ is positive definite on $\mathbb{B}_d \times \mathbb{B}_d$. For the related

details one can refer to [AMcC]. Note that the Arveson space H_d^2 and the Dirichlet space $\mathcal{D}_d$ defined by the kernel

$$k(z, w) = -\frac{1}{\langle z, w \rangle} \log(1 - \langle z, w \rangle)$$

have $\mathcal{U}$ -invariant complete Nevanlinna-Pick kernels.

The following is our main result:

Theorem 1.1. *If $k = k_v$ or k is a $\mathcal{U}$ -invariant complete Nevanlinna-Pick kernel, then the d -shift $\bar{S}$ on H_d^k has the QW-property.*

We point out in passing that not on all Hilbert spaces H_1^k with the kernels stated in Theorem 1.1 do the shift operators S have the W-property. H. Hedenmalm and K. Zhu ([HZh]) showed that the W-property can fail in certain weighted Bergman spaces $L_a^2(\mathbb{D}, (1-|z|^2)^v dA)$ for some $v > 1$, where dA is the normalized area measure on $\mathbb{D}$. In addition, for a $\mathcal{U}$ -invariant complete NP kernel $k(z, w) = (1 - \frac{1}{17}z\bar{w} - \frac{16}{17}z^2\bar{w}^2)^{-1}$ with $z, w \in \mathbb{D}$, S has no W-property on the induced Hilbert space H_1^k ([McCT]). So the QW-property is quite different from the W-property.

2. THE PROOF OF THE MAIN RESULT

We first recall some properties concerning the $\mathcal{U}$ -invariant reproducing analytic Hilbert space H_d^k . For more details one can refer to [GHX]. A routine verification shows that for a $\mathcal{U}$ -invariant kernel k on $\mathbb{B}_d$, there exists a unique power series $\sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ with nonnegative coefficients $\{a_k\}$ satisfying

$$(2.1) \quad k(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n, \quad z, w \in \mathbb{B}_d.$$

Here as usual $\langle z, w \rangle = \sum_{i=1}^d z_i \bar{w}_i$. Conversely, it is easy to check that if a power series $\sum_{n=0}^{\infty} a_n z^n$ has nonnegative coefficients, then k defined by (2.1) defines a $\mathcal{U}$ -invariant reproducing kernel on $\mathbb{B}_d$. In addition,

$$(2.2) \quad \left\{ \sqrt{a_{|\alpha|}} \frac{|\alpha|!}{\alpha!} z^\alpha : a_{|\alpha|} \neq 0 \right\}$$

forms a canonical orthonormal basis for the Hilbert space H_d^k . Here $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index of nonnegative integers and

$$|\alpha| = \alpha_1 + \dots + \alpha_d, \quad \alpha! = \alpha_1! \dots \alpha_d!, \quad z^\alpha = z_1^{\alpha_1} \dots z_d^{\alpha_d}.$$

We assume that $a_n \neq 0$ for $n \geq 0$ and $\sup_n \frac{a_n}{a_{n+1}} < \infty$, so that the d -shift $\bar{S}$ is bounded on H_d^k .

Let k be as (2.1). Since $K(0, 0) = 1$, then we have

$$\frac{1}{k(z, w)} = \sum_{n=0}^{\infty} c_n \langle z, w \rangle^n$$

in some open neighborhood U of zero. The multinomial formula gives that

$$k(z, w) = \sum_{|\alpha|=0}^{\infty} a_{|\alpha|} \frac{|\alpha|!}{\alpha!} z^\alpha \bar{w}^\alpha, \quad \frac{1}{k(z, w)} = \sum_{|\beta|=0}^{\infty} c_{|\beta|} \frac{|\beta|!}{\beta!} z^\beta \bar{w}^\beta$$

for $(z, w) \in U$. Note that the coefficients $\{a_n\}$ and $\{c_n\}$ satisfy

$$(2.3) \quad c_0 a_0 = 1, \quad \sum_{\beta \leq \alpha} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} = 0, \quad \alpha \neq 0,$$

where and in what follows $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for all $i = 1, \dots, d$, and

$$\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d).$$

Note that $a_0 = 1$, so $c_0 = 1$, and $c_1 = -c_0 a_1 / a_0 = -a_1 < 0$. In addition, $\{c_n\}$ is a sequence of real numbers.

For the d -shift $\bar{S} = (S_1, \dots, S_d)$ on the Hilbert space H_d^k , we set

$$Q_n = \sum_{|\beta| \leq n} c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^\beta \bar{S}^{*\beta}, \quad Q = \sum_{|\beta|=0}^{\infty} c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^\beta \bar{S}^{*\beta},$$

where

$$\bar{S}^\beta = S_1^{\beta_1} \dots S_d^{\beta_d}, \quad \bar{S}^{*\beta} = S_1^{*\beta_1} \dots S_d^{*\beta_d}.$$

Proposition 2.1. *For every polynomial p ,*

$$Q(p) = p(0).$$

Hence, Q can be extended continuously on H_d^k as the evaluation functional at zero.

Proof. For every monomial z^α , using the reproducing property of k , we get

$$\begin{aligned} S_i^{*\beta_i} [z^\alpha](z) &= \langle S_i^{*\beta_i} w^\alpha, k(\cdot, z) \rangle_{H_d^k} = \langle w^\alpha, w_i^{\beta_i} k(\cdot, z) \rangle_{H_d^k} \\ &= a_{|\alpha-\beta_i e_i|} \langle w^\alpha, w^\alpha \rangle_{H_d^k} \frac{|\alpha-\beta_i e_i|!}{(\alpha_i-\beta_i)!} z^{\alpha-\beta_i e_i} \end{aligned}$$

for $\beta_i \leq \alpha_i$, where $\alpha - \beta_i e_i = (\alpha_1, \dots, \alpha_i - \beta_i, \dots, \alpha_d)$. Then by (2.2),

$$S_i^{*\beta_i} z^\alpha = \frac{\alpha_i!}{|\alpha|!} \frac{|\alpha-\beta_i e_i|!}{(\alpha_i-\beta_i)!} \frac{a_{|\alpha-\beta_i e_i|}}{a_{|\alpha|}} z^{\alpha-\beta_i e_i}$$

for $\beta_i \leq \alpha_i$ and $S_i^{*\beta_i} z^\alpha = 0$ for $\beta_i > \alpha_i$. Hence, the same arguments give that

$$(2.4) \quad \bar{S}^\beta \bar{S}^{*\beta}(z^\alpha) = \begin{cases} \frac{\alpha!}{|\alpha|!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \frac{a_{|\alpha-\beta|}}{a_{|\alpha|}} z^\alpha, & \beta \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $Q(1) = c_0 = 1$ and for every $\alpha \neq 0$, we have

$$\begin{aligned} Q(z^\alpha) &= \sum_{|\beta|=0}^{\infty} c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^\beta \bar{S}^{*\beta}(z^\alpha) = \sum_{\beta \leq \alpha} c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^\beta \bar{S}^{*\beta}(z^\alpha) \\ &= \sum_{\beta \leq \alpha} c_{|\beta|} \frac{|\beta|!}{\beta!} \frac{\alpha!}{|\alpha|!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \frac{a_{|\alpha-\beta|}}{a_{|\alpha|}} z^\alpha \\ &= \frac{\alpha!}{|\alpha|! a_{|\alpha|}} z^\alpha \sum_{\beta \leq \alpha} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} = 0 \end{aligned}$$

by (2.3). Thus, for every polynomial p , we obtain $Qp = p(0)$, as desired. $\square$

The following lemma gives a sufficient condition for when Q_n converges to Q in SOT.

Lemma 2.2. *If there exists a positive integer $N(\geq 2)$ such that $c_i \geq 0$ or $c_i \leq 0$ for all $i > N$, then $Q_n \rightarrow Q$ (SOT).*

Proof. Fix each $f(z) = \sum_{|\alpha|=0}^{\infty} b_{\alpha} z^{\alpha}$. It follows from (2.3) and (2.4) that we have

$$\begin{aligned} Q_n f - Qf &= \sum_{|\alpha|=1}^{\infty} b_{\alpha} \sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^{\beta} \bar{S}^{*\beta}(z^{\alpha}) \\ &= \sum_{|\alpha| > n} b_{\alpha} \sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^{\beta} \bar{S}^{*\beta}(z^{\alpha}) \\ &= \sum_{|\alpha| > n} b_{\alpha} z^{\alpha} \frac{\alpha!}{|\alpha|! a_{|\alpha|}} \sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!}. \end{aligned}$$

Hence we get

(2.5)

$$\|Q_n f - Qf\|_{H_d^k}^2 = \sum_{|\alpha| > n} |b_{\alpha}|^2 \|z^{\alpha}\|_{H_d^k}^2 \left[\frac{\alpha!}{|\alpha|! a_{|\alpha|}} \sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \right]^2.$$

We claim that there is a constant $C > 0$ such that

$$(2.6) \quad \left| \frac{\alpha!}{|\alpha|! a_{|\alpha|}} \sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \right| \leq C$$

for $|\alpha| \geq N$ and $n \geq N$. Indeed, we first assume that when $n > N$ all $c_n \leq 0$. Note that $c_0 = 1, c_1 < 0$. It follows from (2.3) that

$$\sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \geq 0$$

for $|\alpha| \geq N$ and $n \geq N$, which gives

$$\begin{aligned} & \left| \frac{\alpha!}{|\alpha|! a_{|\alpha|}} \sum_{\beta \leq \alpha, |\beta| \leq n} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \right| \\ & \leq \frac{\alpha!}{|\alpha|! a_{|\alpha|}} \sum_{\beta \leq \alpha, |\beta| \leq N} c_{|\beta|} a_{|\alpha-\beta|} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{(\alpha-\beta)!} \\ & = \sum_{\beta \leq \alpha, |\beta| \leq N} c_{|\beta|} \frac{a_{|\alpha-\beta|}}{a_{|\alpha|}} \frac{|\beta|!}{\beta!} \frac{|\alpha-\beta|!}{|\alpha|!} \frac{\alpha!}{(\alpha-\beta)!} \\ & \leq \max_{|\beta| \leq N} \frac{|\beta|!}{\beta!} |c_{|\beta|}| \sup_{\beta \leq \alpha, |\beta| \leq N} \frac{a_{|\alpha-\beta|}}{a_{|\alpha|}} \sup_{\beta \leq \alpha, |\beta| \leq N} \frac{\alpha!}{(\alpha-\beta)!} \end{aligned}$$

for $|\alpha| \geq N$ and $n \geq N$. Note that $\sup_n \frac{a_n}{a_{n+1}} < \infty$; thus it is easy to see that the claim follows. For the case when for $n > N$ all $c_n \geq 0$, the reasoning of the claim is similar.

It follows from (2.5) and (2.6) that

$$\|Q_n f - Qf\|_{H_d^k}^2 \leq C^2 \sum_{|\alpha| > n} |b_{\alpha}|^2 \|z^{\alpha}\|_{H_d^k}^2 \rightarrow 0$$

as $n \rightarrow \infty$ because of $\|f\|_{H_d^k}^2 = \sum_{|\alpha|=0}^{\infty} |b_{\alpha}|^2 \|z^{\alpha}\|_{H_d^k}^2 < \infty$. This means that Q_n converges to Q in SOT. The proof is complete. $\square$

The following lemma is also useful in our proof.

Lemma 2.3. *Let M be a closed subspace of a Hilbert space H and $\bar{T} = (T_1, \dots, T_d)$ be a d -tuple of commuting operators acting on H . Then*

$$M \ominus [P_M \bar{T} M^\perp]_{\bar{T}} = H \ominus [M^\perp]_{\bar{T}}.$$

In particular, $[P_M \bar{T} M^\perp]_{\bar{T}} = M$ if and only if $[M^\perp]_{\bar{T}} = H$.

Proof. Suppose $f \in H \ominus [M^\perp]_{\bar{T}}$. Then

$$f \perp T_1^{i_1} \cdots T_d^{i_d} M^\perp$$

for all $i_1, \dots, i_d \geq 0$. Obviously $f \in M$. Since

$$\begin{aligned} T_1^{i_1} \cdots T_d^{i_d} P_M T_j M^\perp &\subset T_1^{i_1} \cdots T_d^{i_d} T_j M^\perp - T_1^{i_1} \cdots T_d^{i_d} P_{M^\perp} T_j M^\perp \\ &\subset T_1^{i_1} \cdots T_d^{i_d} T_j M^\perp - T_1^{i_1} \cdots T_d^{i_d} M^\perp, \end{aligned}$$

we have $f \perp T_1^{i_1} \cdots T_d^{i_d} P_M T_j M^\perp$ for all $i_1, \dots, i_d \geq 0$ and $1 \leq j \leq d$. Hence $f \in M$ and $f \perp [P_M \bar{T} M^\perp]_{\bar{T}}$, that is, $f \in M \ominus [P_M \bar{T} M^\perp]_{\bar{T}}$.

For the converse, suppose $f \in M \ominus [P_M \bar{T} M^\perp]_{\bar{T}}$. Then

$$(2.7) \quad f \perp T_1^{i_1} \cdots T_d^{i_d} P_M T_j M^\perp, \quad 1 \leq j \leq d, i_1, \dots, i_d \geq 0.$$

Noting that

$$\begin{aligned} T_1^{i_1} \cdots T_d^{i_d} M^\perp &\subset T_1^{i_1} \cdots T_d^{i_d-1} P_M T_d M^\perp + T_1^{i_1} \cdots T_d^{i_d-1} P_{M^\perp} T_d M^\perp \\ &\subset T_1^{i_1} \cdots T_d^{i_d-1} P_M T_d M^\perp + T_1^{i_1} \cdots T_d^{i_d-1} M^\perp, \end{aligned}$$

by induction, we have

$$\begin{aligned} T_1^{i_1} \cdots T_d^{i_d} M^\perp &\subset \sum_{i=1}^{i_d} T_1^{i_1} \cdots T_d^{i_d-i} P_M T_d M^\perp \\ &\quad + \sum_{i=1}^{i_d-1} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}-i} P_M T_{d-1} M^\perp + \cdots \\ &\quad + \sum_{i=1}^{i_1} T_1^{i_1-i} P_M T_1 M^\perp + M^\perp. \end{aligned}$$

From (2.7) we obtain that $f \perp T_1^{i_1} \cdots T_d^{i_d} M^\perp$ for every $i_1, \dots, i_d \geq 0$. So $f \in H \ominus [M^\perp]_{\bar{T}}$, and the proof is complete. $\square$

We are now ready to prove the main result.

Proof of Theorem 1.1. We first claim that if $Q_n \rightarrow Q$ (SOT or WOT), then $\bar{S}$ has the QW-property on H_d^k . In fact, if the claim is not true, then there is a nontrivial invariant subspace M of H_d^k for $\bar{S}$ such that $M \neq [P_M \bar{S} M^\perp]_{\bar{S}}$. Thus by Lemma 2.3, there exists $f \in M$ with $f \neq 0$ such that $f \perp S_1^{i_1} \cdots S_d^{i_d} M^\perp$ for every $i_1, \dots, i_d \geq 0$. It follows that $\bar{S}^{*\alpha} f \in M$ for all multi-indexes α .

We write $f(z) = \sum_{|\alpha|=0}^\infty b_\alpha z^\alpha$ with some $b_{\alpha'} \neq 0$ and let $h = \bar{S}^{*\alpha'} f$. Then $h \in M$ and $\bar{S}^{*\alpha} h \in M$ for every multi-index α , so $\bar{S}^\alpha \bar{S}^{*\alpha} h \in M$ for every α . Note that M is strongly (and weakly) closed. Hence by the condition and Proposition 2.1,

$$\sum_{|\beta|=0}^\infty c_{|\beta|} \frac{|\beta|!}{\beta!} \bar{S}^\beta \bar{S}^{*\beta} h = h(0) = b_{\alpha'} \in M,$$

which implies that $M = H_d^k$, a contradiction. So the claim holds.

Now we only need to show that the two kinds of reproducing kernels in the theorem satisfy the condition of Lemma 2.2. In fact, if $k = k_v$ and $\frac{1}{k_v(z,w)} = 1 + \sum_{n=1}^{\infty} c_n \langle z, w \rangle^n$, then

$$c_n = (-1)^n \frac{v(v-1) \cdots (v-n+1)}{n!}, \quad n \geq 1.$$

It is easy to see that there exists a positive integer $N_v (\geq 2)$ such that $c_n \geq 0$ or $c_n \leq 0$ for all $n > N_v$. If k is a complete Nevanlinna-Pick kernel and $\frac{1}{k(z,w)} = 1 + \sum_{n=1}^{\infty} c_n \langle z, w \rangle^n$, then [AMcC, Theorem 7.33] tells us that all $c_n \leq 0$ for $n \geq 1$. The proof is complete. $\square$

Remark 2.4. Let $H_0 = \{f \in H_d^k : f(0) = 0\}$ and P_0 be a projection onto H_0 . Then

$$P_0 = I - Q = \sum_{|\beta|=1}^{\infty} (-c_{|\beta|}) \frac{|\beta|!}{\beta!} \bar{S}^{\beta} \bar{S}^{*\beta},$$

where I is the identity operator on H_d^k . If k is a complete Nevanlinna-Pick kernel, then $c_{|\beta|} \leq 0$ for all $|\beta| \geq 1$. Therefore, the right side of the above equality is the infinite sum of positive operators and so it is easy to see that the sum converges in WOT (also see [McCT, Lemma 1.4]).

The following corollary is a direct application of the main result. We call a d -tuple of operators $\bar{S}^* = (S_1^*, \dots, S_d^*)$ the backward d -shift on H_d^k . Thus M is an invariant subspace for the d -shift $\bar{S}$ if and only if $M^{\perp}$ is an invariant subspace for $\bar{S}^*$. So the following corollary tells us that two nontrivial invariant subspaces of H_d^k for the backward d -shift $\bar{S}^*$ are never orthogonal, a result which generalizes Theorem 1 of [Gar] using different techniques.

Corollary 2.5. *Let M_1 and M_2 be two invariant subspaces for $\bar{S}$ of H_d^k with $k = k_v$ or k is a $\mathcal{U}$ -invariant complete Nevanlinna-Pick kernel. If $M_1^{\perp} \perp M_2^{\perp}$, then $M_1 = H_d^k$ or $M_2 = H_d^k$.*

Proof. If $M_1 \neq H_d^k$, then by Theorem 1.1 and Lemma 2.3, $[M_1^{\perp}]_{\bar{S}} = H_d^k$. Since the condition means that $M_1^{\perp} \subset M_2$, we have $M_2 = H_d^k$, and the proof follows. $\square$

We finish this paper with the following natural question:

Question 2.6. Does the d -shift $\bar{S}$ on each reproducing analytic Hilbert space H_d^k with $\mathcal{U}$ -invariant kernel k have the QW-property?

For a general operator T rather than the shift operator S , the question is not the case. For example, let $T = S^2$ and $M = H_0$, defined as in Remark 2.4. Then one easily sees that

$$M \ominus [P_M S^2 M^{\perp}]_{S^2} = \overline{\text{span}}\{z, z^3, z^5, \dots\} \neq \{0\}.$$

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