HILBERT-PERELMAN'S FUNCTIONAL AND LAGRANGE MULTIPLIERS

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(Communicated by Varghese Mathai)

ABSTRACT. We use the Hilbert-Perelman functional over a suitable subspace of its domain and rederive the extremal Kähler flow equation by using the ensuing Lagrange multipliers. We modify the said functional appropriately to present this pseudo-differential equation as a gradient flow and, as a consequence, to show that some of Perelman's monotonicity results hold in this context also.

1. INTRODUCTION

One of the most prominent technical contributions of G. Perelman to the analysis of the Ricci flow was the introduction of monotonicity considerations [15]. Certain geometric quantities associated to the evolving metrics are nondecreasing. On the basis of that, by using gauge transformations in the same role that test functions play in distribution theory, Perelman obtained his nonlocal volume collapsing theorem I, the first of his major contributions to Hamilton's approach to the geometrization conjecture.

If we take $(s_g/n)g - r_g$ as the velocity vector of a path of evolving metrics, r_g and s_g the Ricci and scalar curvature, respectively, we would obtain a backwards heat equation for the evolution of the scalar curvature s_g . In defining his Ricci flow, Hamilton [10] resolved this problem in an ad hoc manner, replacing the velocity above by the tensor $(\bar{s}_g/n)g - r_g$ instead, where now $\bar{s}_g$ is the average of s_g . In making this change, the right side of his famous geometric flow lost the property of being the gradient of the Hilbert functional under variations of the metric. By an appropriate modification of the Hilbert functional using a gauge changing function, Perelman showed how to gain this property back. The resulting gauge-modified Ricci flow became a gradient-like flow, and this makes the evolution of certain geometric quantities associated to the evolving metrics monotonic in time.

In the Kähler context, the Ricci flow may be used when seeking Kähler-Einstein metrics representing the first Chern class, a priori assumed to be signed. The resulting flow equation has a solution defined for all time [5]. When seeking canonical representatives of more general Kähler classes, the *extremal* metrics of Calabi [3, 4], it is necessary to use the *extremal heat flow* instead [18]. If we seek constant scalar curvature representatives of the class, it is perhaps more natural to use the recently defined *pseudo-Calabi flow* [7]. Both of these two other flows had been presented in an ad hoc manner also, although at least the monotonicity of the (generalized) K -energy [16] has been shown to hold. Technically speaking, they differ from

Received by the editors September 20, 2010.

2010 *Mathematics Subject Classification.* Primary 53C55, 58E11; Secondary 53C21.

the Ricci flow in that they are genuine nonlinear pseudo-differential equations. The main goal of the present note is to show that if we make use of the Hilbert-Perelman functional over a suitable subspace, the resulting Lagrange multipliers allow us to derive the extremal and pseudo-Calabi flows from a variational principle. A suitable modification of the functional allows us to present them as gradient flows also. As a consequence, several of the monotonicity results of Perelman apply (almost) verbatim to these pseudo-differential flows.

2. THE EXTREMAL FLOW

Let (M, J, Ω) be an n -dimensional closed complex manifold of Kähler type polarized by a positive class Ω . Let g be a Kähler metric on M ; thus, $\omega(X, Y) := g(JX, Y)$ is a closed form. We call ω the *Kähler form* and the cohomology class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ the *Kähler class* of the metric g , respectively. Since g can be recovered from ω , usually we refer to the two as equivalent to each other. Thus, we may reexpress the polarizing assumption on the class Ω by saying that there exist Kähler metrics ω such that $\Omega = [\omega]$. We let $\mathfrak{M}_\Omega$ be the set of all Kähler metrics whose Kähler forms represent Ω .

Since the group of diffeomorphisms of M acts naturally on the space of Riemannian metrics on M , we normally study the space of metrics modulo this action. The group of diffeomorphisms also acts on the space of complex structures. In the study of complex manifolds, it is of additional importance to further distinguish the elements of the group of diffeomorphisms that act trivially on J , that is to say, that fix the complex structure. In order to explain the details of this, we introduce some basic definitions.

Definition 1. A complex deformation of J is a family of complex structures J_t such that $J_{t_0} = J$ for some particular value of the parameter. The complex deformation is said to be trivial if, and only if, J_t is isomorphic to J for each value of the parameter t .

An almost complex structure is complex if its Nijenhuis tensor N_J vanishes identically [14]. Thus, an infinitesimal complex deformation of J is given by a tensor T of type $(1, 1)$ that satisfies the relations

$$TJ + JT = 0, \quad (DN)_J(T) = 0,$$

where $(DN)_J$ is the Fréchet derivative of N at J .

Given a vector field X on M , we denote by $\mathcal{L}_X$ the Lie derivative in the direction of X .

Definition 2. An infinitesimal complex deformation of the form $T = \mathcal{L}_X J$ for some vector field X is said to be trivial. Given a metric g , an infinitesimal complex deformation T is said to be essential if, and only if, T is orthogonal to the space of trivial deformations.

Given a trivial infinitesimal deformation that is integrable, the resulting complex deformation J_t of J will be given by a one-parameter group of diffeomorphisms.

Among all trivial infinitesimal deformations $\mathcal{L}_X J$ of J , we have those given by vector fields X that are holomorphic. If for a given Kähler metric g we have that $X = \nabla_g f$ is holomorphic, then this vector field will leave J unchanged. Thus, when studying Kähler metrics, we disregard trivial deformations and consider only those

diffeomorphisms generated by the gradient of functions f that are J -invariant and orthogonal to the space of holomorphy potentials. We describe this next.

Let us fix a maximal compact subgroup G of the automorphism group of (M, J) . We denote its Lie algebra by $\mathfrak{g}$, and the center of this algebra by $\mathfrak{z}$. We define $\mathfrak{M}_{\Omega, G}$ to be the space of all G -invariant elements of $\mathfrak{M}_{\Omega}$. Given any $g \in \mathfrak{M}_{\Omega, G}$, we let G_g be the Green's operator of g acting on functions, and let $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$, where $\mathfrak{g}_0 \subset \mathfrak{g}$ is the ideal of Killing fields with zeroes. If $\{X_1, \dots, X_m\}$ is a basis for $\mathfrak{z}_0$, the set of functions,

$$(1) \quad \begin{aligned} p_0(\omega_g) &= 1, \\ p_j(\omega_g) &= 2iG_g\bar{\partial}_g^*((JX_j + iX_j) \lrcorner \omega_g), \quad j = 1, \dots, m, \end{aligned}$$

spans the space of G -invariant real-holomorphy potentials, real-valued functions whose gradients are holomorphic vector fields. We define π_g to be the L^2 -projection onto this space and Π_g to be the natural lift of this projection to the level of $(1, 1)$ -forms (see [16, Proposition 1]).

Definition 3. Given a metric $g \in \mathfrak{M}_{\Omega, G}$, the space of holomorphy potentials $\mathcal{H}_g$ is the subspace of $C^\infty(M)$ spanned by the functions in (1).

The space of holomorphy potentials $\mathcal{H}_g$ constitutes the range of the projection operator π_g , and the one-dimensional subspace of constants is always contained in it.

We now turn our attention to the problem of finding canonical representatives of the polarizing class Ω . These are the critical points of the squared L^2 -norm of the scalar curvature, defined as a functional over $\mathfrak{M}_{\Omega}$. Such metrics have come to be known as extremal, after the term coined by Calabi [2, 3], who characterized them by the fact that the gradient of their scalar curvature s_g is a holomorphic vector field. We have stated, and advocated [17, 18, 20], that the condition for a Kähler metric g to be extremal can be formulated equivalently by the condition

$$(2) \quad s_g = \pi_g s_g \iff \rho_g = \Pi_g \rho_g,$$

where ρ_g is the Ricci form of g . Here, the group G in use arises from the fact that extremal metrics must be invariant under a maximal compact subgroup of $\text{Aut}(M, J)$, as proven by Calabi [4], and that all such are conjugate to each other, as observed in [12]. Thus, in the search for extremal representatives of Ω , it suffices to fix one maximal compact subgroup G of $\text{Aut}(M, J)$ and look for this type of metric among the elements of $\mathfrak{M}_{\Omega, G}$, the G -invariant elements of $\mathfrak{M}_{\Omega}$. This characterization of the extremal condition that then (2) provides sets the concept in a manner analogous to the way Einstein metrics are defined, making it possible to use techniques developed for their analysis in the study of extremal metrics.

We define $C_G^\infty(M)$ to be the space of smooth functions that are invariant under G . Then we have the following elementary result:

Lemma 4. *The space $\mathfrak{M}_{\Omega, G}$ is an infinite-dimensional manifold whose tangent space at g is given by*

$$T_g \mathfrak{M}_{\Omega, G} = i\partial\bar{\partial} \left\{ \varphi \in C_G^\infty(M) : \int \varphi \psi d\mu_g = 0 \text{ for all } \psi \in \mathcal{H}_g \right\}.$$

Remark 5. When G is discrete, we recover the commonly used characterization of $\mathfrak{M}_{\Omega, G}$, with its tangent space at g consisting of functions that are g -orthogonal to the constants.

Let us consider the larger spaces $\mathfrak{M}_G$ of all Kähler metrics that are invariant under G , and $\mathfrak{M}_{G,v}$, the subset of metrics in $\mathfrak{M}_G$ of volume v . By the Hodge decomposition theorem, we may identify the cohomology group $H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$ with the space $\mathcal{H}_g^{1,1}(M)$ of real-valued g -harmonic $(1, 1)$ -forms on M . We have that

Lemma 6. *The space $\mathfrak{M}_G$ is an infinite-dimensional manifold whose tangent space at g is given by*

$$T_g \mathfrak{M}_G = \mathcal{H}_{g,0}^{1,1} \oplus \mathbb{R} \omega_g \oplus i\partial\bar{\partial} \left\{ \varphi \in C_G^\infty(M) : \int \varphi \psi d\mu_g = 0 \text{ for all } \psi \in \mathcal{H}_g \right\}.$$

Here, ω_g is the Kähler form of g , and $\mathcal{H}_{g,0}^{1,1}$ is the subspace of trace-free g -harmonic $(1, 1)$ -forms. Further,

$$T_g \mathfrak{M}_{G,v} = \mathcal{H}_{g,0}^{1,1} \oplus i\partial\bar{\partial} \left\{ \varphi \in C_G^\infty(M) : \int \varphi \psi d\mu_g = 0 \text{ for all } \psi \in \mathcal{H}_g \right\}.$$

Remark 7. Obviously, a fact that plays a role below, the orthogonal complement of $T_g \mathfrak{M}_{\Omega,G}$ in $T_g \mathfrak{M}_G$ consists of $\mathcal{H}_g^{1,1} \oplus i\partial\bar{\partial} \{ \varphi : \varphi \in C_G^\infty(M) \cap \mathcal{H}_g \}$.

2.1. The extremal flow as the gradient of the Hilbert-Perelman functional. Let us consider the functional

$$(3) \quad \begin{aligned} \mathfrak{M}_{\Omega,G} \times C_G^\infty(M) &\xrightarrow{\mathcal{F}} \mathbb{R} \\ (g, f) &\mapsto \int_M (s_g + |\nabla_g f|^2) e^{-f} d\mu_g, \end{aligned}$$

where $d\mu_g$ is the measure defined by the metric g . We use variational formulas [1] in order to analyze its deformation over suitable submanifolds of metrics.

Indeed, if a metric g is deformed infinitesimally in the direction of the symmetric two-tensor h , then the scalar curvature and volume form of the metric vary infinitesimally according to the expressions

$$(4) \quad \begin{aligned} \frac{d}{dt} s_{g+th} \big|_{t=0} &= \Delta_g(\text{trace } h) + \delta_g(\delta_g h) - (r_g, h)_g, \\ \frac{d}{dt} d\mu_{g+th} \big|_{t=0} &= \frac{1}{2} \text{trace } h d\mu_g. \end{aligned}$$

Here, δ_g is the divergence operator on symmetric tensors, and the g -trace of h is just the pointwise inner product $(g, h)_g$, which we call $\bar{h}$ below, for convenience. A simple dualization argument [15] yields that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}_{(g,f)}(h_{ij}, k) &= \int e^{-f} \left[\Delta_g \bar{h} + \delta_g \delta_g h - (r_g, h)_g - (h, \nabla_g f \otimes \nabla_g f)_g \right. \\ &\quad \left. + 2(\nabla_g f, \nabla_g k)_g + (s_g + |\nabla_g f|^2) \left(\frac{1}{2} \bar{h} - k \right) \right] d\mu_g \\ &= \int e^{-f} \left[- (h, r_g \right. \\ &\quad \left. + \nabla_g^2 f)_g + \left(\frac{1}{2} \bar{h} - k \right) (-2\Delta_g f - |\nabla_g f|^2 + s_g) \right] d\mu_g. \end{aligned}$$

Proposition 8. *The Euler-Lagrange equation of (3) over the subspace of $\mathfrak{M}_{\Omega,G} \times C_G^\infty(M)$ that fixes the measure $e^{-f} d\mu_g$ is given, up to the action of diffeomorphisms on both the metric and the complex structure, by the extremal equation*

$$\rho_g = \Pi_g \rho_g.$$

Proof. If the said measure is fixed, the only contribution to the variational formula above is given by

$$\frac{d}{dt} \mathcal{F}_{(g,f)}(h_{ij}, k) = - \int e^{-f} (h, r_g + \nabla_g^2 f)_g d\mu_g.$$

Since our metrics are Kähler, the $(1, 1)$ -tensor h is J -invariant. Hence, the Euler-Lagrange equation says that the J -invariant component of the tensor $r_g + \nabla_g^2 f$ must be orthogonal to $T_g \mathfrak{M}_{\Omega, G}$. The $(1, 1)$ -form associated to the J -invariant component of $\nabla_g^2 f$ is $i\partial\bar{\partial}f$, while we know that r_g is J -invariant. We now use Lemma 4 and Lemma 6 (see also Remark 7) to express the Euler-Lagrange equation in terms of $(1, 1)$ -forms. We conclude that

$$\rho_g + i\partial\bar{\partial}f = \rho_g^h + i\partial\bar{\partial}\psi,$$

for some $(1, 1)$ -form ρ_g^h that is harmonic, and some function ψ that is a holomorphy potential. Since for a harmonic form we have that $\Pi_g \rho_g^h = \rho_g^h$, we conclude that $\psi - f = -G_g(\pi_g s_g) - G_g((1 - \pi_g)\phi)$ for the G -invariant function $\phi = s_g$. Therefore,

$$\rho_g + i\partial\bar{\partial}G_g((1 - \pi_g)\phi) = \Pi_g \rho_g,$$

which represents a deformation of the equation (2) by the diffeomorphism generated by the gradient of $G_g((1 - \pi_g)\phi)$. This finishes the proof. $\square$

Corollary 9. *The extremal heat equation is given by the diffeomorphism modified extremal flow equation*

$$(5) \quad \frac{d\omega_g}{dt} = \Pi_g \rho_g - \rho_g - i\partial\bar{\partial}f,$$

where the function f evolves according to the backwards heat equation

$$(6) \quad \frac{df}{dt} = \frac{1}{2} \Delta_g f + \frac{1}{2} (s_g - \pi_g s_g).$$

Proof. Abusing notation, let us rename as f the function $G_g((1 - \pi_g)\phi)$ in the argument above. Then we have the diffeomorphism modified extremal equation

$$\frac{d\omega_g}{dt} = \Pi_g \rho_g - \rho_g - i\partial\bar{\partial}f,$$

as stated. Since the variations are taken over functions that fix the measure $e^{-f} d\mu_g$, we see that the generating function of the diffeomorphism evolves according to the equation

$$\frac{d}{dt} f = \frac{1}{2} \Delta_g f + \frac{1}{2} (\pi_g s_g - s_g),$$

a perturbation of the backward heat equation by the scalar factor $(\pi_g s_g - s_g)/2$. $\square$

Thus, the resulting gauged extremal equation changes the metric ω_g and the complex structure J by the one-parameter group of diffeomorphisms generated by the gradient of a function $f \in C_g^\infty(M)$, a function that solves the evolution equation (6).

Remark 10. Suppose that we remove the group G in all the considerations made above and change the role of $\mathcal{H}_g$ to the space spanned by the constants only. By varying the Hilbert-Perelman functional on the enlarged domain $\mathfrak{M}_\Omega \times C^\infty(M)$, we would then obtain the equation

$$\rho_g + i\partial\bar{\partial}G_g((1 - \bar{\pi}_g)\phi) = \rho_g^h,$$

where $\bar{\pi}_g$ is the orthogonal projection onto the constants. The resulting geometric flow is the pseudo-Calabi flow of Chen and Zheng [7]. It is the case that if Ω carries a constant scalar curvature representative, then G must be reductive [13]. So in the search for a constant scalar representative of Ω , we might want to use the pseudo-Calabi flow on G -invariant metrics for a reductive G .

The main point of rederiving our extremal flow [18] as the critical evolution equation of the functional (3) on a suitable subspace of metrics is to show that Perelman's monotonicity theorems also apply to it. As indicated above, these monotonicity results were derived by taking advantage of the gradient nature of the Ricci flow over the entire space of metrics, not just submanifolds of it. We overcome this seeming flaw in our presentation by modifying suitably the function (3).

Given a Kähler metric g , let $\Pi_g r_g$ be the symmetric two-tensor defined by $\Pi_g r_g(X, Y) = \Pi_g \rho_g(X, JY)$. Given a $(1, 1)$ -form α , let $g(t)$ denote the infinitesimal deformation of g given by $g(t) = g + th$, where $h(\cdot, \cdot) = \alpha(\cdot, J\cdot)$. We define the function p_g to be such that

$$\frac{d}{dt} p_{g(t)} \big|_{t=0} = -2(\Pi_g \rho_g, \alpha) = -(\Pi_g r_g, h).$$

Since $\pi_g s_g = 2(\Pi_g \rho_g, \omega_g)$, p_g is equal to $\pi_g s_g - f_g$, where f_g has Fréchet derivative given by $2(\frac{d}{dt}(\Pi_{g(t)} \rho_{g(t)}) \big|_{t=0}, \omega_g)$, and so, for instance, in the generic case when (M, J) carries no nontrivial holomorphic vector fields, it is easy to see that p_g coincides with $\pi_g s_g$. We now consider the functional

$$(7) \quad \begin{aligned} \mathfrak{M}_G \times C_G^\infty(M) &\xrightarrow{\tilde{\mathcal{F}}} \mathbb{R} \\ (g, f) &\mapsto \int_M (s_g - p_g + |\nabla_g f|^2) e^{-f} d\mu_g, \end{aligned}$$

whose derivative is given by

$$\begin{aligned} \frac{d}{dt} \tilde{\mathcal{F}}_{(g,f)}(h_{ij}, k) &= \int e^{-f} \left[-(h, r_g - \Pi_g r_g + \nabla_g^2 f)_g \right. \\ &\quad \left. + \left(\frac{1}{2} \bar{h} - k \right) (-2\Delta_g f - |\nabla_g f|^2 + s_g - p_g) \right] d\mu_g. \end{aligned}$$

Thus, the gradient flow of $\tilde{\mathcal{F}}$ among variations of (g, f) in $\mathfrak{M}_G \times C_G^\infty(M)$ that fix the measure $e^{-f} d\mu_g$ is precisely the extremal flow system (5), (6). Since ρ and $\Pi\rho$ represent the same class, the Kähler class of the path of metrics solving (5) is fixed and equals the class of the initial data.

With this unconstrained derivation of the extremal flow presented above, we can now almost directly use the monotonicity idea of Perelman in its analysis. We describe two of these results next. This provides additional justifications [18] for our earlier statement that the characterization of extremality given by (2) puts these metrics on the same footing with Einstein metrics, making it plausible that techniques used in the analysis of the latter can be used in the understanding of the former.

Theorem 11 (Perelman [15]). *The lowest eigenvalue of $4\Delta_g + s_g - p_g$ given by*

$$\lambda(g) = \inf_{f \in C_G^\infty(M): \int e^{-f} d\mu_g = 1} \tilde{\mathcal{F}}(g, f)$$

is monotone along solutions to (5).

In our context, Perelman's scaled-invariant version of the functional $\mathcal{F}$ becomes

$$\mathcal{W}(g, f, \tau) = \int_M (\tau(s_g - p_g + |\nabla_g f|^2) + f - n) \left(\frac{1}{4\pi\tau} \right)^{\frac{n}{2}} e^{-f} d\mu_g,$$

considered over the domain $\mathfrak{M}_G \times C_G^\infty(M) \times \mathbb{R}$. For functions in $C_G^\infty(M)$ that satisfy the condition

$$(8) \quad \int \left(\frac{1}{4\pi\tau} \right)^{\frac{n}{2}} e^{-f} d\mu = 1,$$

we may then define

$$\mu(g, \tau) = \inf_{f: (8) \text{ holds}} \mathcal{W}(g, f, \tau)$$

and get the monotonicity statement for the quantity $\nu(g) = \inf \mu(g, \tau)$ (over all positive τ) as well (see [15], page 8). We then have the nonlocal volume collapsing for solutions to the extremal flow that have finite lifespan T :

Theorem 12 (Perelman [15]). *If the lifespan T of the extremal flow is finite, then the family of metrics $g_{ij}(t) \in \mathfrak{M}_{\Omega, G}$ is not locally volume collapsing at T .*

When $\Omega = c_1(M, J)$ and the Futaki invariant vanishes (see §3), the extremal flow coincides with the Kähler-Ricci flow. In that case, by Cao's theorem [5], the flow equation has a solution for all time. No general long-time existence is known for the extremal flow. If the manifold does not carry nontrivial holomorphic vector fields, the extremal and pseudo-Calabi flow agree with each other. In that case, Chen and Zheng [7] have some very interesting global existence and stability results.

3. EXTREMAL MONGE-AMPÈRE EQUATION

Given a complex manifold (M, J) of Kähler type, we denote by $\mathfrak{h} = \mathfrak{h}(M, J)$ and $\mathcal{K}$ the algebra of holomorphic vector fields and Kähler cone, respectively. The *Futaki character* $\mathfrak{F} : \mathfrak{h}(M, J) \times \mathcal{K} \rightarrow \mathbb{C}$ is defined to be the map

$$(9) \quad \mathfrak{F}(\Xi, \Omega) = 2 \int_M \Xi(\psi_g) d\mu_g = -2 \int_M \Xi(G_g s_g) d\mu_g,$$

where the integral on the right side is computed using any Kähler representative g of Ω [9, 4]. In this expression, ψ_g is the Ricci potential of the metric g , a function defined up to a constant by the Hodge decomposition

$$\rho_g = \rho_g^h + i\partial\bar{\partial}\psi_g$$

of the Ricci form.

Let c_1 be the first Chern class of (M, J) . If $g \in \mathfrak{M}_\Omega$, the projection of the scalar curvature s_g onto the constants is given by

$$(10) \quad s_\Omega = 4n\pi \frac{c_1 \cup \Omega^{n-1}}{\Omega^n},$$

a function that depends only on Ω (and J). There exists a vector field $X_\Omega \in \mathfrak{h}$ such that

$$(11) \quad g \ni \mathfrak{M}_\Omega \rightarrow \int_M s_g^2 d\mu_g \geq E(\Omega) := s_\Omega^2 \frac{\Omega^n}{n!} - \mathfrak{F}(X_\Omega, \Omega),$$

and the equality is achieved if, and only if, the metric is extremal. This lower bound was known earlier [11, 19] for G -invariant metrics representing Ω . It was proven by

Chen [6] to hold in general, for any metric in $\mathfrak{M}_\Omega$. If the metric representing Ω is extremal, the Futaki invariant is the obstruction to its being one of constant scalar curvature.

The vector field X_Ω may depend on the choice of a maximal compact subgroup G of the automorphism group of (M, J) , but the value $\mathfrak{F}(X_\Omega, \Omega)$ does not, and given any metric g in $\mathfrak{M}_{\Omega, G}$, we have that $X_\Omega = \partial_g^\#(\pi_g s_g)$, where $\partial_g^\#$ is the operator defined by the identity $g(\partial_g^\# f, \cdot) = \bar{\partial} f$ for any function f . Hence,

$$(12) \quad E(\Omega) = \int (\pi_g s_g)^2 d\mu_g.$$

By Calabi's characterization, the condition (2) that defines a metric $g \in \mathfrak{M}_{\Omega, G}$ to be extremal can be reexpressed as

$$(13) \quad \rho_g = \rho_g^h - i\partial\bar{\partial}G_g(\pi_g s_g),$$

where ρ_g^h is the g -harmonic component of ρ_g . In other words, a metric is extremal if the Ricci potential $\psi_g = -G_g(\pi_g s_g)$ of the metric is the image under the Green's operator of the holomorphy potential $\pi_g s_g$. In order to compare this condition to the equation Yau thoroughly analyzed in solving one of Calabi's conjectures [21], we exhibit it next as a fully nonlinear second-order equation in the potential function φ of the metric.

Given a Kähler metric g , we set $\Omega = [\omega_g]$ and $v = \Omega^n/n!$. The Ricci form ρ_g decomposes as

$$(14) \quad \rho = \frac{s_\Omega}{2n}\omega + \rho_0 + i\partial\bar{\partial}f,$$

where s_Ω is given by (10), ρ_0 is a g -harmonic (1,1)-form of zero trace, and f is a smooth real-valued function perpendicular to the constants. Since the cohomology class represented by the Ricci form of any Kähler metric is fixed [8], given a deformation $\omega_{\tilde{g}} = \omega_g + i\partial\bar{\partial}\varphi$ of g that represents Ω also, the corresponding decomposition of the Ricci form $\rho_{\tilde{g}}$ is given by

$$\rho_{\tilde{g}} = \frac{s_\Omega}{2n}\omega_{\tilde{g}} + \tilde{\rho}_0 + i\partial\bar{\partial}\tilde{f},$$

where $\tilde{\rho}_0 = \rho_0 + 2i\partial\bar{\partial}G_{\tilde{g}}(\text{trace}_{\tilde{g}}\rho_0)$, $G_{\tilde{g}}$ the Green's operator of the metric $\omega_{\tilde{g}}$. So if we have the metrics ω_g and $\omega_{\tilde{g}} = \omega_g + i\partial\bar{\partial}\varphi$, and F is the function defined by

$$(15) \quad \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^F,$$

by these observations we then have that

$$\begin{aligned} \rho_{\tilde{g}} &= \rho_g - i\partial\bar{\partial}F \\ &= \frac{s_\Omega}{2n}\omega_g + \rho_0 + i\partial\bar{\partial}f - i\partial\bar{\partial}F \\ &= \frac{s_\Omega}{2n}\omega_{\tilde{g}} + \rho_0 + i\partial\bar{\partial}f - \frac{s_\Omega}{2n}i\partial\bar{\partial}\varphi - i\partial\bar{\partial}F, \end{aligned}$$

and computing the trace of this expression, we obtain that

$$s_{\tilde{g}} = s_\Omega + 2\text{trace}_{\tilde{g}}\rho_0 + \Delta_{\tilde{g}}\left(\frac{s_\Omega}{2n}\varphi + F - f\right).$$

If the metric $\tilde{g}$ is to be extremal, then $s_{\tilde{g}} = \pi_{\tilde{g}} s_{\tilde{g}}$, that is to say, $s_{\tilde{g}}$ must be a linear combination of the set of functions (1) spanning the space of real holomorphy potentials. Thus, the deformation potential φ gives rise to an extremal metric if

$$(16) \quad s_{\tilde{g}} = \pi_{\tilde{g}} s_{\tilde{g}} = \sum_{j=0}^m c_j p_j(\omega_{\tilde{g}}) = s_{\Omega} + 2\text{trace}_{\tilde{g}} \rho_0 + \Delta_{\tilde{g}} \left(\frac{s_{\Omega}}{2n} \varphi + F - f \right),$$

for some (real) constants $\{c_0, \dots, c_m\}$. These constants are themselves functions of the potential φ . Applying the Green's operator, we arrive at the following:

Proposition 13. *Let (M, J, Ω) be a compact polarized Kähler manifold of dimension n , and g be a metric representing the class Ω that is invariant under a maximal compact subgroup G of the biholomorphism group of (M, J) . The metric*

$$\tilde{g}_{i\bar{k}} = g_{i\bar{k}} + \varphi_{i\bar{k}}$$

is extremal if, and only if, φ is a G -invariant solution of the equation

$$\frac{\det(g_{i\bar{k}} + \varphi_{i\bar{k}})}{\det(g_{i\bar{k}})} = e^{-\frac{s_{\Omega}}{2n} \varphi + f + G_{\tilde{g}}(\pi_{\tilde{g}} s_{\tilde{g}} - 2\text{trace}_{\tilde{g}} \rho_0) + c}.$$

In this expression, $G_{\tilde{g}}$ is the Green's operator of the extremal metric, the function f and (1,1)-form ρ_0 are those appearing in the decomposition (14) for the Ricci tensor of g , and c is a constant (which, when $s_{\Omega} \neq 0$, can be absorbed into φ and thus taken to be zero).

The Monge-Ampère equation for a constant scalar curvature metric is obtained if we merely change the role of $\pi_{\tilde{g}} s_{\tilde{g}}$ above by the constant s_{Ω} .

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