A VOLTERRA-TYPE INTEGRATION OPERATOR ON FOCK SPACES

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(Communicated by Richard Rochberg)

ABSTRACT. We study certain spectral properties and the invariant subspaces for some classes of integration operators of Volterra type on the Fock space.

1. INTRODUCTION

For analytic functions f, g we consider the Volterra-type integration operator given by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta.$$

The boundedness and compactness, as well as some spectral properties (such as Schatten class membership) of T_g acting on various spaces of analytic functions of the unit disc $\mathbb{D}$ in $\mathbb{C}$ have been extensively investigated (see [2, 6] for Hardy spaces, [3, 5, 8, 17, 19] for weighted Bergman spaces, and [9, 10] for Dirichlet spaces, or the surveys [1, 20] and the references therein). Furthermore, the spectrum of T_g on weighted Bergman spaces was recently characterized in [3].

In this paper we consider the operator T_g acting on the Fock spaces $\mathcal{F}^p$, $p > 0$ (here, the symbol g is an entire function with $g(0) = 0$). Recall that, for $p > 0$, the Fock space $\mathcal{F}^p$ consists of entire functions f for which

$$\|f\|_p = \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z) e^{-\frac{|z|^2}{2}}|^p dA(z) \right)^{\frac{1}{p}} < \infty.$$

The boundedness and compactness of T_g follow by classical methods, and we include a sketch of their proofs in Section 3 for the sake of completeness. It turns out that, if $p \leq q$, $T_g : \mathcal{F}^p \rightarrow \mathcal{F}^q$ is bounded if and only if the symbol g is a polynomial of degree ≤ 2 , while the necessary and sufficient condition for compactness is: g is a polynomial of degree ≤ 1 . For $p > q$, the operator T_g is bounded if and only if it is compact, and these are equivalent to the condition

$$q > \frac{2p}{p+2} \quad \text{and} \quad g(z) = az \quad \text{for some } a \in \mathbb{C}.$$

In particular, this shows that the primitive of a function in $\mathcal{F}^p$ is in $\mathcal{F}^q$ for $q > 2p/(p+2)$, and this result is sharp. In Section 4 we turn to the spectral properties of T_g . Regarding the Schatten class membership of $T_g|_{\mathcal{F}^2}$, we show that, provided it is compact, T_g fails to be Hilbert-Schmidt, but it belongs to all classes $\mathcal{S}^p$ for

Received by the editors May 30, 2011.

2010 *Mathematics Subject Classification*. Primary 30H20, 47B38.

Key words and phrases. Fock spaces, integration operator, spectrum, invariant subspaces.

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$p > 2$. Moreover, we prove that the spectrum of $T_g|_{\mathcal{F}^2}$ is a closed disc centred at the origin whose radius depends on the symbol g .

In the last section of this paper we characterize the invariant subspaces of the Volterra operator

$$Vf(z) = \int_0^z f(\zeta) d\zeta,$$

which is a particular case of the operator T_g , corresponding to $g(z) = z$. A complete description of the invariant subspaces of V , when acting on various classical spaces of analytic functions on the unit disc (Hardy spaces, weighted Bergman spaces, Dirichlet spaces) was obtained in [4]. We prove the Fock space analogue of these results, showing that the proper invariant subspaces of $V : \mathcal{F}^p \rightarrow \mathcal{F}^p$, $p > 0$, are given by

$$A_N^p = \overline{\text{Span}\{z^k : k \geq N+1\}}^{\mathcal{F}^p},$$

where N is an arbitrary nonnegative integer.

2. PRELIMINARIES

Recall that the point evaluation functionals are bounded on $\mathcal{F}^p$, $p > 0$. More precisely, for every $f \in \mathcal{F}^p$, we have

$$(1) \quad |f(z)| \leq e^{|z|^2/2} \|f\|_p, \quad z \in \mathbb{C}.$$

Moreover, the monomials $z^n/\sqrt{n!}$, $n \geq 0$, form an orthonormal basis for $\mathcal{F}^2$, and therefore the reproducing kernel of $\mathcal{F}^2$ is given by $K_\lambda(z) = e^{\bar{\lambda}z}$, i.e.

$$f(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} f(z) e^{\lambda \bar{z}} e^{-|z|^2} dA(z), \quad \lambda \in \mathbb{C}, \quad f \in \mathcal{F}^2.$$

The following straightforward fact will be used in our further considerations: for an entire function f , the maximum principle and the subharmonicity of $|f|^p$ ensure that

$$(2) \quad \begin{aligned} \int_{\mathbb{C}} \frac{|f(z)|^p}{(1+|z|)^p} e^{-p|z|^2/2} dA(z) &\sim \int_{|z|>1} \frac{|f(z)|^p}{(1+|z|)^p} e^{-p|z|^2/2} dA(z) \\ &\sim \int_{|z|>1} \left| \frac{f(z)}{z} \right|^p e^{-p|z|^2/2} dA(z). \end{aligned}$$

Moreover, if $f(0) = 0$, the above quantities are comparable to

$$\int_{\mathbb{C}} \left| \frac{f(z)}{z} \right|^p e^{-p|z|^2/2} dA(z).$$

The next proposition can be easily deduced from a result in [7]. We include a sketch of its proof for the sake of completeness.

Proposition 1. *For an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ and $p > 0$, the following holds:*

$$(3) \quad \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2/2} dA(z) \sim |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-p|z|^2/2} dA(z).$$

Proof. It is enough to prove the claim for entire functions f with $f(0) = 0$. Indeed, the result for a general entire function f follows by applying (3) to $z \mapsto f(z) - f(0)$.

Suppose now that $f(z) = zg(z)$, where g is entire. Then by Lemmas 5-6 in [7] we have

$$\int_{\mathbb{C}} |zg(z)|^p e^{-p|z|^2/2} dA(z) \sim |g(0)|^p + \int_{\mathbb{C}} |g'(z)|^p e^{-p|z|^2/2} dA(z).$$

This is equivalent to

$$(4) \quad \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2/2} dA(z) \sim |f'(0)|^p + \int_{\mathbb{C}} \left| \frac{zf'(z) - f(z)}{z^2} \right|^p e^{-p|z|^2/2} dA(z).$$

The integral on the right side is comparable to

$$\int_{|z|>1} \left| \frac{zf'(z) - f(z)}{z^2} \right|^p e^{-p|z|^2/2} dA(z).$$

Using this fact, relation (2) and the obvious inequality $(a+b)^p \leq 2^p(|a|^p + |b|^p)$, we now obtain

$$\int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-p|z|^2/2} dA(z) \lesssim \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^2/2} dA(z).$$

Hence one inequality in (3) is proven. We prove the reverse inequality by contradiction. Assume there exists a sequence of entire functions f_n with $f_n(0) = 0$ such that $\int_{\mathbb{C}} |f_n(z)|^p e^{-p|z|^2/2} dA(z) = 1$ and

$$(5) \quad \int_{\mathbb{C}} \frac{|f'_n(z)|^p}{(1+|z|)^p} e^{-p|z|^2/2} dA(z) \leq \frac{1}{n}, \quad n \geq 1.$$

The expression $(1+|z|)$ is bounded above and below on any compact set in $\mathbb{C}$. Then by the subharmonicity of $|f'_n(z)|^p$ and relation (5) we deduce that the sequence $f'_n(z)$ converges to zero uniformly on the compact sets in $\mathbb{C}$ and, since $f_n(0) = 0$, the same holds for f_n . We now write relation (4) for each f_n and we use (2) to get

$$\begin{aligned} 1 &\sim |f'_n(0)|^p + \int_{\mathbb{C}} \left| \frac{zf'_n(z) - f_n(z)}{z^2} \right|^p e^{-p|z|^2/2} dA(z) \\ &\lesssim |f'_n(0)|^p + \int_{|z|>1} \left| \frac{f'_n(z)}{z} \right|^p e^{-p|z|^2/2} dA(z) + \int_{|z|>1} \left| \frac{f_n(z)}{z^2} \right|^p e^{-p|z|^2/2} dA(z) \\ &\lesssim |f'_n(0)|^p + \frac{1}{n} + \int_{R>|z|>1} \left| \frac{f_n(z)}{z^2} \right|^p e^{-p|z|^2/2} dA(z) \\ &\quad + \frac{1}{R^2} \int_{|z|>R} |f_n(z)|^p e^{-p|z|^2/2} dA(z) \\ &\leq |f'_n(0)|^p + \frac{1}{n} + \int_{R>|z|>1} \left| \frac{f_n(z)}{z^2} \right|^p e^{-p|z|^2/2} dA(z) + \frac{1}{R^2}, \end{aligned}$$

for any $R > 1$. Letting first $n \rightarrow \infty$ and then $R \rightarrow \infty$ above we obtain a contradiction. Thus (3) holds. $\square$

3. BOUNDEDNESS AND COMPACTNESS

The next theorem characterizes the boundedness and the compactness of T_g .

Theorem 1. (i) For $0 < p \leq q$ the operator T_g is bounded from $\mathcal{F}^p$ to $\mathcal{F}^q$ if and only if $g(z) = az^2 + bz$ for some $a, b \in \mathbb{C}$. Moreover, $T_g : \mathcal{F}^p \rightarrow \mathcal{F}^q$ is compact if and only if $g(z) = az$ for some $a \in \mathbb{C}$.

(ii) Assume $0 < q < p$ and $g \not\equiv 0$. Then the operator T_g is bounded from $\mathcal{F}^p$ to $\mathcal{F}^q$ if and only if $q > \frac{2p}{p+2}$ and $g(z) = az$, for some $a \in \mathbb{C} \setminus 0$. Furthermore, if $T_g : \mathcal{F}^p \rightarrow \mathcal{F}^q$ is bounded, then it is compact.

Proof. (i) We start by proving the sufficiency part of the statements in (i). Assume $g(z) = az^2 + bz$. Then Proposition 1 gives

$$\begin{aligned} \|T_g f\|_q^q &\sim \int_{\mathbb{C}} \frac{|fg'|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ &= \int_{\mathbb{C}} |f|^q \frac{|2az+b|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ &\lesssim \|f\|_q^q \lesssim \|f\|_p^q, \end{aligned}$$

and the boundedness of T_g follows from the inclusion $\mathcal{F}^p \subseteq \mathcal{F}^q$.

Let us now show that if g' is constant, then T_g is compact. To this end, let f_n be a sequence of entire functions such that $\sup_n \|f_n\|_p < \infty$ and $f_n \rightarrow 0$ uniformly on compact sets in $\mathbb{C}$. Also, let $R > 0$. An application of Proposition 1 yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|T_g f_n\|_q^q &\sim \limsup_{n \rightarrow \infty} \int_{\mathbb{C}} |f_n(z)|^q \frac{|g'(z)|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ (6) \quad &\sim \limsup_{n \rightarrow \infty} \int_{|z| \leq R} \frac{|f_n(z)|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ &\quad + \int_{|z| > R} \frac{|f_n(z)|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ &\leq \frac{1}{(1+R)^q} \limsup_{n \rightarrow \infty} \|f_n\|_p^q. \end{aligned}$$

Since $\sup_n \|f_n\|_p < \infty$, we now let $R \rightarrow \infty$ in the above relation to deduce that $T_g f_n \xrightarrow{\mathcal{F}^q} 0$ as $n \rightarrow \infty$. Thus T_g is compact.

Let us now prove the necessity. If T_g is bounded, Proposition 1 yields

$$\begin{aligned} (7) \quad \|K_\lambda\|_p^q &\gtrsim \|T_g K_\lambda\|_q^q \sim \int_{\mathbb{C}} \frac{|g'(z)|^q}{(1+|z|)^q} |K_\lambda|^q e^{-q|z|^2/2} dA(z) \\ &\geq \int_{D(\lambda,1)} \frac{|g'(z)|^q}{(1+|z|)^q} e^{-q|z-\lambda|^2/2} e^{q|\lambda|^2/2} dA(z) \\ &\gtrsim e^{q|\lambda|^2/2} \int_{D(\lambda,1)} \frac{|g'(z)|^q}{(1+|z|)^q} dA(z) \gtrsim e^{q|\lambda|^2/2} \frac{|g'(\lambda)|^q}{(1+|\lambda|)^q}, \quad \lambda \in \mathbb{C}, \end{aligned}$$

by subharmonicity and the fact that $(1+|z|) \sim (1+|\lambda|)$ for $|z-\lambda| < 1$. Since $\|K_\lambda\|_p = e^{|\lambda|^2/2}$, the last relation above implies that

$$|g'(\lambda)| \lesssim 1 + |\lambda|, \quad \lambda \in \mathbb{C},$$

and hence g is an analytic polynomial of degree ≤ 2 .

Now, if T_g is compact, let $k_\lambda(z) = e^{-|\lambda|^2/2} K_\lambda(z) = e^{\bar{\lambda}z - |\lambda|^2/2}$. Notice that $k_\lambda \rightarrow 0$ uniformly on compact sets as $|\lambda| \rightarrow \infty$. Moreover, $\|k_\lambda\|_p = 1$. Hence $\|T_g k_\lambda\|_q \rightarrow 0$ as $|\lambda| \rightarrow \infty$. On the other hand, relation (7) shows that

$$\|T_g k_\lambda\|_q \gtrsim \frac{|g'(\lambda)|}{1+|\lambda|}, \quad \lambda \in \mathbb{C},$$

which, in view of the previous considerations, forces g' to be constant.

(ii) The proof of the necessity follows a similar approach to the one in [14]. Assume T_g is bounded and let $(\lambda_i) \subset \mathbb{C}$ be a uniformly separated sequence, i.e. $\inf_{i \neq j} |\lambda_i - \lambda_j| > 0$. Then, for any fixed radius $R > 0$, there exists a positive integer $\mathcal{I}(R) > 0$ such that any disc of radius R contains at most $\mathcal{I}(R)$ points λ_i . We also assume that (λ_i) is ε -dense for some $\varepsilon > 0$, i.e. every point in $\mathbb{C}$ is within ε -distance of some λ_i . Then, for ε sufficiently small (see Theorem 8.2 in [13] for $p \geq 1$ and Theorem 3.1 in [21] for $0 < p < 1$), it follows that for any sequence $(c_i) \in l^p$ the function defined by

$$f(z) = \sum_{i \geq 1} c_i e^{\bar{\lambda}_i z - |\lambda_i|^2/2}, \quad z \in \mathbb{C},$$

belongs to $\mathcal{F}^p$ and $\|f\|_p \lesssim \|(c_i)\|_{l^p}$. By Khinchine's inequality we obtain

$$\left(\sum_{i \geq 1} |c_i|^2 e^{2 \operatorname{Re}(\bar{\lambda}_i z) - |\lambda_i|^2} \right)^{q/2} \lesssim \int_0^1 \left| \sum_{i \geq 1} r_i(t) c_i e^{\bar{\lambda}_i z - |\lambda_i|^2/2} \right|^q dt,$$

where the r_i are the Rademacher functions on $[0, 1]$ (see [14]). Now denote

$$d\mu(z) := \frac{|g'(z)|^q}{(1 + |z|)^q} e^{-q|z|^2/2} dA(z)$$

and use Fubini's theorem together with Proposition 1 to deduce

$$\begin{aligned} (8) \quad \int_{\mathbb{C}} \left(\sum_{i \geq 1} |c_i|^2 e^{2 \operatorname{Re}(\bar{\lambda}_i z) - |\lambda_i|^2} \right)^{q/2} d\mu(z) &\lesssim \int_0^1 \int_{\mathbb{C}} \left| \sum_{i \geq 1} r_i(t) c_i e^{\bar{\lambda}_i z - |\lambda_i|^2/2} \right|^q d\mu(z) dt \\ &\sim \int_0^1 \left\| T_g \left(\sum_{i \geq 1} r_i(t) c_i e^{\bar{\lambda}_i z - |\lambda_i|^2/2} \right) \right\|_q^q dt \\ &\leq \int_0^1 \|T_g\|^q \left\| \sum_{i \geq 1} r_i(t) c_i e^{\bar{\lambda}_i z - |\lambda_i|^2/2} \right\|_p^q dt \\ &\lesssim \left(\sum_{i \geq 1} |c_i|^p \right)^{q/p}. \end{aligned}$$

Since (λ_i) is ε -dense, it follows that the discs $D(\lambda_i, 2\varepsilon)$ cover $\mathbb{C}$, and therefore

$$\begin{aligned} &\int_{\mathbb{C}} \left(\sum_{i \geq 1} |c_i|^2 e^{2 \operatorname{Re}(\bar{\lambda}_i z) - |\lambda_i|^2} \right)^{q/2} d\mu(z) \\ &\geq \mathcal{I}^{-1}(2\varepsilon) \sum_{i \geq 1} \int_{D(\lambda_i, 2\varepsilon)} |c_i|^q e^{q \operatorname{Re}(\bar{\lambda}_i z) - q|\lambda_i|^2/2} d\mu(z) \\ &\gtrsim \sum_{i \geq 1} |c_i|^q \int_{D(\lambda_i, 2\varepsilon)} \frac{|g'(z)|^q}{(1 + |z|)^q} dA(z). \end{aligned}$$

Since $(c_i) \in l^p$ was arbitrarily chosen, by the previous relation together with (8) we get

$$(9) \quad \left(\int_{D(\lambda_i, 2\varepsilon)} \frac{|g'(z)|^q}{(1 + |z|)^q} dA(z) \right) \in l^{p/(p-q)}.$$

By subharmonicity we have

$$\frac{|g'(z)|^q}{(1+|z|)^q} \lesssim \int_{D(\lambda_i, 2\varepsilon)} \frac{|g'(\zeta)|^q}{(1+|\zeta|)^q} dA(\zeta), \quad z \in D(\lambda_i, 3\varepsilon/2), \quad i \geq 1,$$

with constants depending on ε . We now raise both sides of the above inequality to the power $p/(p-q)$, then we integrate on $D(\lambda_i, 3\varepsilon/2)$ and sum over i to get

$$\begin{aligned} \int_{\mathbb{C}} \left| \frac{g'(z)}{(1+|z|)} \right|^{pq/(p-q)} dA(z) &\leq \sum_{i \geq 1} \int_{D(\lambda_i, 3\varepsilon/2)} \left| \frac{g'(z)}{(1+|z|)} \right|^{pq/(p-q)} dA(z) \\ &\lesssim \sum_{i \geq 1} \left(\int_{D(\lambda_i, 2\varepsilon)} \frac{|g'(z)|^q}{(1+|z|)^q} dA(z) \right)^{p/(p-q)} < \infty. \end{aligned}$$

Thus g' is constant and $q > \frac{2p}{p+2}$.

Now let us prove that g' constant and $q > 2p/(p+2)$ ensure the compactness of T_g . As before, let f_n be a sequence of entire functions such that $\sup_n \|f_n\|_p < \infty$ and $f_n \rightarrow 0$ uniformly on compact sets in $\mathbb{C}$. Then, as in (6), for $R > 0$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|T_g f_n\|_q^q &\sim \limsup_{n \rightarrow \infty} \int_{|z| \leq R} \frac{|f_n(z)|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ &\quad + \int_{|z| > R} \frac{|f_n(z)|^q}{(1+|z|)^q} e^{-q|z|^2/2} dA(z) \\ &\leq \left(\int_{|z| > R} \frac{1}{(1+|z|)^{pq/(p-q)}} dA(z) \right)^{(p-q)/p} \limsup_{n \rightarrow \infty} \|f_n\|_p^q, \end{aligned}$$

where the last step above follows by Hölder's inequality. As $\sup_n \|f_n\|_p < \infty$, we now let $R \rightarrow \infty$ in the above relation to deduce that $T_g f_n \xrightarrow{\mathcal{F}^q} 0$ for $n \rightarrow \infty$. Thus T_g is compact and the proof is complete. $\square$

4. SCHATTEN CLASS MEMBERSHIP AND THE SPECTRUM OF T_g ON $\mathcal{F}^2$

Throughout this section, we shall consider the operator T_g acting on $\mathcal{F}^2$. Recall that an operator T acting on a Hilbert space belongs to the Schatten class $\mathcal{S}^p$, $p > 0$, if the sequence of eigenvalues of $(T^*T)^{1/2}$ belongs to l^p .

Theorem 2. *Suppose T_g is compact. Then T_g belongs to the Schatten class $\mathcal{S}^p$ for all $p > 2$, but it fails to be Hilbert-Schmidt unless g is constant.*

Proof. By Theorem 1, the operator T_g is compact if and only if g' is constant. Let us first show that T_g fails to be Hilbert-Schmidt if $g' \not\equiv 0$. Given an orthonormal basis (e_n) in $\mathcal{F}^2$, we have $K_z(z) = e^{-|z|^2} = \sum_{n \geq 1} |e_n(z)|^2$. We use this together with Proposition 1 to infer

$$\sum_{n \geq 1} \|T_g e_n\|_2^2 \sim \sum_{n \geq 1} \int_{\mathbb{C}} \frac{|e_n(z)|^2}{(1+|z|)^2} e^{-|z|^2} dA(z) = \int_{\mathbb{C}} \frac{K_z(z) e^{-|z|^2}}{(1+|z|)^2} dA(z) = \infty,$$

which shows that T_g is not Hilbert-Schmidt.

Given $p > 2$, a bounded linear operator A on a Hilbert space belongs to $\mathcal{S}^p$ if and only if $\sum_n \|A e_n\|^p < \infty$ for any orthonormal basis (e_n) of the Hilbert space

(see [11]). For an orthonormal basis (e_n) of $\mathcal{F}^2$ we have, as before,

$$\begin{aligned} \sum_{n \geq 1} \|T_g e_n\|_2^p &\sim \sum_{n \geq 1} \left(\int_{\mathbb{C}} \frac{|e_n(z)|^2}{(1+|z|)^2} e^{-|z|^2} dA(z) \right)^{p/2} \\ &\leq \sum_{n \geq 1} \left(\int_{\mathbb{C}} |e_n(z)|^2 e^{-|z|^2} dA(z) \right)^{(p-2)/2} \int_{\mathbb{C}} \frac{|e_n(z)|^2 e^{-|z|^2}}{(1+|z|)^p} dA(z) \\ &= \sum_{n \geq 1} \int_{\mathbb{C}} \frac{|e_n(z)|^2 e^{-|z|^2}}{(1+|z|)^p} dA(z) = \int_{\mathbb{C}} \frac{1}{(1+|z|)^p} dA(z) < \infty, \end{aligned}$$

by Hölder's inequality and since (e_n) is an orthonormal basis. The proof is now complete. $\square$

Let us now turn to the spectrum of $T_g : \mathcal{F}^2 \rightarrow \mathcal{F}^2$. We assume T_g is bounded, and hence $g(z) = az^2 + bz$, for some $a, b \in \mathbb{C}$. Note first that for $\lambda \in \mathbb{C} \setminus \{0\}$, the equation

$$f - \frac{1}{\lambda} T_g f = h$$

has the unique analytic solution

$$(10) \quad f(z) = R_{\lambda, g} h(z) = h(0) e^{g(z)/\lambda} + e^{g(z)/\lambda} \int_0^z e^{-g(\zeta)/\lambda} h'(\zeta) d\zeta, \quad z \in \mathbb{C}.$$

Thus the resolvent set of T_g consists precisely of those points $\lambda \in \mathbb{C}$ for which $R_{\lambda, g}$ is a bounded operator on $\mathcal{F}^2$. Furthermore, the investigation of the spectrum of T_g is related to the behavior of the exponentials $e^{g/\lambda}$, $\lambda \in \mathbb{C} \setminus \{0\}$, as was first pointed out by Pommerenke [18]. Notice that for $h \equiv 1$ in the last relation above we get $R_{\lambda, g} 1 = e^{g/\lambda}$. Hence $e^{g/\lambda} \in \mathcal{F}^2$ whenever λ belongs to the resolvent set of T_g . This provides a sufficient condition for a point λ to belong to the spectrum of T_g . A necessary and sufficient condition will be easily obtained once we have proved the next lemma.

Lemma 1. Assume $g(z) = az^2 + bz$ and $|\lambda| > 2a$. Then, for any entire function f , we have

$$\int_{\mathbb{C}} |e^{g/\lambda} f|^2 e^{-|z|^2} dA(z) \lesssim |f(0)|^2 + \int_{\mathbb{C}} \frac{|e^{g/\lambda} f'|^2}{(1+|z|)^2} e^{-|z|^2} dA(z).$$

Proof. Suppose first that $f(0) = f'(0) = 0$. A simple calculation based on the fact that the $\mathcal{F}^2$ -norm of an entire function with Taylor series $\sum_{n \geq 0} a_n z^n$ is given by $(\sum_{n \geq 0} |a_n|^2 n!)^{1/2}$ shows that

$$\|f\|_2 \leq \left\| \frac{f'}{z} \right\|_2.$$

Replacing f by $e^{g/\lambda} f$ above we obtain

$$\|e^{g/\lambda} f\|_2 \leq \left\| e^{g/\lambda} \left(\frac{2az+b}{\lambda z} f + \frac{f'}{z} \right) \right\|_2 \leq \left\| e^{g/\lambda} \frac{2az+b}{\lambda z} f \right\|_2 + \left\| e^{g/\lambda} \frac{f'}{z} \right\|_2,$$

and hence

$$(11) \quad \left(1 - \frac{|2a|}{|\lambda|} \right) \|e^{g/\lambda} f\|_2 \leq \left\| e^{g/\lambda} \frac{bf}{\lambda z} \right\|_2 + \left\| e^{g/\lambda} \frac{f'}{z} \right\|_2.$$

We claim that this implies that $\|e^{g/\lambda} f\|_2 \lesssim \|e^{g/\lambda} f'/z\|_2$. Indeed, if this did not hold, there would exist a sequence of entire functions (f_n) with $f_n(0) = f'_n(0) = 0$

such that $\|e^{g/\lambda}f_n\|_2 = 1$ and $\|e^{g/\lambda}f'_n/z\|_2 \leq 1/n$ for $n \geq 1$. The last inequality together with (1) show that the sequence $f'_n(z)$ converges to zero uniformly on the compact sets in $\mathbb{C}$, and since $f_n(0) = 0$, the same holds for f_n/z . On the other hand, for $R > 0$, we deduce from (11) that

$$\begin{aligned} 1 - \frac{|2a|}{|\lambda|} &\leq \left(\int_{|z| \leq R} \left| e^{g/\lambda} \frac{bf_n}{\lambda z} \right|^2 e^{-|z|^2} dA(z) \right)^{1/2} \\ &\quad + \frac{|b|}{|\lambda|R} \left(\int_{|z| > R} |e^{g/\lambda} f_n|^2 e^{-|z|^2} dA(z) \right)^{1/2} + \frac{1}{n} \\ &\leq \left(\int_{|z| \leq R} \left| e^{g/\lambda} \frac{bf_n}{\lambda z} \right|^2 e^{-|z|^2} dA(z) \right)^{1/2} + \frac{|b|}{|\lambda|R} + \frac{1}{n}. \end{aligned}$$

We first let $n \rightarrow \infty$, and subsequently $R \rightarrow \infty$, in the above relation to obtain a contradiction. Thus

$$\|e^{g/\lambda}f\|_2 \lesssim \|e^{g/\lambda}f'/z\|_2 \sim \left(\int_{\mathbb{C}} \frac{|e^{g/\lambda}f'|^2}{(1+|z|^2)} e^{-|z|^2} \right)^{1/2}.$$

The desired result for a general f can be easily deduced by applying the above inequality to $z \mapsto f(z) - f(0) - f'(0)z$. $\square$

Theorem 3. Suppose $g(z) = az^2 + bz$ for some $a, b \in \mathbb{C}$. Then the spectrum of T_g is the closed disc centred at the origin and of radius $2|a|$.

Proof. Let $\rho(T_g)$ denote the resolvent set of T_g . As mentioned before, if $\lambda \in \rho(T_g)$, then $e^{g/\lambda} \in \mathcal{F}^2$. A straightforward calculation using polar coordinates shows that, if $|\lambda| < 2|a|$, then $e^{g/\lambda} \notin \mathcal{F}^2$, and hence $\overline{D(0, 2|a|)}$ is contained in the spectrum of T_g . Let us now prove that if $|\lambda| > 2|a|$, the formal resolvent $R_{\lambda, g}$, given by (10), is a bounded operator. If $|\lambda| > 2|a|$, then clearly $e^{g/\lambda} \in \mathcal{F}^2$. We first apply Lemma 1, and subsequently Proposition 1, to deduce

$$\begin{aligned} \|R_{\lambda, g}f\|_2 &\leq |f(0)| \|e^{g(z)/\lambda}\|_2 + \left\| e^{g(z)/\lambda} \int_0^z e^{-g(\zeta)/\lambda} f'(\zeta) d\zeta \right\|_2 \\ &\lesssim \|f\|_2 + \left(\int_{\mathbb{C}} \frac{|f'(z)|^2}{(1+|z|)^2} e^{-|z|^2} dA(z) \right)^{1/2} \lesssim \|f\|_2, \end{aligned}$$

which shows that $R_{\lambda, g}$ is bounded. Thus $\rho(T_g) = \{\lambda \in \mathbb{C} : |\lambda| > 2|a|\}$. $\square$

5. INVARIANT SUBSPACES OF THE VOLTERRA OPERATOR

In the particular case $g(z) = z$, the operator T_g becomes the Volterra operator

$$Vf(z) = \int_0^z f(\zeta) d\zeta, \quad z \in \mathbb{C}, \quad f \in \mathcal{F}^p, \quad p > 0.$$

By Theorem 1, the operator $V : \mathcal{F}^p \rightarrow \mathcal{F}^p$ is compact. The aim of this section is to characterize the invariant subspaces of V . A closed subspace $\mathcal{M}$ in $\mathcal{F}^p$ is called invariant for V if $V\mathcal{M} \subset \mathcal{M}$. We start with the following simple observation.

Remark 1. The spectrum of the Volterra operator is the set $\{0\}$.

Proof. Notice that V has no eigenvalues, since for any $f \in \mathcal{F}^p \setminus \{0\}$ the function Vf has a zero at $z = 0$ of one order greater than the corresponding order for f . Since V is compact and has no eigenvalues, its spectrum must be $\{0\}$. $\square$

Remark 2. For $x_0 = p > 0$ we consider the sequence defined inductively by

$$x_{n+1} = \frac{2x_n}{x_n + 2}, \quad n \geq 0.$$

Note that (x_n) is decreasing and $\lim_{n \rightarrow \infty} x_n = 0$. Then, for any nonnegative integer M , it follows by Theorem 1 (ii) that the operator V^M is bounded from $\mathcal{F}^p$ to $\mathcal{F}^q$ if $q > x_M$.

Theorem 4. *Assume $p > 0$. Then $\mathcal{M} \subset \mathcal{F}^p$ is a proper invariant subspace of $V : \mathcal{F}^p \rightarrow \mathcal{F}^p$ if and only if there exists a nonnegative integer N such that*

$$\mathcal{M} = \{f \in \mathcal{F}^p : f^{(k)}(0) = 0 \text{ for } 0 \leq k \leq N\} = \overline{\text{Span}\{z^k : k \geq N+1\}}^{\mathcal{F}^p}.$$

Proof. Clearly, the sets

$$A_N^p := \{f \in \mathcal{F}^p : f^{(k)}(0) = 0 \text{ for } 0 \leq k \leq N\}$$

are invariant subspaces for V .

Let us now prove that these are all the invariant subspaces of V . For $p = 2$ this follows directly from a result of Nikol'skiĭ (see [15, 16]) about invariant subspaces of weighted shifts. Notice that, with respect to the standard orthonormal basis $e_n(z) = (n!)^{-1/2}z^n$, $n \geq 0$, of $\mathcal{F}^2$, the operator V is a weighted forward shift. More precisely,

$$Ve_n = \frac{1}{(n+1)^{1/2}}e_{n+1}, \quad n \geq 0.$$

If the weight sequence of a weighted forward shift belongs to some l^s with $s < \infty$, then Nikol'skiĭ's theorem implies that its only invariant subspaces are the trivial ones, i.e.

$$\mathcal{N}_k = \overline{\text{Span}\{e_n : n \geq k\}}, \quad k \geq 0.$$

This is certainly the case for our operator; therefore the assertion follows.

We are now going to show that the result for $p \neq 2$ can be deduced via the case $p = 2$. Assume first that $0 < p < 2$ and let $\mathcal{M}$ be an invariant subspace for $V : \mathcal{F}^p \rightarrow \mathcal{F}^p$. From the inclusion $\mathcal{F}^p \subset \mathcal{F}^2$ we deduce that $\overline{\mathcal{M}}^{\mathcal{F}^2}$ is an invariant subspace of $V : \mathcal{F}^2 \rightarrow \mathcal{F}^2$ and hence

$$\overline{\mathcal{M}}^{\mathcal{F}^2} = A_N^2,$$

for some $N \geq 0$. In particular, this implies that $\mathcal{M} \subseteq A_N^p$. We let $f \in A_N^2$. Then, by the above, there exists a sequence $(f_n) \subset \mathcal{M}$ such that $f_n \xrightarrow{\mathcal{F}^2} f$ as $n \rightarrow \infty$. Now choose M large enough such that $x_M < p$, where (x_n) is the sequence defined in Remark 2 with $x_0 = 2$. Therefore $V^M : \mathcal{F}^2 \rightarrow \mathcal{F}^p$ is bounded and

$$V^M f_n \xrightarrow{\mathcal{F}^p} V^M f.$$

By the invariance of $\mathcal{M}$ we get $V^M f_n \in \mathcal{M}$, and hence $V^M f \in \mathcal{M}$ for all $f \in A_N^2$. This last fact guarantees that the monomials x^k , $k \geq M + N + 1$, belong to $\mathcal{M}$, and then we have

$$(12) \quad A_{N+M}^p \subseteq \mathcal{M} \subseteq A_N^p.$$

If $\mathcal{M} \neq A_{N+M}^p$, let k be the smallest nonnegative integer such that there exists $f \in \mathcal{M}$ with $f^{(N+1+k)}(0) \neq 0$ (clearly $0 \leq k < M$). But, for this particular f , we have $V^{M-k-1}f \in \mathcal{M}$, which in view of the first inclusion in (12) implies $z^{N+M} \in \mathcal{M}$, and therefore $A_{N+M-1} \subseteq \mathcal{M}$. We repeat this procedure until we

obtain $A_{N+k}^p \subseteq \mathcal{M}$, and then the choice of k forces $\mathcal{M} = A_{N+k}^p$, so that the proof is done for $0 < p < 2$.

It remains to prove the assertion for $p > 2$. Suppose $\mathcal{M}$ is an invariant subspace for $V : \mathcal{F}^p \rightarrow \mathcal{F}^p$. It follows by Theorem 1 that $V : \mathcal{F}^p \rightarrow \mathcal{F}^2$ is bounded. Then $\overline{V\mathcal{M}}^{\mathcal{F}^2}$ is an invariant subspace for $V : \mathcal{F}^2 \rightarrow \mathcal{F}^2$, and by Nikol'skiĭ's theorem,

$$\overline{V\mathcal{M}}^{\mathcal{F}^2} = A_N^2,$$

for some $N \geq 0$. From the above relation we obtain $\mathcal{M} \subseteq A_{N-1}^p$. On the other hand, since $\mathcal{F}^2 \subset \mathcal{F}^p$ we have

$$A_N^2 = \overline{V\mathcal{M}}^{\mathcal{F}^2} \subset \overline{V\mathcal{M}}^{\mathcal{F}^p} \subseteq \mathcal{M},$$

and then the monomials $z^k, k \geq N+1$ belong to $\mathcal{M}$. Hence $A_N^p \subseteq \mathcal{M} \subseteq A_{N-1}^p$. As in the case $0 < p < 2$, it now follows that either $\mathcal{M} = A_N^p$ or $\mathcal{M} = A_{N-1}^p$, which completes the proof. $\square$

ACKNOWLEDGEMENTS

While preparing this paper for submission, the author learned that results similar to those in Section 3, in the setting of several variables, were obtained recently (and independently) by Z. J. Hu [12]. The author would also like to express her gratitude to the anonymous referee for constructive suggestions.

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