QUADRATIC TRANSFORMS INSIDE THEIR GENERIC INCARNATIONS

SHREERAM S. ABHYANKAR

(Communicated by Lev Borisov)

ABSTRACT. The generic incarnation of a quadratic transform is obtained by adjoining an indeterminate. This leads to the singularity tree of a generic member of a pencil of curves.

1. INTRODUCTION

Given any two-dimensional regular local domain R , let $Q(R)$ be the set of all two-dimensional iterated quadratic transforms of R , and let L be the quotient field of R . For the terminology to be used in this paper, see Section 2 and Remark (4.0) of [Ab11]; for further notation and terminology, see Section 1 of my book [Ab3] and pages 146-161 and 552-577 of my book [Ab8]; more background material can be found in [Ab9, Ab10, Ab12]. By [Ab1] (also see [Ab2]) $Q(R)$ coincides with the set of all two-dimensional regular local domains whose quotient field is L and which dominate R . Consider the rational function field $L(t)$ in an indeterminate t over L . For every subring S of L let S^t be the localization of $S[t]$ at the multiplicative set of all those members of $S[t]$ whose coefficients generate the unit ideal in S . Let $Q^t(R) = \{S^t : S \in Q(R)\}$. We shall show that $Q^t(R) \subset Q(R^t)$. We may refer to $Q^t(R)$ as the generic incarnation of $Q(R)$. Given any nonzero element F in the maximal ideal $M(R)$ of R , we visualize R as the local ring of a simple point P of an algebraic or arithmetic surface and $F = 0$ as the local equation of a curve C on the surface passing through P . Now the singularity tree of C at P is a subset of $Q(R)$. For singularity trees, see Figure 5 on page 426 of [Ab4], Figure 18.2 on page 132 of [Ab6], and the material of [Ab5]. Let $G = 0$ be the local equation of a curve having no common component with $F = 0$ at P . We are interested in the singularity tree of a generic member of the pencil generated by F and G . As a concrete illustration, take R to be the localization of the bivariate polynomial ring $k[X, Y]$ over a field k at the maximal ideal generated by (X, Y) , and take F and G to be nonconstant polynomials $F(X, Y)$ and $G(X, Y)$ having no nonconstant common factor. Now R^t is the localization of $k(t)[X, Y]$ at the maximal ideal generated by (X, Y) and we let $\Phi = \Phi(X, Y) = F(X, Y) + tG(X, Y) \in k(t)[X, Y]$. Then the local equation of a generic member of the pencil generated by F and G is $\Phi = 0$, and we study its singularity tree. Elsewhere we shall relate this tree to the dicritical divisors of the said pencil.

Received by the editors May 25, 2011.

2010 *Mathematics Subject Classification*. Primary 14A05.

Key words and phrases. Pencil, singularity tree, generic member.

©2012 American Mathematical Society
 Reverts to public domain 28 years from publication

2. TERMINOLOGY

Consider the rational function field $L(t)$ in an indeterminate t over a field L . We define the t -extension S^t of any subring S of L by putting

$$S^t = \begin{cases} \text{the localization of the polynomial ring } S[t] \\ \text{at the multiplicative set of all those members of } S[t] \\ \text{whose coefficients generate the unit ideal in } S. \end{cases}$$

Note that $L^t = L(t)$ and, in Nagata's notation (page 18 of [Nag]), $S^t = S(t)$. Clearly

$$\begin{cases} S \mapsto S^t \text{ gives a map } \rho^t : \rho(L) \rightarrow \rho(L^t), \text{ where } \rho(L) \text{ and } \rho(L^t) \\ \text{are the sets of all subrings of } L \text{ and } L^t \text{ respectively.} \end{cases}$$

To supplement Section 2 of [Ab11] we introduce some more terms. So given any two-dimensional regular local domain R we put:

$$\begin{aligned} D_j(R)^\Delta &= \{o(S) : S \in Q_j(R)\} = j\text{-th layer of the set of all prime divisors of } R. \\ D(R)^\Delta &= \coprod_{j \in \mathbb{N}} D_j(R)^\Delta = \text{set of all prime divisors of } R. \\ Q_j(R) &= j\text{-th layer of the **quadratic tree** of } R. \\ Q(R) &= \coprod_{j \in \mathbb{N}} Q_j(R) = \text{the **quadratic tree** of } R. \\ Q(R)_j^\Delta &= Q_j(R) \amalg D(R)_j^\Delta = j\text{-th layer of the **full quadratic tree** of } R. \\ Q(R)^\Delta &= \coprod_{j \in \mathbb{N}} Q(R)_j^\Delta = \text{the **full quadratic tree** of } R. \\ R^t &= t\text{-extension of } R. \\ Q_j^t(R) &= \{S^t : S \in Q_j(R)\} = j\text{-th layer of } t\text{-extension of the quadratic tree of } R. \\ Q^t(R) &= \{S^t : S \in Q(R)\} = \coprod_{j \in \mathbb{N}} Q_j^t(R) = t\text{-extension of the quadratic tree of } R. \end{aligned}$$

Note that $Q^t(R)$ as well as its complement in $Q(R^t)$ are both infinite sets. They are sort of analogues of rational and irrational points of the real line. Actually they are more analogues to the k -algebraic and k -transcendental points of the projective line over $k(t)$, where k is a subfield of L . At any rate, the said two infinite sets have the same cardinality.

In Section 2 of [Ab11], for any $T \in Q(R)$ and any nonzero ideal I in R , we have defined the concept of the (R, T) -transform of I , which we denote by $(R, T)(I)$. Now we define:

$$\begin{aligned} \mathfrak{Q}_j(R, I) &= \{T \in Q_j(R) : (R, T)(I) \text{ is not principal}\} = j\text{-th layer of the **ideal tree** of } I \text{ in } R, \\ \mathfrak{Q}(R, I) &= \{T \in Q(R) : (R, T)(I) \text{ is not principal}\} = \coprod_{j \in \mathbb{N}} \mathfrak{Q}_j(R, I) = \text{the **ideal tree** of } I \text{ in } R, \\ \mathfrak{P}_j(R, I) &= \{T \in Q_j(R) : (R, T)(I) \neq T\} = j\text{-th layer of the **singularity tree** of } I \text{ in } R, \\ \mathfrak{P}(R, I) &= \{T \in Q(R) : (R, T)(I) \neq T\} = \coprod_{j \in \mathbb{N}} \mathfrak{P}_j(R, I) = \text{the **singularity tree** of } I \text{ in } R, \\ \text{and for any nonzero element } z \text{ in the quotient field of } R \text{ we define:} \\ \mathfrak{Q}_j(R, z) &= \mathfrak{Q}_j(R, J_R(z)) = j\text{-th layer of the **ideal tree** of } z \text{ in } R, \\ \mathfrak{Q}(R, z) &= \mathfrak{Q}(R, J_R(z)) = \coprod_{j \in \mathbb{N}} \mathfrak{Q}_j(R, z) = \text{the **ideal tree** of } z \text{ in } R, \\ \text{and for any nonzero element } F \text{ in } R \text{ we define:} \end{aligned}$$

$\mathfrak{P}_j(R, F) = \mathfrak{P}_j(R, FR) = j$ -th layer of the **singularity tree** of F in R ,
 $\mathfrak{P}(R, F) = \mathfrak{P}(R, FR) = \coprod_{j \in \mathbb{N}} \mathfrak{P}_j(R, F) =$ the **singularity tree** of F in R .

(2A) Transforms of elements. Given any $T \in Q(R)$ and any nonzero ideal I in R , we shall now extend the idea of the transform of I to the idea of the transform of specific elements of I . Given any generators x, y of $M(R)$ and any coefficient set κ of R , as explained in Section 2 of [Ab11], there is a unique finite QDT sequence $(R_j, x_j, y_j, \kappa_j)_{0 \leq j \leq \nu}$ of (R, x, y, κ) with $T = R_\nu$. We put

$$(R, T)(x, y, \kappa) = (x_\nu, y_\nu, \kappa_\nu)$$

and we call $(x_\nu, y_\nu, \kappa_\nu)$ the (R, T) -**transform** of (x, y, κ) . For $0 \leq j \leq \nu$ let

$$I_j = (R, R_j)(I) \quad \text{and} \quad d_j = \text{ord}_{R_j} I_j \quad \text{and} \quad z_j = \begin{cases} y_j & \text{if } j \neq 0 \text{ and } x_{j-1}/y_{j-1} \in M(R_j) \\ x_j & \text{if } j \neq 0 \text{ and } x_{j-1}/y_{j-1} \notin M(R_j) \\ x_j & \text{if } j = 0. \end{cases}$$

Note that then

$$I_0 = I \quad \text{and} \quad I_{j-1} R_j = I_j (z_j^{d_j} R_j) \quad \text{for } 1 \leq j \leq \nu.$$

Given any finite number of nonzero elements $f^{(1)}, \dots, f^{(p)}$ in I , clearly there exist unique nonzero elements $f_j^{(1)}, \dots, f_j^{(p)}$ in I_j for $0 \leq j \leq \nu$ such that

$$f_0^{(1)} = f^{(1)}, \dots, f_0^{(p)} = f^{(p)} \quad \text{and} \quad f_{j-1}^{(1)} = f_j^{(1)} z_j^{d_j}, \dots, f_{j-1}^{(p)} = f_j^{(p)} z_j^{d_j} \quad \text{for } 1 \leq j \leq \nu.$$

We put

$$(R, x, y, \kappa, T, I)(f^{(1)}, \dots, f^{(p)}) = (f_\nu^{(1)}, \dots, f_\nu^{(p)})$$

and we call $(f_\nu^{(1)}, \dots, f_\nu^{(p)})$ the (R, x, y, κ, T, I) -**transform** of $(f^{(1)}, \dots, f^{(p)})$. Note that for $0 \leq j \leq \nu$ we clearly have

$$(2.1) \quad \begin{cases} (R, R_j)(x, y, \kappa) = (x_j, y_j, \kappa_j) \text{ with } (R_j, T)(x_j, y_j, \kappa_j) = (x_\nu, y_\nu, \kappa_\nu), \\ \text{and } (R, R_j)(I) = I_j \text{ with } (R_j, T)(I_j) = I_\nu, \\ \text{and } (R, x, y, \kappa, R_j, I)(f^{(1)}, \dots, f^{(p)}) = (f_j^{(1)}, \dots, f_j^{(p)}) \\ \text{with } (R_j, x_j, y_j, \kappa_j, T, I_j)(f_j^{(1)}, \dots, f_j^{(p)}) = (f_\nu^{(1)}, \dots, f_\nu^{(p)}); \\ \text{moreover, if } I = (f^{(1)}, \dots, f^{(p)})R, \text{ then } I_j = (f_j^{(1)}, \dots, f_j^{(p)})R_j. \end{cases}$$

(2B) Tangent directions. Given any generators x, y of $M(R)$ and any coefficient set κ of R , picturing R as the local ring of a simple point on a surface, we can describe the tangent directions at that point thus. Let $K = H(R) = R/M(R)$, and let X, Y be indeterminates. Given any $0 \neq F \in M(R)$ we can write

$$(1) \quad F = \sum_{i+j=d} \tilde{F}_{ij} x^i y^j \quad \text{with} \quad d = \text{ord}_R F \quad \text{and} \quad \tilde{F}_{ij} \in R.$$

We can think of $F = 0$ as the local equation of a curve on the surface at that point. As in Remark (4.0) of [Ab11], we define the **initial form**

$$\text{info}(F) = \text{info}_{(R, x, y)} F$$

of F (relative to R, x, y) to be the nonzero homogeneous polynomial

$$\overline{F} = \overline{F}(X, Y) \in K[X, Y] = \text{grad}(R)$$

of degree d with coefficients in K obtained by putting

$$(2) \quad \overline{F}(X, Y) = \sum_{i+j=d} H_R(\tilde{F}_{ij}) X^i Y^j.$$

From that remark, recall the definitions of $\deg_X f$, $\deg_Y f$, $\deg_{X,Y} f$, $\text{ord}_X f$, $\text{ord}_Y f$, $\text{ord}_{X,Y} f$, $\text{info}_{X,Y} f$, with $f = f(X, Y) \in S[X, Y]$, where S is any ring. For any subset $\overline{S}$ of S such that 0 and 1 belong to $\overline{S}$, by $\overline{S}[X, Y]$ we **denote** the set of all $f = f(X, Y) \in S[X, Y]$ whose coefficients belong to $\overline{S}$. By $\overline{S}[X, Y]^h$ we **denote** the set of all $\lambda = \lambda(X, Y) \in \overline{S}[X, Y]$ which are homogeneous of some degree $\omega \in \mathbb{N}$, i.e.,

$$\lambda(X, Y) = \sum_{0 \leq i \leq \omega} \lambda_i X^i Y^{\omega-i} \quad \text{with } \lambda_i \in \overline{S}.$$

By $\overline{S}[X, Y]^{hm}$ we **denote** the set of all $\lambda \in \overline{S}[X, Y]^h$ which are Y -monic; i.e., for a (unique) nonnegative integer $\iota \leq \omega$ we have $\lambda_\iota = 1$ and $\lambda_i = 0$ for all $0 \leq i < \iota$. Note that $\iota = \text{ord}_X \lambda$. In the case of $\overline{S} = \kappa \subset R = S$, by $\kappa[X, Y]^{hmi}$ we **denote** the set of all $\lambda = \lambda(X, Y) \in \kappa[X, Y]^{hm}$ such that

$$H_R(\lambda) = H_R(\lambda)(X, Y) = \sum_{0 \leq i \leq \omega} H_R(\lambda_i) X^i Y^{\omega-i}$$

is irreducible in $K[X, Y]$. Note that this is so iff either $\iota = \omega = 1$ with $\lambda(X, Y) = X$ or $\iota = 0 < \omega \in \mathbb{N}_+$ and

$$\lambda(1, Z) = Z^\omega + \sum_{1 \leq i \leq \omega} H_R(\lambda_i) Z^{\omega-i}$$

is irreducible in the univariate polynomial ring $K[Z]$. Note that

$$\lambda(X, Y) \mapsto \mu(Z) = \lambda(1, Z) \quad \text{gives a bijection } \kappa[X, Y]^{hmi} \setminus \{X\} \rightarrow \kappa[Z]^{mi},$$

where $\kappa[Z]^{mi}$ is the set of all $\mu(Z)$ described in Section 2 of [Ab11]. The material around that description yields the bijection

$$(3) \quad \delta : \kappa[X, Y]^{hmi} \rightarrow Q_1(R), \quad \text{where } \delta(\lambda) = R' \in Q_1(R) \quad \text{with } M(R') = (x', y')R'$$

given by

$$(x', y') = \begin{cases} (\lambda(x/y, 1), y) & \text{if } y/x \notin R' = \delta(\lambda), \text{ i.e., equivalently if } \lambda = X, \\ (x, \lambda(1, y/x)) & \text{if } y/x \in R' = \delta(\lambda), \text{ i.e., equivalently if } \lambda \neq X. \end{cases}$$

Let κ' be the coefficient set of R' described in the said material. We call (R', x', y', κ') the QDT of (R, x, y, κ) in the **tangent direction** λ . We call δ the **tangential bijection**.

For any $\lambda \in \kappa[X, Y]^{hmi}$ let $R' = \delta(\lambda)$. Then obviously

$$(2.2) \quad \begin{cases} \text{upon letting } \omega(\lambda) = \deg_{X,Y} H_R(\lambda), \\ \text{we have } \omega(\lambda) = [R'/M(R') : K], \\ \text{where we regard } R'/M(R') \text{ to be an overfield of } K. \end{cases}$$

Let

$$(z, \zeta) = \begin{cases} (y, x/y) & \text{if } \lambda = X \\ (x, y/x) & \text{if } \lambda \neq X. \end{cases}$$

Then clearly

$$(2.3) \quad \left\{ \begin{array}{l} z \in M(R') \setminus M(R')^2 \text{ with } M(R)R' = zR' \\ \text{and upon letting } F' = F/z^d \\ \text{we have } (R, x, y, \kappa, R', FR)(F) = F' \in R' \setminus zR', \\ \text{and upon letting } d' = \text{ord}_{R'} F' \text{ with } e' = \text{ord}_{V'} \alpha(F'), \\ \text{where } \alpha : R' \rightarrow V' = R'/zR' \text{ is the residue class epimorphism,} \\ \text{we have that } d', e' \text{ are nonnegative integers with } d' \leq e' \\ \text{such that: } d' = 0 \Leftrightarrow e' = 0. \\ \text{Note that } e' = \text{ord}_{(R'/zR')} F' \text{ according to Remark (4.0) of [Ab11].} \end{array} \right.$$

Next let us prove the following:

$$(2.4) \quad \left\{ \begin{array}{l} \text{Let } e \text{ be the unique nonnegative integer such that} \\ (4) \quad \overline{F}(X, Y) \in (H_R(\lambda)^e K[X, Y]) \setminus (H_R(\lambda)^{e+1} K[X, Y]). \\ \text{Then } e' = e. \end{array} \right.$$

By symmetry the case of $\lambda = X$ follows from the case of $\lambda = Y$, and hence in proving (2.4) we may assume that $\lambda \neq X$. Now $(z, \zeta) = (x, y/x)$ and, in view of the references to [Ab3] and [Ab8] given at the beginning of this Section 2, there is a unique epimorphism $\theta : A = R[\zeta] \rightarrow K[Z]$ with $\ker(\theta) = M(R)A = zA$ such that $\theta(\zeta) = Z$ and $\theta(r) = H_R(r)$ for all $r \in R$. Equations (1) to (4) tell us that

$$(5) \quad F' \in A \text{ with } \theta(F') \in (\mu(Z)^e K[Z]) \setminus (\mu(Z)^{e+1} K[Z]).$$

Let $\beta : A \rightarrow R' = A_{(z, \mu(\zeta))A}$ and let $\gamma : K[Z] \rightarrow V = K[Z]_{\mu(Z)K[Z]}$ be the localization injections. Then by (2.3) and (5) we have

$$(6) \quad \text{ord}_{V'} \alpha(\beta(F')) = e' \text{ and } \text{ord}_V \gamma(\theta(F')) = e.$$

Now the permutability of localization and surjection given in (T11) on page 139 of [Ab8] yields a unique isomorphism $\gamma' : V \rightarrow V'$ such that $\gamma'\gamma\theta = \alpha\beta$, i.e., such that the diagram

$$\begin{array}{ccccc} R' & \xlongequal{\quad} & A_{(z, \mu(\zeta))A} & \xrightarrow{\alpha} & V' & \xlongequal{\quad} & R'/zR' \\ \beta \uparrow & & & & \uparrow \gamma' & & \\ A & \xrightarrow{\theta} & K[Z] & \xrightarrow{\gamma} & V & \xlongequal{\quad} & K[Z]_{\mu(Z)K[Z]} \end{array}$$

commutes. Therefore by (6) we get $e' = e$.

(2C) Tangents of curves. To get the tangents to the curve C whose equation is $F = 0$ at the point P whose local ring is R , by factoring $\overline{F}$ we obtain

$$\overline{F}(X, Y) = \widehat{F} \prod_{1 \leq i \leq h} H_R(\overline{F}_i)(X, Y)^{e_i}$$

with

$$\widehat{F} \in K^\times \text{ and } \deg_{X,Y} H_R(\overline{F}_i) = \omega_i \in \mathbb{N}_+,$$

where $h, e_1, \dots, e_h$ are positive integers and $\overline{F}_1, \dots, \overline{F}_h$ are pairwise distinct members of $\kappa[X, Y]^{hmi}$. These members may be viewed as the required **tangents**,

the positive integers $e_1, \dots, e_h$ may be called the corresponding **tangential multiplicities**, and the positive integers $\omega_1, \dots, \omega_h$ may be called the corresponding **tangential degrees**. Clearly

$$(2.5) \quad e_1\omega_1 + \dots + e_h\omega_h = d.$$

For $1 \leq i \leq h$ we let

$$z_i = \begin{cases} y & \text{if } \overline{F}_i = X \\ x & \text{if } \overline{F}_i \neq X \end{cases}$$

with

$$F'_i = F/z_i^d \quad \text{and} \quad R'_i = \delta(\overline{F}_i),$$

and upon letting

$$\alpha_i : R'_i \rightarrow V'_i = R'_i/z_i R'_i$$

be the residue class epimorphism and observing that

$$F'_i \in R'_i = \text{a two-dimensional regular local domain}$$

with

$$V'_i = \text{a one-dimensional regular local domain}$$

we let

$$d'_i = \text{ord}_{R'_i} F'_i \quad \text{and} \quad e'_i = \text{ord}_{V'_i} \alpha_i(F'_i).$$

Recalling that $\mathfrak{P}_1(R, F) = \{T \in Q_1(R) : (R, T)(FR) \neq T\}$, by (2.2) to (2.4) we get the following:

$$(2.6) \quad \begin{cases} R'_1, \dots, R'_h \text{ are exactly all the distinct members of } \mathfrak{P}_1(R, F) \\ \text{and for } 1 \leq i \leq h \text{ we have that } (R, x, y, \kappa, R'_i, FR)(F) = F'_i \\ \text{and } M(R)R'_i = z_i R'_i \text{ with } [R'_i/M(R'_i) : K] = \omega_i \\ \text{and } d'_i, e'_i \text{ are positive integers with } d'_i \leq e'_i = e_i. \end{cases}$$

By (2.5) and (2.6) we see that

$$(2.7) \quad \text{for } 1 \leq i \leq h \text{ we have } d'_i \leq d$$

and

$$(2.8) \quad \begin{cases} \text{if } e_i = 1 \text{ for some } i \in \{1, \dots, h\}, \\ \text{then for that } i \text{ we have } M(R'_i) = (F'_i, z_i)R'_i \end{cases}$$

and

$$(2.9) \quad \begin{cases} \text{if } d'_i \geq d \text{ for some } i \in \{1, \dots, h\}, \\ \text{then } h = 1 = \omega_1 = [R'_1/M(R'_1) : K] = 1 \\ \text{with } d = d'_1 = e'_1 = e_1. \end{cases}$$

(2D) Geometric language. Referring to pages 131-143 of my engineering book [Ab6], assertion (2.6) to (2.9) can be paraphrased geometrically thus. The total transform of C , obtained by applying a QDT centered at P , consists of the **exceptional line** E together with the **(proper) transform** C' of C . The exceptional line E cuts the proper transform C' at points $P_1, \dots, P_h$ with intersection multiplicities $e'_1 = e_1, \dots, e'_h = e_h$. Letting $d'_1, \dots, d'_h$ be the multiplicities of C' at $P_1, \dots, P_h$, we have $d'_1 \leq e'_1, \dots, d'_h \leq e'_h$ with $e'_1\omega_1 + \dots + e'_h\omega_h = d$. After suitable relabelling, $R'_1, \dots, R'_h$ may be regarded as the local rings of $P_1, \dots, P_h$, and then $z_1 = 0, \dots, z_h = 0$ become the equations of E at $P_1, \dots, P_h$, whereas

$F'_1 = 0, \dots, F'_h = 0$ become the equations C' at $P_1, \dots, P_h$, respectively. Assertion (2.9) corresponds to the second paragraph on page 135 of [Ab6] saying that: “Thus, if after one QDT there is no drop in multiplicity, then ... and the line $X = 0$ is not tangent to C' at P_1 .”

(2E) Magnitude. The above cited pages 131-143 constitute Lecture 18 of the book [Ab6]. The rest of that Lecture 18 is a concrete version my Arcata paper [Ab5], which itself was streamlined in my PAMS paper [Ab7]. The main idea is to introduce the concept of the magnitude, abbreviated as mag, of the nonzero principal ideal $J = FR$ in R . We allow J to be the unit ideal, i.e., d to be zero. The magnitude will be a nonnegative rational number or ∞ ; its integral part will tell us how many QDTs will be needed to decrease the order d of J . Here is a concise summary of the matter.

Expanding in the completion $\widehat{R}$ of R we write

$$F = \sum_{(i,j) \in \mathbb{N} \times \mathbb{N}} F_{ij} x^i y^j \quad \text{with } F_{ij} \in \kappa.$$

For any $q \in \mathbb{Q}_+$ = the set of all positive rationals, we define the **weighted order** and the **weighted initial form** by putting

$$\text{word}_{(R,x,y,1,q)} F = W,$$

where

$$W = \min\{i + jq : F_{ij} \neq 0\} \in \mathbb{Q}_+$$

and

$$\text{winfo}_{(R,x,y,1,q)} F = \sum_{(i,j) \in U} H_R(F_{ij}) X^i Y^j \in K[X, Y],$$

where

$$U = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i + jq = W\}.$$

Upon letting

$$D = \text{ord}_{R/xR} J \in \mathbb{N} \cup \{\infty\}$$

by Section 7 of [Ab5] we see that if $D \neq \infty$ and $J \neq y^d R$, then for a unique $q \in \mathbb{Q}_+$ we have $(0, D) \in U \neq \{(0, D)\}$. We let

$$\epsilon = \begin{cases} q & \text{if } D \neq \infty \text{ and } J \neq y^d R \\ \infty & \text{if } D \neq \infty \text{ and } J = y^d R \\ 0 & \text{if } D = \infty \end{cases}$$

and we define $\text{mag}(R, J, x, y)$ by putting

$$\text{mag}(R, J, x, y) = \epsilon.$$

If $D \neq \infty$ with $J \neq y^d R$, then we define the **normalized initial form** by putting

$$\text{ninfo}_{(R,x,y)} J = H_R(F_{0D})^{-1} \text{winfo}_{(R,x,y,1,\epsilon)} F.$$

We say that $\text{ninfo}_{(R,x,y)} J$ is a **monic power** to mean that

$$D \neq \infty \text{ with } J \neq y^d R \text{ and } \text{ninfo}_{(R,x,y)} J = (Y - p(X))^D \text{ with } p(X) \in K[X]^\times.$$

Note that mag and ninfo are independent of the quantities κ and F , and hence these quantities are not included in the notation. Likewise, winfo is independent of κ and hence κ is not included in the notation. Using (2.6) and Section 7 of [Ab5] we can show that:

Proposition (2.10). *In the above notation we have the following.*

(2.10.1) $\epsilon \in \{0\} \cup \mathbb{Q}_+ \cup \{\infty\}$.

(2.10.2) Assume that $D \neq \infty$ with $J \neq y^d R$. Then $D \in \mathbb{N}_+$ with $d \in \mathbb{N}_+$ and $\epsilon \in \mathbb{Q}_+$ with $D!\epsilon \in \mathbb{N}_+$. Moreover, $\epsilon < 1 \Leftrightarrow D \neq d \Leftrightarrow D > d$.

(2.10.3) Assume that $\text{ninfo}_{(R,x,y)} J$ is a monic power. Then $D = d \in \mathbb{N}_+$ with $\epsilon \in \mathbb{N}_+$ and $\text{ninfo}_{(R,x,y)} J = (Y - H_R(c))^d$ for a unique nonzero $c \in \kappa$. Moreover, ϵ increases by changing y to $y - cx$, i.e., $(x, y - cx)R = M(R)$ with $\text{mag}(R, J, x, y - cx) > \epsilon$.

(2.10.4) Assume that $\epsilon > 1$. Then $\mathfrak{P}_1(R, J)$ contains exactly one member R' , and upon letting (R', x', y', κ') be the corresponding QDT of (R, x, y, κ) we have $(x', y', \kappa') = (x, y/x, \kappa)$, and upon letting $J' = (R, R')(J)$ with $\epsilon' = \text{mag}(R', J', x', y')$ we have $\epsilon' = \epsilon - 1$ with the convention that $\infty \pm \theta = \infty$ for all $\theta \in \mathbb{Q}$, and upon letting

$$F' = (R, x, y, \kappa, R')(F) \text{ with } d' = \text{ord}_{R'} J' \text{ and } D' = \text{ord}_{R'/x'R'} J'$$

we have $F' = F/x^d$ with $D' = d \geq d'$, and if $\epsilon \neq \infty$, then upon letting

$$\Omega(X, Y) = \text{winfo}_{(R,x,y,1,\epsilon)} F \in K[X, Y]$$

and

$$\Omega'(X, Y) = \text{winfo}_{(R',x',y',1,\epsilon')} F' \in K[X, Y]$$

with

$$\Lambda(X, Y) = \text{ninfo}_{(R,x,y)} J \in K[X, Y] \text{ and } \Lambda'(X, Y) = \text{ninfo}_{(R',x',y')} J' \in K[X, Y]$$

we have

$$\Omega(X, XY) = X^d \Omega'(X, Y) \text{ with } \Lambda(X, XY) = X^d \Lambda'(X, Y)$$

and

$$\text{ninfo}_{(R,x,y)} J \text{ is a monic power} \Leftrightarrow \text{ninfo}_{(R',x',y')} J' \text{ is a monic power.}$$

Inspired by (2.9) and the definition of ϵ , we make the following definitions. We say that (R, J) is **resolved** to mean that $J = y^{*d} R$ for some $y^* \in M(R) \setminus M(R)^2$. We say that (R, J) is **unitangent** to mean that $J = (y^{*d} + \tilde{y})R$ for some $y^* \in M(R) \setminus M(R)^2$ and $\tilde{y} \in M(R)^{d+1}$. Recall that a ring A is said to be **pseudogeometric** if A is Noetherian and, for any prime ideal P in A , the integral closure of A/P in any finite algebraic field extension of the quotient field of A/P is a finite (A/P) -module.

We let

$$B(R, J) = \{(x^*, y^*) : (x^*, y^*)R = M(R) \text{ and } \text{ord}_{R/x^*} = \text{ord}_R J\}$$

and

$$\widehat{e} = \begin{cases} \max\{\text{mag}(R, J, x^*, y^*) : (x^*, y^*) \in B(R, J)\} & \text{if } B(R, J) \neq \emptyset \\ 1 & \text{if } B(R, J) = \emptyset, \end{cases}$$

where $\max$ denotes the least upper bound, and we define $\text{mag}[R](J)$ by putting

$$\text{mag}[R](J) = \widehat{e}.$$

We define the **integral magnitude** $\text{inmag}[R](J)$ to be the maximum $s \in \mathbb{N}_+ \cup \{\infty\}$ for which there exists a sequence $(R_j, J_j)_{0 \leq j < s}$ with $(R_0, J_0) = (R, J)$ such that for $1 \leq u < s$ we have $R_j \in Q_1(R_{j-1})$ with $J_j = (R_{j-1}, R_j)(J_{j-1})$ and $\text{ord}_{R_j} J_j = d$. By the **integral part** $[q]$ of any $q \in \mathbb{Q}$ we mean the largest integer $\leq q$; we also put $[\infty] = \infty$.

Using (2.10) and Sections 2 to 9 of [Ab5] we can show that:

Proposition (2.11). *In the above notation with $\text{inmag}[R](J) = s$ we have the following.*

- (2.11.1) $1 \leq \widehat{\epsilon} \in \mathbb{Q}_+ \cup \{\infty\}$.
- (2.11.2) $d = 0 \Rightarrow (R, J)$ is resolved $\Rightarrow s = \widehat{\epsilon} = \infty$.
- (2.11.3) $\widehat{\epsilon} > 1 \Leftrightarrow \widehat{\epsilon} \neq 1 \Leftrightarrow (R, J)$ is unitangent.
- (2.11.4) $\widehat{\epsilon} \neq \infty \Rightarrow d! \widehat{\epsilon} \in \mathbb{N}_+$.
- (2.11.5) $\widehat{\epsilon} < 2 \Rightarrow \text{ord}_{R'}(R, R')(J) < d$ for all $R' \in Q_1(R)$.
- (2.11.6) Assume that $\widehat{\epsilon} \geq 2$. Then $\mathfrak{P}_1(R, J)$ contains exactly one member R' , and upon letting $J' = (R, R')(J)$ we have $B(R', J') \neq \emptyset$ and $\text{mag}[R'](J') = \widehat{\epsilon} - 1$ with the convention that $\infty \pm \theta = \infty$ for all $\theta \in \mathbb{Q}$.
- (2.11.7) $\epsilon \geq \widehat{\epsilon} \neq \infty \Rightarrow \epsilon = \widehat{\epsilon}$ and $\text{ninfo}_{(R, x, y)} J$ is not a monic power.
- (2.11.8) $1 \leq \epsilon < \widehat{\epsilon} \Rightarrow \text{ninfo}_{(R, x, y)} J$ is a monic power.
- (2.11.9) $1 < \widehat{\epsilon} \neq \infty \Rightarrow \text{mag}(R, J, x^*, y^*) = \widehat{\epsilon}$ for some $(x^*, y^*) \in B(R, J)$.
- (2.11.10) $\widehat{\epsilon} = \infty \Rightarrow (\widehat{R}, \widehat{J}\widehat{R})$ is resolved.
- (2.11.11) R is pseudogeometric and $\widehat{\epsilon} = \infty \Rightarrow (R, J)$ is resolved.
- (2.11.12) $d \neq 0 \Rightarrow s = [\widehat{\epsilon}]$. Also see (9.4) and (9.14) of [Ab5].

3. BUSHES AND THICKETS

Let R be a two-dimensional regular local domain. We shall now introduce several types of subsets of $Q(R)$.

Let $B \subset Q(R)$. We define the **breadth** of B by putting

$$\beta(B) = \max\{|Q_j(R) \cap B| : j \in \mathbb{N}\}$$

and we note that $\beta(B) \in \mathbb{N} \cup \{\infty\}$ and $\beta(B) = 0 \Leftrightarrow B = \emptyset$. By a **pine** of B we mean an infinite QDT sequence $S = (S_j)_{0 \leq j < \infty}$ of some member S_0 of B such that for all $j \in \mathbb{N}$ we have $Q_j(S_1) \cap B = \{S_{j+1}\}$; we call S_0, S_1, S_j the **base**, **subbase**, **j -th component** of S respectively. By a **subpine** of the pine S we mean a pine $S' = (S'_j)_{0 \leq j < \infty}$ of B such that for some $j' \in \mathbb{N}$ we have $S'_j = S_{j'+j}$ for all $j \in \mathbb{N}$; if $j' \neq 0$, then we call S' a **proper subpine** of S . By a **maximal pine** of B we mean a pine of B which is not a proper subpine of any pine of B . All these definitions are relative to R .

Let $B \subset Q(R)$ and let I be a nonzero ideal in R . A pine $S = (S_j)_{0 \leq j < \infty}$ of B is said to be **stationary** at (R, I) if there exist (unique) nonnegative integers

$$r = \text{rlen}(R, S, I) \quad \text{and} \quad u = \text{uord}(R, S, I),$$

which we respectively call the **reduced length** and **ultimate order** of (R, S, I) , such that

$$\begin{cases} \text{for all } r \leq j \in \mathbb{N} \text{ we have} \\ [S_j/M(S_j) : S_r/M(S_r)] = 1 \text{ and } \text{ord}_{S_j}(R, S_j)(I) = u \end{cases}$$

and

$$\begin{cases} \text{for all } r > j \in \mathbb{N} \text{ we have} \\ \text{either } [S_r/M(S_r) : S_j/M(S_j)] > 1 \text{ or } \text{ord}_{S_j}(R, S_j)(I) > u. \end{cases}$$

When the above conditions are satisfied, we define the **solidified subpine** $S' = (S'_j)_{0 \leq j < \infty}$ of S by putting $S'_j = S_{j+r}$ for all $j \in \mathbb{N}$. B is said to be **stable** at (R, I) if every pine of B is stationary at (R, I) . B is said to be **strongly square free** at (R, I) if for every pine S of B which is stationary at (R, I) we have $\text{uord}(R, S, I) = 1$.

Finally, we say that (R, I) is **strongly square free** if $\mathfrak{P}(R, I)$ is strongly square free at (R, I) .

By a **bush** at R we mean a union of finite QDT sequences of R . By a **thicket** at R we mean a union of infinite QDT sequences of R . Equivalently, a thicket at R is a bush B at R such that for every $T \in B$ we have $Q_1(T) \cap B \neq \emptyset$. Formally speaking, a bush is a subset B of $Q(R)$ such that $B = \{R_{ij} : i \in \Omega \text{ and } 0 \leq j \leq \nu_i\}$ for some set Ω of finite QDT sequences $(R_{ij})_{0 \leq j \leq \nu_i}$ of R with $i \in \Omega$. Likewise, a thicket is a subset B of $Q(R)$ such that $B = \{R_{ij} : i \in \Omega \text{ and } 0 \leq j < \infty\}$ for some set Ω of infinite QDT sequences $(R_{ij})_{0 \leq j < \infty}$ of R with $i \in \Omega$; now upon letting $R_{ilj} = R_{ij}$ we get a finite QDT sequence $(R_{ilj})_{0 \leq j \leq l}$ of R for all $(i, l) \in \Omega \times \mathbb{N}$ with $B = \{R_{ilj} : i \in \Omega, l \in \mathbb{N}, 0 \leq j \leq l\}$ showing that every thicket is a bush. Note that a thicket of finite nonzero breadth β has exactly β maximal pines.

In Figure 18.2 on page 132 of [Ab6], the finite tree emanating from P is a bush but not a thicket. Figure 5 on page 426 of [Ab4] with the dotted branches going to infinity is a thicket and hence also a bush; the dotted branches extended downwards give the maximal pines.

By a **stable thicket** at (R, I) we mean a thicket B at R such that B is stable at (R, I) . By (2.5) and (2.9) together with Section 2 of [Ab11] we get:

Theorem (3.1). *For any $0 \neq F \in R$, $\mathfrak{P}(R, FR)$ is a stable thicket at (R, FR) and its breadth is finite. For any $M(R)$ -primary ideal J in R , $\mathfrak{Q}(R, J)$ is a finite bush at R .*

Note (3.2). A direct characterization of a thicket B of finite nonzero breadth can be given by saying that B is subset of $Q(R)$ such that: (i) $R \in B$, (ii) for every $T \in B$ we have $Q_1(T) \cap B \neq \emptyset$, (iii) for every finite QDT sequence $(R_j)_{0 \leq j \leq \nu}$ of R with $R_\nu \in B$ we have $R_j \in B$ for $0 \leq j < \nu$, and (iv) there exists a (unique) positive integer $\beta = \beta(B)$, called the **breadth** of B , together with a (unique) nonnegative integer $\mu = \mu(B)$, called the **height** of B , for which $|(Q_0(R) \cap B| \leq |(Q_1(R) \cap B| \leq \dots \leq |(Q_{\mu-1}(R) \cap B| < \beta$ and $|Q_\mu \cap B| = |Q_{\mu+1}(R) \cap B| = \dots = \beta$.

4. QDTs AND GENERIC QDTs

As in the beginning of Section 2, let $L^t = L(t)$, where t is an indeterminate over a field L , and let $\rho^t : \rho(L) \rightarrow \rho(L^t)$ be the map given by $S \mapsto S^t$, where $\rho(L)$ and $\rho(L^t)$ are the sets of all subrings of L and L^t respectively. In view of (T30) on page 233 of [Ab8] we see that:

$$(4.1) \quad \begin{cases} \text{If } S \in \rho(L) \text{ is a local domain, then } S^t \text{ is a local domain with} \\ \dim(S^t) = \dim(S) \text{ and } M(S^t) = M(S)S^t \text{ and } S^t \cap L = S. \\ \text{If } S \in \rho(L) \text{ is a regular local domain, then } S^t \text{ is a regular local domain.} \end{cases}$$

Now let R be a two-dimensional regular local domain with quotient field L , and let F, G be nonzero elements in $M(R)$ which generate an $M(R)$ -primary ideal J in R . Taking over the setup of Subsections (2B) and (2C), and extending it to G , we can write

$$(1G) \quad G = \sum_{i+j=d(G)} \tilde{G}_{ij} x^i y^j \quad \text{with} \quad d(G) = \text{ord}_R G \quad \text{and} \quad \tilde{G}_{ij} \in R$$

and

$$(2G) \quad \overline{G} = \overline{G}(X, Y) = \text{info}(G) = \text{info}_{(R, x, y)} G = \sum_{i+j=d(G)} H_R(\tilde{G}_{ij}) X^i Y^j.$$

For any $\lambda \in \kappa[X, Y]^{hmi}$, upon letting $R' = \delta(\lambda)$ and (z, ζ) be as in Subsection (2B), by (2.3) and (2.4) we see that:

$$(3G) \quad \left\{ \begin{array}{l} z \in M(R') \setminus M(R')^2 \text{ with } M(R)R' = zR' \\ \text{and upon letting } G' = G/z^{d(G)} \\ \text{we have } (R, x, y, \kappa, R', GR)(G) = G' \in R' \setminus zR' \\ \text{and upon letting } d'(G) = \text{ord}_{R'} G' \text{ with } e'(G) = \text{ord}_{(R'/zR')} G' \\ \text{we have that } d'(G), e'(G) \text{ are nonnegative integers with } d'(G) \leq e'(G) \\ \text{such that } d'(G) = 0 \Leftrightarrow e'(G) = 0 \end{array} \right.$$

and

$$(4G) \quad \left\{ \begin{array}{l} \text{Let } e(G) \text{ be the unique nonnegative integer such that} \\ \overline{G}(X, Y) \in (H_R(\lambda)^{e(G)} K[X, Y]) \setminus (H_R(\lambda)^{e(G)+1} K[X, Y]). \\ \text{Then } e'(G) = e(G). \end{array} \right.$$

By factoring $\overline{G}$ we obtain

$$\overline{G}(X, Y) = \widehat{G} \prod_{1 \leq i \leq h(G)} H_R(\overline{G}_i)(X, Y)^{e_i(G)}$$

with

$$\widehat{G} \in K^\times \quad \text{and} \quad \deg_{X, Y} H_R(\overline{G}_i) = \omega_i(G) \in \mathbb{N}_+,$$

where $h(G), e_1(G), \dots, e_{h(G)}(G)$ are positive integers and $\overline{G}_1, \dots, \overline{G}_{h(G)}$ are pairwise distinct members of $\kappa[X, Y]^{hmi}$. Clearly

$$(5G) \quad e_1(G)\omega_1(G) + \dots + e_h(G)\omega_h(G) = d(G).$$

For $1 \leq i \leq h(G)$ we let

$$z_i(G) = \begin{cases} y & \text{if } \overline{G}_i = X \\ x & \text{if } \overline{G}_i \neq X \end{cases}$$

with

$$G'_i = G/z_i(G)^{d(G)} \quad \text{and} \quad R'_i(G) = \delta(\overline{G}_i),$$

and upon letting

$$\alpha_i(G) : R'_i(G) \rightarrow V'_i(G) = R'_i(G)/z_i(G)R'_i(G)$$

be the residue class epimorphism and observing that

$$G'_i \in R'_i(G) \text{ is a two-dimensional regular local domain}$$

with

$$V'_i(G) \text{ is a one-dimensional regular local domain}$$

we let

$$d'_i(G) = \text{ord}_{R'_i(G)} G'_i \quad \text{and} \quad e'_i(G) = \text{ord}_{V'_i(G)} \alpha_i(G)(G'_i).$$

By (2G) to (4G) we get the following:

$$(6G) \quad \begin{cases} R'_1(G), \dots, R'_{h(G)}(G) \text{ are exactly all the distinct members of } \mathfrak{P}_1(R, G) \\ \text{and for } 1 \leq i \leq h(G) \text{ we have that } (R, x, y, \kappa, R'_i(G), GR)(G) = G'_i \\ \text{and } M(R)R'_i(G) = z_i(G)R'_i(G) \text{ with } [R'_i(G)/M(R'_i(G)) : K] = \omega_i(G) \\ \text{and } d'_i(G), e'_i(G) \text{ are positive integers with } d'_i(G) \leq e'_i(G) = e_i(G). \end{cases}$$

By (5G) and (6G) we see that

$$(7G) \quad \text{for } 1 \leq i \leq h(G) \text{ we have } d'_i(G) \leq d(G)$$

and

$$(8G) \quad \begin{cases} \text{if } e_i(G) = 1 \text{ for some } i \in \{1, \dots, h(G)\}, \\ \text{then for that } i \text{ we have } M(R'_i(G)) = (G'_i, z_i(G))R'_i(G) \end{cases}$$

and

$$(9G) \quad \begin{cases} \text{if } d'_i(G) \geq d(G) \text{ for some } i \in \{1, \dots, h(G)\}, \\ \text{then } h(G) = 1 = \omega_1(G) = [R'_1(G)/M(R'_1(G)) : K] = 1 \\ \text{with } d(G) = d'_1(G) = e'_1(G) = e_1(G). \end{cases}$$

Turning to the two-dimensional regular local domain R^t with quotient field L^t and maximal ideal $M(R^t) = (x, y)R^t$, we get a coefficient set $\kappa(t)$ of R^t , where $\kappa(t)$ is the set of all members of $L(t)$ which, when written in reduced form with monic denominators, has for its numerator and denominator, polynomial expressions in t with coefficients in κ . Note that the residue field $H(R^t)$ can be identified with $K(\tau)$, where the element $\tau = H_{R^t}(t)$ is transcendental over K . For the graded ring of R^t we have $\text{grad}(R^t) = K(\tau)[X, Y]$. Now $\kappa(t)[X, Y]^{hmi}$ is the set of all $\lambda = \lambda(X, Y) \in \kappa(t)[X, Y]^{hmi}$ such that $H_{R^t}(\lambda) = H_{R^t}(\lambda(X, Y))$ is irreducible in $K(\tau)[X, Y]$. By changing (R, κ, δ) to $(R^t, \kappa(t), \delta^t)$ in (2B)(3) we get the **tangential bijection** $\delta^t : \kappa(t)[X, Y]^{hmi} \rightarrow Q_1(R^t)$ as described there. This also gives the QDT (R', x', y', κ') of $(R^t, x, y, \kappa(t))$ in the **tangent direction** $\lambda \in \kappa(t)[X, Y]^{hmi}$. Clearly

$$(4.2) \quad \begin{cases} \kappa[X, Y]^{hmi} \subset \kappa(t)[X, Y]^{hmi} \text{ and} \\ \text{for all } \lambda \in \kappa[X, Y]^{hmi} \text{ we have } \delta^t(\lambda) = \delta(\lambda)^t. \end{cases}$$

Given any $0 \neq \Phi \in M(R^t)$, we can write

$$(1\Phi) \quad \Phi = \sum_{i+j=d(\Phi)} \tilde{\Phi}_{ij} x^i y^j \quad \text{with} \quad d(\Phi) = \text{ord}_{R^t} \Phi \quad \text{and} \quad \tilde{\Phi}_{ij} \in R$$

and

$$(2\Phi) \quad \overline{\Phi} = \overline{\Phi}(X, Y) = \text{info}(\Phi) = \text{info}_{(R^t, x, y)} \Phi = \sum_{i+j=d(\Phi)} H_{R^t}(\tilde{\Phi}_{ij}) X^i Y^j.$$

For any $\lambda \in \kappa(t)[X, Y]^{hmi}$, upon letting $R' = \delta^t(\lambda)$ and (z, ζ) be as in Subsection (2B), by (2.3) and (2.4) we see that:

$$(3\Phi) \quad \begin{cases} z \in M(R') \setminus M(R')^2 \text{ with } M(R^t)R' = zR' \\ \text{and upon letting } \Phi' = \Phi/z^{d(\Phi)} \\ \text{we have } (R, x, y, \kappa(t), R', \Phi R^t)(\Phi) = \Phi' \in R' \setminus zR' \\ \text{and upon letting } d'(\Phi) = \text{ord}_{R'} \Phi' \text{ with } e'(\Phi) = \text{ord}_{(R'/zR')} \Phi' \\ \text{we have that } d'(\Phi), e'(\Phi) \text{ are nonnegative integers with } d'(\Phi) \leq e'(\Phi) \\ \text{such that } d'(\Phi) = 0 \Leftrightarrow e'(\Phi) = 0 \end{cases}$$

and

$$(4\Phi) \quad \begin{cases} \text{Let } e(\Phi) \text{ be the unique nonnegative integer such that} \\ \overline{\Phi}(X, Y) \in (H_{R^t}(\lambda)^{e(\Phi)} K[X, Y]) \setminus (H_{R^t}(\lambda)^{e(\Phi)+1} K(\tau)[X, Y]). \\ \text{Then } e'(\Phi) = e(\Phi). \end{cases}$$

By factoring $\overline{\Phi}$ we obtain

$$\overline{\Phi}(X, Y) = \widehat{\Phi} \prod_{1 \leq i \leq h(\Phi)} H_{R^t}(\overline{\Phi}_i)(X, Y)^{e_i(\Phi)}$$

with

$$\widehat{\Phi} \in K(\tau)^\times \text{ and } \deg_{X,Y} H_{R^t}(\overline{\Phi}_i) = \omega_i(\Phi) \in \mathbb{N}_+,$$

where $h(\Phi), e_1(\Phi), \dots, e_{h(\Phi)}(\Phi)$ are positive integers and $\overline{\Phi}_1, \dots, \overline{\Phi}_{h(\Phi)}$ are pairwise distinct members of $\kappa(t)[X, Y]^{hmi}$. Clearly

$$(5\Phi) \quad e_1(\Phi)\omega_1(\Phi) + \dots + e_{h(\Phi)}(\Phi)\omega_{h(\Phi)}(\Phi) = d(\Phi).$$

For $1 \leq i \leq h(\Phi)$ we let

$$z_i(\Phi) = \begin{cases} y & \text{if } \overline{\Phi}_i = X \\ x & \text{if } \overline{\Phi}_i \neq X \end{cases}$$

with

$$\Phi'_i = \Phi/z_i(\Phi)^{d(\Phi)} \text{ and } R'_i(\Phi) = \delta^t(\overline{\Phi}_i),$$

and upon letting

$$\alpha_i(\Phi) : R'_i(\Phi) \rightarrow V'_i(\Phi) = R'_i(\Phi)/z_i(\Phi)R'_i(\Phi)$$

be the residue class epimorphism and observing that

$$\Phi'_i \in R'_i(\Phi) \text{ is a two-dimensional regular local domain}$$

with

$$V'_i(\Phi) \text{ is a one-dimensional regular local domain}$$

we let

$$d'_i(\Phi) = \text{ord}_{R'_i(\Phi)} \Phi'_i \text{ and } e'_i(\Phi) = \text{ord}_{V'_i(\Phi)} \alpha_i(\Phi)(\Phi'_i).$$

By (2Φ) to (4Φ) we get the following:

$$(6\Phi) \quad \begin{cases} R'_1(\Phi), \dots, R'_{h(\Phi)}(\Phi) \text{ are exactly all the distinct members of } \mathfrak{P}_1(R^t, \Phi) \\ \text{and for } 1 \leq i \leq h(\Phi) \text{ we have that } (R^t, x, y, \kappa(t), R'_i(\Phi), \Phi R^t)(\Phi) = \Phi'_i \\ \text{and } M(R^t)R'_i(\Phi) = z_i(\Phi)R'_i(\Phi) \text{ with } [R'_i(\Phi)/M(R'_i(\Phi)) : K(\tau)] = \omega_i(\Phi) \\ \text{and } d'_i(\Phi), e'_i(\Phi) \text{ are positive integers with } d'_i(\Phi) \leq e'_i(\Phi) = e_i(\Phi). \end{cases}$$

By (5Φ) and (6Φ) we see that

$$(7\Phi) \quad \text{for } 1 \leq i \leq h(\Phi) \text{ we have } d'_i(\Phi) \leq d(\Phi)$$

and

$$(8\Phi) \quad \begin{cases} \text{if } e_i(\Phi) = 1 \text{ for some } i \in \{1, \dots, h(\Phi)\}, \\ \text{then for that } i \text{ we have } M(R'_i(\Phi)) = (\Phi'_i, z_i(\Phi))R'_i(\Phi) \end{cases}$$

and

$$(9\Phi) \quad \begin{cases} \text{if } d'_i(\Phi) \geq d(\Phi) \text{ for some } i \in \{1, \dots, h(\Phi)\}, \\ \text{then } h(\Phi) = 1 = \omega_1(\Phi) = [R'_1(\Phi)/M(R'_1(\Phi)) : K(\tau)] = 1 \\ \text{with } d(\Phi) = d'_1(\Phi) = e'_1(\Phi) = e_1(\Phi). \end{cases}$$

Henceforth **assume that** $\Phi = F + tG$ **and recall that** $J = (F, G)R$. Now clearly

$$d, d(G), d(\Phi) \text{ are positive integers with } d(\Phi) = \min(d, d(G))$$

and

$$\overline{\Phi} = \begin{cases} \overline{F} & \text{if } d < d(G) \\ \tau \overline{G} & \text{if } d > d(G) \\ \overline{F} + \tau \overline{G} & \text{if } d = d(G). \end{cases}$$

Therefore we have the following three exhaustive but mutually disjoint cases:

- case (E) when $d = d(G)$ with $\overline{F}/\overline{G} \notin K$;
- case (F) when either $d = d(G)$ with $\overline{F}/\overline{G} \in K$ or $d < d(G)$;
- case (G) when $d > d(G)$.

Calculation (4.3). Assume case (E). Recalling the exact definition of GCD from Remark (4.0) of [Ab7] let

$$F^E = \text{GCD}(\overline{F}, \overline{G}) \text{ with } F^D = \overline{F}/F^E \text{ and } G^D = \overline{G}/F^E.$$

Then

$$\overline{F} = F^E F^D \text{ and } \overline{G} = F^E G^D,$$

where F^E, F^D, G^D are nonzero homogeneous polynomials in $K[X, Y]$ whose (X, Y) -degree is $d(E), d(D), d(D)$ respectively, where

$$d(E) \in \mathbb{N} \text{ and } d(D) = d - d(E) \in \mathbb{N}_+.$$

The factorizations of $\overline{F}$ and $\overline{G}$ can be relabelled so that

$$\begin{cases} \text{for some } a, b, c, h^* \text{ in } \mathbb{N} \text{ with } a + b + c = h^* \leq h \\ \text{and } e_1^*, \dots, e_{h^*}^* \text{ in } \mathbb{N}_+ \text{ with } e_1^* = e_1 = e_1(G), \dots, e_a^* = e_a = e_a(G) \\ \text{and } e_{a+1}^* = e_{a+1} < e_{a+1}(G), \dots, e_{a+b}^* = e_{a+b} < e_{a+b}(G) \\ \text{and } e_{a+b+1}^* = e_{a+b+1}(G) < e_{a+b+1}, \dots, e_{a+b+c}^* < e_{a+b+c}(G) < e_{a+b+c} \end{cases}$$

we have

$$H_R(\overline{F}_1) = H_R(\overline{G}_1), \dots, H_R(\overline{F}_{h^*}) = H_R(\overline{G}_{h^*})$$

and

$$F^E = \prod_{1 \leq i \leq h^*} H_R(\overline{F}_i)(X, Y)^{e_i^*}$$

with

$$F^D = \widehat{F} \prod_{a+b+1 \leq i \leq a+b+c} H_R(\overline{F}_i)(X, Y)^{e_i^* - e_i} \prod_{h^*+1 \leq i \leq h} H_R(\overline{F}_i)(X, Y)^{e_i}$$

and

$$G^D = \widehat{G} \prod_{a+1 \leq i \leq a+b} H_R(\overline{G}_i)(X, Y)^{e_i^* - e_i(G)} \prod_{h^*+1 \leq i \leq h} H_R(\overline{G}_i)(X, Y)^{e_i(G)}.$$

For $1 \leq i \leq h^*$ let

$$R_i^* = \lambda(\overline{F}_i) \text{ with } F_i^* = F/z_i^d \text{ and } G_i^* = G/z_i^d.$$

Then in view of (2.1) and (2.3) we see that

$$\begin{cases} R_1^*, \dots, R_{h^*}^* \text{ are exactly all the distinct members of } \mathfrak{Q}_1(R, J) \\ \text{and for } 1 \leq i \leq h^* \text{ we have } (R, x, y, \kappa, R_i^*, J)(F, G) = (F_i^*, G_i^*). \end{cases}$$

We have

$$\overline{\Phi} = F^E(F^D + \tau G^D)$$

and hence $h(\Phi) = h^* + 1$ and the factorization of Φ can be relabelled so that for $1 \leq i \leq h^*$ we have $\overline{\Phi}_i = \overline{F}_i$ with $e_i(\Phi) = e_i^*$. Now $e_{h^*+1}(\Phi) = 1$ and there is a unique $\Phi^D \in \kappa(t)[X, Y]^{hmi}$ with $\widehat{\Phi}\Phi^D = F^D + tG^D$. By (1 Φ) to (8 Φ) we conclude that

$$\begin{cases} \text{we have (6}\Phi\text{) and in that setup:} \\ \text{for } 1 \leq i < h(\Phi) \text{ we have } R'_i(\Phi) = (R_i^*)^t \text{ with } \Phi'_i = F_i^* + tG_i^*; \\ \text{for } i = h(\Phi) \text{ we have } \overline{\Phi}_i = \Phi^D \text{ with } d'_i = 1 \text{ and } M(R'_i(\Phi)) = (\Phi'_i, z_i(\Phi))R'_i(\Phi); \\ \text{moreover, } R'_1, \dots, R'_{h(\Phi)-1} \text{ are exactly all the distinct members of } \mathfrak{P}_1(R^t, \Phi) \\ \text{which are of the form } \delta(\lambda)^t \text{ for some } \lambda \in \kappa[X, Y]^{hmi}. \end{cases}$$

Calculation (4.4). Assume case (F). Now $d(\Phi) = d$ with $h(\Phi) = h$ and hence the factorization of $\overline{\Phi}$ can be relabelled so that for $1 \leq i \leq h$ we have $\overline{\Phi}_i = \overline{F}_i$ and $e_i(\Phi) = e_i$. For $1 \leq i \leq h$ let $R_i^* = \lambda(\overline{F}_i)$ with $F_i^* = F/z_i^d$ and $G_i^* = G/z_i^d$. Then in view of (2.1) and (2.3) we see that

$$\begin{cases} R_1^*, \dots, R_h^* \text{ are exactly all the distinct members of } \mathfrak{Q}_1(R, J) \\ \text{and for } 1 \leq i \leq h \text{ we have } (R, x, y, \kappa, R_i^*, J)(F, G) = (F_i^*, G_i^*) \end{cases}$$

and by (1 Φ) to (6 Φ) we conclude that

$$\begin{cases} \text{we have (6}\Phi\text{) and in that setup:} \\ \text{for } 1 \leq i \leq h(\Phi) \text{ we have } R'_i(\Phi) = (R_i^*)^t \text{ with } \Phi'_i = F_i^* + tG_i^*, \\ \text{and all the members of } \mathfrak{P}_1(R^t, \Phi) \text{ are of the form } \delta(\lambda)^t \text{ for some } \lambda \in \kappa[X, Y]^{hmi}. \end{cases}$$

Calculation (4.5). Assume case (G). Now $d(\Phi) = d(G)$ with $h(\Phi) = h(G)$ and hence the factorization of $\overline{\Phi}$ can be relabelled so that for $1 \leq i \leq h(G)$ we have $\overline{\Phi}_i = \overline{G}_i$ and $e_i(\Phi) = e_i(G)$. For $1 \leq i \leq h(G)$ let $R_i^* = \lambda(\overline{G}_i)$ with $F_i^* = F/z_i^{d(G)}$ and $G_i^* = G/z_i^{d(G)}$. Then in view of (2.1) and (3G) we see that

$$\begin{cases} R_1^*, \dots, R_{h(G)}^* \text{ are exactly all the distinct members of } \mathfrak{Q}_1(R, J) \\ \text{and for } 1 \leq i \leq h(G) \text{ we have } (R, x, y, \kappa, R_i^*, J)(F, G) = (F_i^*, G_i^*) \end{cases}$$

and by (1 Φ) to (6 Φ) we conclude that

$$\begin{cases} \text{we have (6}\Phi\text{) and in that setup:} \\ \text{for } 1 \leq i \leq h(\Phi) \text{ we have } R'_i(\Phi) = (R_i^*)^t \text{ with } \Phi'_i = F_i^* + tG_i^*, \\ \text{and all the members of } \mathfrak{P}_1(R^t, \Phi) \text{ are of the form } \delta(\lambda)^t \text{ for some } \lambda \in \kappa[X, Y]^{hmi}. \end{cases}$$

The above three calculations yield:

Theorem (4.6). Recall that $J = (F, G)R$ is an $M(R)$ -primary ideal in R and $\Phi = F + tG$. Let $R_1^*, \dots, R_{h^*}^*$ be all the distinct members of $\mathfrak{Q}_1(R, J)$ and for $1 \leq i \leq h^*$ let $\lambda_i^* = \delta^{-1}(R_i^*)$ and $R_i' = (R_i^*)^t$ and $(F_i^*, G_i^*) = (R, x, y, \kappa, R_i^*, J)(F, G)$. Then we have the following.

(4.6.1) $R_1', \dots, R_{h^*}'$ are exactly all those distinct members of $\mathfrak{P}_1(R^t, \Phi)$ which are of the form $\delta(\lambda)^t$ for some $\lambda \in \kappa[X, Y]^{hmi}$. $\lambda_1^*, \dots, \lambda_{h^*}^*$ are exactly all the distinct tangent directions of Φ in $\kappa[X, Y]^{hmi}$. For $1 \leq i \leq h^*$ we have $(R^t, x, y, \kappa(t), R_i', \Phi R^t)(\Phi) = F_i^* + tG_i^*$ and $\delta^t(\lambda_i^*) = R_i'$.

(4.6.2) If either $\text{ord}_R F \neq \text{ord}_R G$ or $\text{info}(F)/\text{info}(G) \in K$, then $R_1', \dots, R_{h^*}'$ are exactly all the distinct members of $\mathfrak{P}_1(R^t, \Phi)$.

(4.6.3) If $\text{ord}_R F = \text{ord}_R G$ and $\text{info}(F)/\text{info}(G) \notin K$, then there is exactly one member R' of $\mathfrak{P}_1(R^t, \Phi)$ different from $R_1', \dots, R_{h^*}'$. Moreover, upon letting $\Phi' = (R^t, x, y, \kappa(t), R', \Phi R^t)(\Phi)$ we have $\text{ord}_{R'} \Phi' = 1$. More precisely, upon letting $d = \text{ord}_R F$ we have $\Phi' = \Phi/x^d$ with $M(R') = (\Phi', x)R'$, and upon letting $\Phi^D = (\delta^t)^{-1}(R')$ we have

$$\overline{\Phi} = \widehat{\Phi} H_{R^t}(\Phi^D) \prod_{1 \leq i \leq h^*} H_R(\lambda_i^*)^{e_i^*} \text{ with } \widehat{\Phi} \in K(\tau)^\times \text{ and } e_i^* \in \mathbb{N}_+.$$

Note (4.7). Elsewhere, as a consequence of the above theorem, we shall show that $(R^t, \Phi R^t)$ is strongly square free.

REFERENCES

- [Ab1] S. S. Abhyankar, *On the valuations centered in a local domain*, American Journal of Mathematics, vol. 78 (1956), pages 321-348. MR0082477 (18:556b)
- [Ab2] S. S. Abhyankar, *Ramification Theoretic Methods in Algebraic Geometry*, Princeton University Press, 1959. MR0105416 (21:4158)
- [Ab3] S. S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces*, First Edition of 1966, Published by Academic Press; Second Enlarged Edition of 1998, Published by Springer-Verlag. MR1617523 (99c:14021)
- [Ab4] S. S. Abhyankar, *Historical ramblings in algebraic geometry and related algebra*, American Mathematical Monthly, vol. 83 (1976), pages 409-448. MR0401754 (53:5581)
- [Ab5] S. S. Abhyankar, *Desingularization of plane curves*, Singularities, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, vol. 40, Part 1 (1983), pages 1-45. MR713043 (85d:14024)
- [Ab6] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, American Mathematical Society, 1990. MR1075991 (92a:14001)
- [Ab7] S. S. Abhyankar, *Polynomial expansion*, Proceedings of the American Mathematical Society, vol. 126 (1998), pages 1583-1596. MR1443142 (98g:12003)
- [Ab8] S. S. Abhyankar, *Lectures on Algebra I*, World Scientific, 2006. MR2253320 (2007k:13001)
- [Ab9] S. S. Abhyankar, *Inversion and Invariance of Characteristic Terms Part I*, The Legacy of Alladi Ramakrishnan in the Mathematical Sciences (2010), pages 93-168, Springer. MR2744259 (2012b:14057)
- [Ab10] S. S. Abhyankar, *Dicritical divisors and Jacobian problem*, Indian Journal of Pure and Applied Mathematics, vol. 41 (2010), pages 77-97. MR2650101 (2011d:14107)
- [Ab11] S. S. Abhyankar, *Pillars and towers of quadratic transformations*, Proceedings of the American Mathematical Society, vol. 139 (2011), pages 3067-3082. MR2811263
- [Ab12] S. S. Abhyankar, *More about dicriticals*, Proceedings of the American Mathematical Society, vol. 139 (2011), pages 3083-3097. MR2811264
- [Nag] M. Nagata, *Local Rings*, Wiley, 1962. MR0155856 (27:5790)