EQUATIONS AND SYZYGIES OF SOME KALMAN VARIETIES

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ABSTRACT. Given a subspace L of a vector space V , the Kalman variety consists of all matrices of V that have a nonzero eigenvector in L . Ottaviani and Sturmfels described minimal equations in the case that $\dim L = 2$ and conjectured minimal equations for $\dim L = 3$. We prove their conjecture and describe the minimal free resolution in the case that $\dim L = 2$, as well as some related results. The main tool is an exact sequence which involves the coordinate rings of these Kalman varieties and the normalizations of some related varieties. We conjecture that this exact sequence exists for all values of $\dim L$.

INTRODUCTION

Let V be a vector space over a field of arbitrary characteristic. For a subspace $L \subsetneq V$, the associated Kalman variety consists of all matrices that have a nonzero eigenvector in L . A more general definition and basic properties of Kalman varieties are contained in Section 1.1. The algebraic and geometric properties of this variety were studied by Ottaviani and Sturmfels in [OS], and their definition was motivated by Kalman’s observability condition in control theory [Kal].

In particular, Ottaviani and Sturmfels find minimal generators for the prime ideal of the Kalman variety when $\dim L = 2$ and conjecture the number of equations needed when $\dim L = 3$. (When $\dim L = 1$, the Kalman variety is an affine space.) Our main results involve calculating the minimal free resolution in the case $\dim L = 2$ (Theorem 3.3) and proving their conjecture in the case $\dim L = 3$ (Theorem 3.6). We point out that even though the Kalman varieties are of determinantal type in these cases, they are not Cohen–Macaulay varieties when $\dim V - 1 > \dim L > 1$, so the resolution is not obtained from the Eagon–Northcott complex.

The main tool is the geometric approach to free resolutions via sheaf cohomology (Section 1.3). However, it is not a straightforward application because this approach only provides information for the *normalization* of the Kalman variety, and the Kalman variety is not normal whenever $\dim L > 1$. The main insight into this problem is that the Kalman varieties and their higher analogues (defined in Section 1.1) appear to have a certain inductive structure. We prove that this structure exists when $\dim L \leq 3$ (Theorem 3.2) and conjecture that it exists in general (see Conjecture 3.1). As further evidence, we sketch a proof of this conjecture in the case when $\dim V = \dim L + 1$ and the ground field is of characteristic 0 (see Section 3.4).

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This inductive structure should provide a means to study the equations of the Kalman variety when $\dim L > 3$. The validity of Conjecture 3.1 would make the Kalman varieties a good testing ground for studying the equations and free resolutions of nonnormal varieties. In particular, there are very few known instances where the approach described in Section 1.3 works effectively for nonnormal varieties. One particularly important instance where the approach in Section 1.3 is relevant but where the varieties can fail to be normal are the nilpotent orbits in Lie theory [Wey, Chapter 8], so hopefully the insights gained from studying the easier case of Kalman varieties will be useful in more complicated situations.

The outline of the article is as follows. In Section 1, we summarize the properties of Kalman varieties that we will be using, as well as the necessary constructions and theorems needed to use the geometric approach to free resolutions. In Section 2, we prove a few preparatory results on the normalizations of Kalman varieties, which we use in Section 3 to prove our main results.

1. PRELIMINARIES

1.1. Kalman varieties. Fix a field K , a vector space V , and a subspace $L \subsetneq V$. Set $W = (V/L)^*$, and let $\text{End}(V)$ be the space of linear operators on V with coordinate ring $A = \text{Sym}(\text{End}(V)^*)$, which is graded via $\deg \text{End}(V)^* = 1$. Also, let $n = \dim V$, $d = \dim L$ and pick $1 \leq s \leq d$. The **Kalman variety** is

$$\mathcal{K}_{s,d,n} = \{\varphi \in \text{End}(V) \mid \text{there exists } U \subset L \text{ such that } \dim U = s \text{ and } \varphi(U) \subseteq U\}.$$

Equations that define $\mathcal{K}_{s,d,n}$ (at least set-theoretically) can be obtained as follows. Pick an ordered basis for V starting with a basis for L followed by a basis for V/L and write φ in block matrix form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then $\mathcal{K}_{s,d,n}$ is the zero locus of the $(d-s+1) \times (d-s+1)$ minors of the **reduced Kalman matrix**

$$(1.1) \quad \begin{pmatrix} \gamma \\ \gamma\alpha \\ \vdots \\ \gamma\alpha^{d-1} \end{pmatrix}$$

[OS, Theorem 4.5]. These equations are far from minimal, and it is unclear if they define a prime ideal. Note that $\mathcal{K}_{s,d,n}$ carries an action of the group

$$P = \{g \in \mathbf{GL}(V) \mid g(L) = L\},$$

but often we will just use the symmetry provided by the subgroup $G \cong \mathbf{GL}(L) \times \mathbf{GL}(W)$ of P .

Let $\mathbf{Gr}(s, L)$ denote the Grassmannian of s -dimensional subspaces of L . Then $\mathbf{Gr}(s, L)$ has a tautological sequence of vector bundles

$$0 \rightarrow \mathcal{R} \rightarrow L \times \mathbf{Gr}(s, L) \rightarrow \mathcal{Q} \rightarrow 0,$$

where $\text{rank } \mathcal{R} = s$ and $\text{rank } \mathcal{Q} = d - s$. Consider the subbundle $\mathcal{S}$ of $\text{End}(V) \times \mathbf{Gr}(s, L)$ defined by

$$\mathcal{S} = \{(\varphi, U) \mid \varphi(U) \subseteq U\}.$$

The bundle $\mathcal{S}$ is not completely reducible, but it has a filtration whose associated graded bundle is

$$\text{gr } \mathcal{S} = \text{End}(\mathcal{R}) \oplus \text{Hom}(V/\mathcal{R}, V).$$

For later use, set $\xi = ((\text{End}(V) \times \mathbf{Gr}(s, L))/S)^*$. Then

$$\xi = \mathcal{R} \otimes (\mathcal{Q}^* \oplus W).$$

Let $p_1: \text{End}(V) \times \mathbf{Gr}(s, L) \rightarrow \text{End}(V)$ be the projection. Then $p_1(\mathcal{S}) = \mathcal{K}_{s,d,n}$ and $p_1: \mathcal{S} \rightarrow \mathcal{K}_{s,d,n}$ is a projective birational morphism.

For $s = d$, $\mathcal{K}_{d,d,n}$ is isomorphic to affine space and its defining ideal is generated by $L \otimes W \subset A_1$. For $s < d$, we can deduce from the map p_1 that the singular locus and nonnormal locus of $\mathcal{K}_{s,d,n}$ coincide and is $\mathcal{K}_{s+1,d,n}$, and that $\mathcal{K}_{s,d,n}$ is an irreducible subvariety of codimension $s(n-d)$ in $\text{End}(V)$ [OS, Theorem 4.4]. In particular, when $n > d+1$, $\mathcal{K}_{s,d,n}$ is not Cohen–Macaulay by Serre’s criterion for normality. When $s = 1$ and $n = d+1$, $\mathcal{K}_{1,d,d+1}$ is a hypersurface and hence is Cohen–Macaulay, but we do not know what happens when $s > 1$ and $n = d+1$ in general.

1.2. Characteristic-free multilinear algebra. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, let $\ell(\lambda)$ be the largest i such that $\lambda_i \neq 0$. If $\sum_i \lambda_i = n$, we write $\lambda \vdash n$ and $|\lambda| = n$. The dual partition λ' is defined by $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$. The notation a^b means the sequence $(a, \dots, a)$ (b times). Given a partition, we can think of it as a collection of boxes (i, j) where $1 \leq j \leq \lambda_i$. The content of (i, j) is $c(i, j) = j - i$ and the hook length is $h(i, j) = \lambda_i - j + \lambda'_j - i + 1$.

Let R be a commutative ring and let U be a free R -module of finite rank n . We define the determinant of U to be $\det U = \bigwedge^n U$. The Schur and Weyl functors are denoted $\mathbf{L}_\lambda U$ and $\mathbf{K}_\lambda U$, respectively. See [Wey, Chapter 2] for their definition. However, we will change the notation from [Wey, Chapter 2] so that we use $\mathbf{L}_{\lambda'} U$ to mean $\mathbf{L}_\lambda U$. In particular, $\mathbf{L}_d U \cong S^d U$, $\mathbf{L}_{1^d} U \cong \bigwedge^d U \cong \mathbf{K}_{1^d} U$ and $\mathbf{K}_d U \cong D^d U$, where S denotes symmetric powers and D denotes divided powers.

We recall the relevant properties that we need. First, both $\mathbf{K}_\lambda U$ and $\mathbf{L}_\lambda U$ are representations of $\mathbf{GL}(U)$. Both $\mathbf{K}_\lambda U$ and $\mathbf{L}_\lambda U$ are free U -modules of the same rank, and this rank is given by

$$(1.2) \quad \text{rank } \mathbf{L}_\lambda U = \text{rank } \mathbf{K}_\lambda U = \prod_{(i,j) \in \lambda} \frac{n + c(i, j)}{h(i, j)}$$

[Sta, Corollary 7.21.4]. In particular, we have $\mathbf{L}_\lambda(U) = 0$ if and only if $\ell(\lambda) > \text{rank } U$ and similarly with $\mathbf{K}_\lambda(U)$. Also, $\mathbf{L}_{\lambda_1+1, \dots, \lambda_n+1} U = \det U \otimes \mathbf{L}_{\lambda_1, \dots, \lambda_n} U$, and similarly for $\mathbf{K}$, so we can use this to define $\mathbf{L}_\lambda$ and $\mathbf{K}_\lambda$ when λ is a weakly decreasing sequence of integers which are allowed to be negative. There is a canonical isomorphism $\mathbf{L}_\lambda(U^*) = \mathbf{K}_\lambda(U)^*$ [Wey, Proposition 2.1.18]. Also there are isomorphisms $\mathbf{L}_{\lambda_1, \dots, \lambda_n}(U^*) = \mathbf{L}_{-\lambda_n, \dots, -\lambda_1} U$ and similarly for $\mathbf{K}$ [Wey, Exercise 2.18].

The functors $\mathbf{L}_\lambda$ and $\mathbf{K}_\lambda$ are compatible with base change. Hence it makes sense to construct $\mathbf{L}_\lambda \mathcal{U}$ and $\mathbf{K}_\lambda \mathcal{U}$ when $\mathcal{U}$ is a locally free sheaf on a scheme. When R is a $\mathbf{Q}$ -algebra (or $\mathbf{Q}$ -scheme), we have $\mathbf{L}_\lambda U \cong \mathbf{K}_\lambda U$. In this case, we will use the notation $\mathbf{S}_\lambda U$ to make it clear that we are dealing with the characteristic 0 situation. In positive characteristic, they need not be isomorphic, and this is one of the reasons that some of our proofs will only be valid in characteristic 0.

Given two free modules U and U' , the symmetric powers $S^d(U \otimes U')$ have a $\mathbf{GL}(U) \times \mathbf{GL}(U')$ -equivariant filtration whose associated graded module is

$$\text{gr } S^d(U \otimes U') = \bigoplus_{\lambda \vdash d} \mathbf{L}_\lambda U \otimes \mathbf{L}_\lambda U'.$$

Similarly, the exterior powers $\bigwedge^d(U \otimes U')$ have a $\mathbf{GL}(U) \times \mathbf{GL}(U')$ -equivariant filtration whose associated graded module is

$$\mathrm{gr} \bigwedge^d(U \otimes U') = \bigoplus_{\lambda \vdash d} \mathbf{L}_\lambda U \otimes \mathbf{K}_{\lambda'} U'$$

[Wey, Theorem 2.3.2]. These are the **Cauchy identities**. Furthermore, given two partitions λ, μ , the tensor product $\mathbf{L}_\lambda U \otimes \mathbf{L}_\mu U$ has a filtration whose associated graded module is of the form

$$\mathrm{gr} \mathbf{L}_\lambda U \otimes \mathbf{L}_\mu U = \bigoplus_{\nu \vdash |\lambda| + |\mu|} (\mathbf{L}_\nu U)^{\oplus c_{\lambda, \mu}^\nu}$$

[Bof, Theorem 3.7]. When R is a $\mathbf{Q}$ -algebra (or $\mathbf{Q}$ -scheme), the above filtrations become direct sum decompositions.

The $c_{\lambda, \mu}^\nu$ are Littlewood–Richardson coefficients (see [Wey, Theorem 2.3.4]). We will only need to know these numbers when $\lambda = (d)$ or $\lambda = (1^d)$, which we now explain. We say that ν is obtained from μ by adding a horizontal strip of length d if $|\nu| = |\mu| + d$ and we have the inequalities $\nu_i \geq \mu_i \geq \nu_{i+1}$ for all i . In this case, we have $c_{(d), \mu}^\nu = 1$. Otherwise, we have $c_{(d), \mu}^\nu = 0$. For the case $\lambda = (1^d)$, we use the identity $c_{(1^d), \mu}^\nu = c_{(d), \mu'}^{\nu'}$. Alternatively, $c_{(1^d), \mu}^\nu$ is nonzero (and equal to 1) if and only if $|\nu| = |\mu| + d$ and $\mu_i \leq \nu_i \leq \mu_i + 1$. These are the **Pieri rules**.

1.3. The geometric approach to syzygies. Fix a field K . Let X be a projective variety and let U be a vector space, and denote the projections $p_1: U \times X \rightarrow U$ and $p_2: U \times X \rightarrow X$. Let $\mathcal{S} \subset U \times X$ be a subbundle with quotient bundle $\mathcal{T}$ and set $Y = p_1(\mathcal{S}) \subset U$. Also, set $\xi = \mathcal{T}^*$ and let $A = \mathrm{Sym}(U^*)$ be the coordinate ring of U with grading given by $\deg U^* = 1$. The notation $A(-i)$ denotes the ring A with a grading shift so that it is generated in degree i . For all $i \in \mathbf{Z}$, define graded A -modules

$$\mathbf{F}_i = \bigoplus_{j \geq 0} \mathrm{H}^j(X; \bigwedge^{i+j} \xi) \otimes_K A(-i-j).$$

Theorem 1.3. (a) *There exist minimal differentials $d_i: \mathbf{F}_i \rightarrow \mathbf{F}_{i-1}$ of degree 0 so that $\mathbf{F}_\bullet$ is a complex of graded free A -modules such that*

$$\mathrm{H}_{-i}(\mathbf{F}_\bullet) = \mathrm{R}^i(p_1)_* \mathrm{Sym}(\mathcal{S}^*).$$

In particular, if the higher direct images of $\mathrm{Sym}(\mathcal{S}^)$ vanish and p_1 is birational, then $\mathbf{F}_\bullet$ is a minimal A -free resolution of the normalization of Y .*

(b) *Suppose that ξ is a direct sum of locally free sheaves $\xi_1 \oplus \xi_2$. For $r, s \geq 0$, define*

$$\mathbf{F}_i^{\leq r, s} = \bigoplus_{j \geq 0} \bigoplus_{k=i+j-s}^r \mathrm{H}^j(X; \bigwedge^k \xi_1 \otimes \bigwedge^{i+j-k} \xi_2) \otimes_K A(-i-j)$$

with the convention that negative exterior powers are 0. Then $\mathbf{F}_\bullet^{\leq r, s}$ is a sub-complex of $\mathbf{F}_\bullet$.

Proof. (a) is the content of [Wey, Theorems 5.1.2, 5.1.3] and (b) follows from the proof of [Wey, Lemma 5.2.3]. $\square$

Now use the notation from Section 1.1. We consider the case of a Grassmannian $X = \mathbf{Gr}(s, L)$ whose points are the s -dimensional subspaces of L . The cotangent bundle of X is $\mathcal{R} \otimes \mathcal{Q}^*$.

Theorem 1.4 (Kempf vanishing). *Let α, β be two partitions such that $\alpha_{d-s} \geq \beta_1$. Then*

$$H^j(\mathbf{Gr}(s, L); \mathbf{L}_\alpha(\mathcal{R}^*) \otimes \mathbf{L}_\beta(\mathcal{Q}^*)) = \begin{cases} \mathbf{L}_{(\alpha, \beta)}(L^*) & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

Furthermore, if $\alpha_{d-s} < \beta_1$, then $\mathbf{L}_\alpha(\mathcal{R}^*) \otimes \mathbf{L}_\beta(\mathcal{Q}^*)$ has no sections.

Proof. For the first statement, see [BK, Theorem 3.1.1]. For the second statement, the sheaf $\mathbf{L}_\alpha(\mathcal{R}^*) \otimes \mathbf{L}_\beta(\mathcal{Q}^*)$ is the pushforward of a line bundle on the flag variety, and this line bundle has global sections if and only if $\alpha_{d-s} \geq \beta_1$. $\square$

Given a permutation w , we define the **length** of w to be $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$. Also, define $\rho = (d-1, d-2, \dots, 1, 0)$. Given a sequence of integers α , we define $w \bullet \alpha = w(\alpha + \rho) - \rho$.

Theorem 1.5 (Borel–Weil–Bott). *Suppose that the characteristic of K is 0. Let α, β be two partitions and set $\nu = (\alpha, \beta)$. Then exactly one of the following two situations occurs.*

1. *There exists $w \neq \text{id}$ such that $w \bullet \nu = \nu$. Then all cohomology of $\mathbf{S}_\alpha \mathcal{Q} \otimes \mathbf{S}_\beta \mathcal{R}$ vanishes.*
2. *There is a (unique) w such that $\eta = w \bullet \nu$ is a weakly decreasing sequence. Then*

$$H^{\ell(w)}(\mathbf{Gr}(s, L); \mathbf{S}_\alpha \mathcal{Q} \otimes \mathbf{S}_\beta \mathcal{R}) = \mathbf{S}_\eta L$$

and all other cohomology vanishes.

Proof. See [Wey, Corollary 4.1.9]. $\square$

2. NORMALIZATIONS OF KALMAN VARIETIES

Let $\mathcal{O}_{s,d,n}$ denote the coordinate ring of $\mathcal{K}_{s,d,n}$ and let $\tilde{\mathcal{O}}_{s,d,n}$ denote the normalization of $\mathcal{O}_{s,d,n}$. In this section we prove some results on $\mathcal{O}_{s,d,n}$ that will be used in the main results of this article (Theorem 3.3 and Theorem 3.6). Some additional results on the normalizations can be found in Proposition 3.8 and Proposition 3.10. Continue the notation of Section 1.1.

Proposition 2.1. *Over a field of characteristic 0, the higher direct images of $\mathcal{S}$ vanish for all s, d, n . In particular, $\tilde{\mathcal{O}}_{s,d,n}$ has rational singularities and hence is Cohen–Macaulay. The higher direct images also vanish in arbitrary characteristic in the case $s = 1$ and in the case $s = 2, d = 3$. In particular, $\tilde{\mathcal{O}}_{1,d,n}$ and $\tilde{\mathcal{O}}_{2,3,n}$ are flat over $\mathbf{Z}$.*

Combined with Theorem 3.2 we conclude that $\mathcal{O}_{1,d,n}$ and $\mathcal{O}_{2,3,n}$ are also flat over $\mathbf{Z}$.

Proof. First suppose that the characteristic is 0. By Theorem 1.3(a), it is enough to show that $\mathbf{F}_i = 0$ for $i < 0$. The summands of $\bigwedge^q \xi$ are of the form $\mathbf{S}_\lambda \mathcal{R} \otimes \mathbf{S}_\mu \mathcal{Q}^* \otimes \mathbf{S}_\nu W$, where $|\lambda| = q$ and $|\mu| \leq q$. From the description of Borel–Weil–Bott (Theorem 1.5), it is clear that such a sheaf can only have cohomology in degree at most q , which proves the claim.

Now suppose that the characteristic is arbitrary. For $s = 1$, the claim follows from Kempf vanishing (Theorem 1.4) since $\mathrm{gr} \mathcal{S} = \mathcal{O} + \mathrm{Hom}(\mathcal{Q}, V) + \mathrm{Hom}(W^*, V)$, so $\mathrm{Sym}(\mathrm{gr} \mathcal{S}^*)$ has no higher cohomology, and hence the same is true for $\mathrm{Sym}(\mathcal{S}^*)$. The case of $s = 2$ and $d = 3$ will be shown in Proposition 2.5. $\square$

Remark 2.2. We expect that the higher direct images vanish for all s, d, n and in all characteristics, but we are unable to prove this. $\square$

Proposition 2.3. $\tilde{\mathcal{O}}_{1,d,n}$ has (Castelnuovo–Mumford) regularity $d-1$ and the terms of its minimal free resolution $\mathbf{F}_\bullet$ are

$$\begin{aligned} \mathbf{F}_0 &= A \oplus A(-1) \oplus \cdots \oplus A(-d+1) \\ \mathbf{F}_i &= \bigoplus_{a=\max(0, i+2d-1-n)}^{d-1} \mathbf{K}_{(i, 1^{d-a-1})} L \otimes \bigwedge^{i+d-a-1} W \otimes A(-i-d+1) \quad (1 \leq i \leq n-d). \end{aligned}$$

Proof. Use the notation of Section 1.3. We have

$$\bigwedge^q \xi = \bigoplus_{i=0}^{d-1} S^q \mathcal{R} \otimes \bigwedge^i \mathcal{Q}^* \otimes \bigwedge^{q-i} W$$

with the convention that negative exterior powers are 0. For $0 \leq q \leq d-1$, we have

$$H^j(\mathbf{Gr}(1, L); S^q \mathcal{R} \otimes \bigwedge^i \mathcal{Q}^*) = \begin{cases} K & \text{if } q = i = j, \\ 0 & \text{else} \end{cases}$$

[EFS, Proposition 5.5]. For $d \leq q$, we have by Serre duality that

$$\begin{aligned} H^j(\mathbf{Gr}(1, L); S^q \mathcal{R} \otimes \bigwedge^i \mathcal{Q}^*) &= H^{d-1-j}(\mathbf{Gr}(1, L); S^{q-d} \mathcal{R}^* \otimes \bigwedge^i \mathcal{Q})^* \otimes \det L \\ &= H^{d-1-j}(\mathbf{Gr}(1, L); S^{q-d+1} \mathcal{R}^* \otimes \bigwedge^{d-1-i} \mathcal{Q}^*)^*. \end{aligned}$$

By Kempf vanishing (Theorem 1.4), the last term is 0 for $j < d-1$. When $j = d-1$, we get

$$H^0(\mathbf{Gr}(1, L); S^{q-d+1} \mathcal{R}^* \otimes \bigwedge^{d-1-i} \mathcal{Q}^*)^* = \mathbf{L}_{(q-d+1, 1^{d-1-i})}(L^*)^* = \mathbf{K}_{(q-d+1, 1^{d-1-i})} L,$$

and this term contributes to $\mathbf{F}_{q-d+1}$. The rest follows from Section 1.3. $\square$

Corollary 2.4. Let $\mathbf{F}_\bullet$ be the minimal free resolution of $\tilde{\mathcal{O}}_{1,d,n}$. For $i > 1$, the only nonzero components in the differential $\mathbf{F}_i \rightarrow \mathbf{F}_{i-1}$ are the maps

$$\begin{aligned} &\mathbf{K}_{i, 1^{d-a-1}} L \otimes \bigwedge^{i+d-1-a} W \otimes A(-i-d+1) \\ &\rightarrow \mathbf{K}_{i-1, 1^{d-a-1}} L \otimes \bigwedge_{i+d-1-a}^{i+d-2-a} W \otimes A(-i-d+2), \\ &\mathbf{K}_{i-1, 1^{d-a}} L \otimes \bigwedge W \end{aligned}$$

with the convention that a term on the right is 0 if it does not appear in $\mathbf{F}_{i-1}$.

Proof. Consider the Koszul complex of $\mathcal{O}_{\mathcal{S}}$ over the total space of $\text{End}(V) \times \mathbf{Gr}(s, L)$. For simplicity, we work over $\mathbf{Gr}(s, L)$ by pushing forward along the projection (which is an equivalence since $\text{End}(V) \times \mathbf{Gr}(s, L) \rightarrow \mathbf{Gr}(s, L)$ is affine). The degree $i + d - 1$ component of the map above is obtained by applying H^{d-1} to the map of sheaves

$$S^{i+d-1}\mathcal{R} \otimes \bigwedge^a \mathcal{Q}^* \otimes \bigwedge^{i+d-1-a} W \rightarrow S^{i+d-2}\mathcal{R} \otimes \bigwedge^{a-1} \mathcal{Q}^* \otimes \bigwedge^{i+d-2-a} W \otimes \text{End}(V)$$

in this Koszul complex. The equations for $\mathcal{O}_{\mathcal{S}}$ are given by $\xi = \mathcal{R} \otimes (\mathcal{Q}^* \oplus W) \subset \text{End}(V)$. In particular, we can restrict our attention to the map

$$S^{i+d-1}\mathcal{R} \otimes \bigwedge^a \mathcal{Q}^* \otimes \bigwedge^{i+d-1-a} W \rightarrow \begin{array}{c} S^{i+d-2}\mathcal{R} \otimes \bigwedge^{a-1} \mathcal{Q}^* \otimes \bigwedge^{i+d-1-a} W \otimes (\mathcal{R} \oplus \mathcal{Q}^*) \\ S^{i+d-2}\mathcal{R} \otimes \bigwedge^a \mathcal{Q}^* \otimes \bigwedge^{i+d-2-a} W \otimes (\mathcal{R} \oplus W) \end{array}.$$

Using Serre duality, this is the same as taking the dual map of applying H^0 to

$$\begin{array}{c} S^{i-1}\mathcal{R}^* \otimes \bigwedge^{a-1} \mathcal{Q} \otimes \bigwedge^{i+d-1-a} W^* \\ S^{i-1}\mathcal{R}^* \otimes \bigwedge^a \mathcal{Q} \otimes \bigwedge^{i+d-2-a} W^* \otimes W^* \end{array} \rightarrow S^{i-1}\mathcal{R}^* \otimes \bigwedge^a \mathcal{Q} \otimes \bigwedge^{i+d-1-a} W^*.$$

Since the differentials in the Koszul complex are obtained via comultiplication, both of these maps are given by exterior multiplication. Hence the map on sections is surjective, which implies that our desired maps are injective (and hence nonzero).

That there are no other nonzero maps follows from Theorem 1.3(b). $\square$

Proposition 2.5. *If the characteristic of K is 0, then the first few terms of the minimal free resolution $\mathbf{F}_{\bullet}$ of $\tilde{\mathcal{O}}_{2,3,n}$ are:*

$$\begin{aligned} \mathbf{F}_0 &= A \oplus A(-1) \oplus A(-2), \\ \mathbf{F}_1 &= \bigwedge^2_{L \otimes W} L \otimes \bigwedge^2 W \otimes A(-2) \oplus L \otimes W \otimes A(-3), \\ \mathbf{F}_2 &= \bigwedge^3_{S^2 L \otimes \bigwedge^2 W} S_{2,1} L \otimes \bigwedge^3 W \otimes A(-3) \oplus \bigwedge^2_{S^3 L \otimes S_{2,1} W} (L \otimes W) \otimes A(-4), \\ \mathbf{F}_3 &= \bigwedge^4_{S^3 L \otimes \bigwedge^3 W} S_{3,1} L \otimes \bigwedge^4 W \otimes A(-4) \oplus \begin{array}{c} S_{2,1,1} L \otimes S_{2,1,1} W \\ S_{2,1,1} L \otimes S_{2,2} W \\ S^3 L \otimes \bigwedge^3 W \\ S_{2,1} L \otimes S_{2,1} W \end{array} \otimes A(-5). \end{aligned}$$

The ranks of these $\mathbf{F}_i$ are the same for any field. Furthermore, the regularity of $\tilde{\mathcal{O}}_{2,3,n}$ is 2.

Proof. Since $\dim \mathbf{Gr}(2, 3) = 2$, the regularity of $\tilde{\mathcal{O}}_{2,3,n}$ is at most 2 by Theorem 1.3(a). So the above reduces to calculating the cohomology of $\bigwedge^q \xi$ for $0 \leq q \leq 5$, which we first do in characteristic 0. This is a straightforward, although tedious, application of the Cauchy identity, Pieri rule, and Borel–Weil–Bott theorem (all explained in Section 1.2), which we omit.

Now assume that the field has characteristic $p > 0$. If $p > 5$, then we may still use Borel–Weil–Bott to calculate the cohomology of $\bigwedge^q \xi$ with $q \leq 5$ (this reduces to the statement that the n th symmetric and divided power functors are naturally isomorphic when $n!$ is invertible). In the remaining cases $p \in \{2, 3, 5\}$, the cohomology calculation can be reduced to a finite calculation with Macaulay 2 [GS], which we explain. First, we have $\xi = \mathcal{R} \otimes (\mathcal{Q}^* \oplus W)$. Since we only go up to $\bigwedge^5 \xi$, we see that the terms which appear in the Cauchy filtration of $\bigwedge^i \xi$ are the same when $\dim W \geq 5$. So we only need to consider the case $\dim W = 5$. Finally, we only need to calculate H^1 and H^2 since we know the Euler characteristic. For $\bigwedge^5 \xi$, we only care about H^2 . We use the following code:

```
A=ZZ/2[z_0,z_1,z_2];
m=matrix{{z_0,z_1,z_2}};
R = sheaf((ker m) ** A^{1});
Q = sheaf(A^{1});
xi = (R ** dual(Q)) ++ (R ++ R ++ R ++ R ++ R);
for i from 1 to 4 do (
    E = exteriorPower(i,xi);
    print (rank HH^1(E), rank HH^2(E)); )
print rank HH^2(exteriorPower(5,xi));
```

This outputs the answers

```
(1, 0)
(45, 1)
(180, 15)
(310, 145)
705
```

which is the expected answer. Then repeat the above with 2 replaced by 3 and 5. $\square$

3. KALMAN VARIETIES

In this section we prove our main results, which include calculating the minimal free resolution of $\mathcal{O}_{1,2,n}$ and the equations of $\mathcal{O}_{1,3,n}$. During the course of our work, we discovered the following conjecture.

Conjecture 3.1. *Fix d . For $s = 1, \dots, d$, let $B_s = \tilde{\mathcal{O}}_{s,d,n}(-\frac{s(s-1)}{2})$. There is a long exact sequence*

$$0 \rightarrow \mathcal{O}_{1,d,n} \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow B_d \rightarrow 0.$$

Furthermore, the ideal of $\mathcal{O}_{1,d,n}$ has minimal generators in degrees $d, d+1, \dots, \frac{d(d+1)}{2}$. The projective dimension of $\mathcal{O}_{1,d,n}$ is $d(n-d) - d + 1$ and its regularity is $\frac{d(d+1)}{2} - 1$.

The rest of the section will imply that this conjecture holds for $d \leq 3$, so we record the result.

Theorem 3.2. *Conjecture 3.1 holds when $d \leq 3$. In particular, there are exact sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{1,2,n} \rightarrow \tilde{\mathcal{O}}_{1,2,n} \rightarrow \mathcal{O}_{2,2,n}(-1) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_{1,3,n} \rightarrow \tilde{\mathcal{O}}_{1,3,n} \rightarrow \tilde{\mathcal{O}}_{2,3,n}(-1) \rightarrow \mathcal{O}_{3,3,n}(-3) \rightarrow 0. \end{aligned}$$

For more precise statements about the number of equations, see Theorem 3.3 and Theorem 3.6.

We expect that the methods used in these cases will extend to any given value of d , but we have been unable to properly organize the combinatorics in the case of general d . However, we are able to prove Conjecture 3.1 in the case $n = d + 1$ and $\text{char } K = 0$. We provide a brief sketch of this case in Section 3.4.

We point out that we were not able to check the conjecture computationally even for the first nontrivial case $d = 4$ and $n = 6$.

3.1. Syzygies for $d = 2$.

Theorem 3.3. *The terms of the minimal free resolution $\mathbf{F}_\bullet$ of $\mathcal{O}_{1,2,n}$ are given by*

$$\begin{aligned} \mathbf{F}_i &= \det L \otimes D^{i-1} L \otimes \bigwedge^{i+1} W \otimes A(-i-1) \\ &\quad \oplus \left(\bigwedge^{i+1} (L \otimes W) \right) / (D^{i+1} L \otimes \bigwedge^{i+1} W) \otimes A(-i-2) \quad (1 \leq i \leq n-3), \\ \mathbf{F}_i &= \bigwedge^{i+1} (L \otimes W) \otimes A(-i-2) \quad (n-2 \leq i \leq 2n-5). \end{aligned}$$

In particular, the projective dimension of $\mathcal{O}_{1,2,n}$ is $2n-5$ and it has regularity 2.

Proof. From Proposition 2.3, $\tilde{\mathcal{O}}_{1,2,n}$ has the following presentation:

$$\begin{array}{ccccc} \bigwedge^2 L \otimes \bigwedge^2 W \otimes A(-2) & \rightarrow & A & \rightarrow & \tilde{\mathcal{O}}_{1,2,n} \rightarrow 0. \\ L \otimes W \otimes A(-2) & & A(-1) & & \end{array}$$

The map $\bigwedge^2 L \otimes \bigwedge^2 W \otimes A(-2) \rightarrow A(-1)$ is 0. We can either appeal to Theorem 1.3(b) or use that no such G -equivariant map exists. Hence the presentation for $\tilde{\mathcal{O}}_{1,2,n}/\mathcal{O}_{1,2,n}$ must be $L \otimes W \otimes A(-2) \rightarrow A(-1)$, and we conclude that the quotient is $\mathcal{O}_{2,2,n}(-1)$.

Let $\tilde{\mathbf{F}}_\bullet$ be the minimal free resolution of $\tilde{\mathcal{O}}_{1,2,n}$ from Proposition 2.3 and let $\mathbf{G}_\bullet$ be the Koszul complex on $L \otimes W$ resolving $\mathcal{O}_{2,2,n}(-1)$. We can lift the quotient map $\tilde{\mathcal{O}}_{1,2,n} \rightarrow \mathcal{O}_{2,2,n}(-1)$ to get a map of complexes $\tilde{\mathbf{F}}_\bullet \rightarrow \mathbf{G}_\bullet$. The i th term of this map is

$$D^i L \otimes \bigwedge^i W \otimes A(-i-1) \rightarrow \bigwedge^i (L \otimes W) \otimes A(-i-1).$$

We claim that the map from $D^i L \otimes \bigwedge^i W$ is an inclusion and the map from $\mathbf{K}_{i,1} L \otimes \bigwedge^{i+1} W$ is 0. By minimality of $\tilde{\mathbf{F}}_\bullet$, the map $D^i L \otimes \bigwedge^i W \rightarrow \tilde{\mathbf{F}}_{i-1}$ is injective, and by Corollary 2.4, the map $\mathbf{K}_{i,1} L \otimes \bigwedge^{i+1} W \rightarrow D^{i-1} L \otimes \bigwedge^{i-1} W \otimes A(-i)$ is zero. By induction on i , we get the claim.

Therefore we know exactly what the minimal cancellations in the comparison map $\tilde{\mathbf{F}}_{\bullet} \rightarrow \mathbf{G}_{\bullet}$ are, which gives the desired resolution $\mathbf{F}_{\bullet}$ via a mapping cone. $\square$

Remark 3.4. In the above proof, we know from general principles that the comparison maps $\tilde{\mathbf{F}}_i \rightarrow \mathbf{G}_i$ must be nonzero since both $\tilde{\mathcal{O}}_{1,2,n}$ and $\mathcal{O}_{2,2,n}$ are Cohen–Macaulay (see the proof of [BEKS, Proposition 2.3]). So one can deduce the required cancellations using just representation theory (at least in characteristic 0) without understanding the differentials. $\square$

3.2. Equations for $s = 1$ and $d = 3$. In Proposition 2.5, we do not know how to write down the $\mathbf{Z}$ -forms for the representations of G involved, so we just switch to the notation $(\lambda; \mu)$ to mean some $\mathbf{Z}$ -form of the module $\mathbf{S}_{\lambda}L \otimes \mathbf{S}_{\mu}W$ and we also write $(-i)$ in place of $\otimes A(-i)$.

Let M be the submodule of $\tilde{\mathcal{O}}_{2,3,n}$ generated by $A \oplus A(-1)$. We will show that there exist short exact sequences

$$0 \rightarrow \mathcal{O}_{1,3,n} \rightarrow \tilde{\mathcal{O}}_{1,3,n} \rightarrow M(-1) \rightarrow 0, \quad 0 \rightarrow M \rightarrow \tilde{\mathcal{O}}_{2,3,n} \rightarrow \mathcal{O}_{3,3,n}(-2) \rightarrow 0,$$

and use a mapping cone to get the equations for $\mathcal{O}_{1,3,n}$.

Proposition 3.5. *The beginning of the minimal A -free resolution of M looks like*

$$\begin{array}{ccccccc} (2, 1; 1^3)(-3) & & & & & & \\ (2; 1^2)(-3) & & & & & & \\ (1^3; 2, 1)(-3) & \rightarrow & (1^2; 1^2)(-2) & \rightarrow & A & \rightarrow & M \rightarrow 0. \\ (1^3; 2, 1)(-4) & & (1; 1)(-2) & \rightarrow & A(-1) & & \\ (1^3; 3)(-5) & & & & & & \end{array}$$

Furthermore, the projective dimension of M is $3n - 10$ and the regularity of M is 3.

Proof. The presentation of $\tilde{\mathcal{O}}_{2,3,n}$ is

$$\begin{array}{ccccc} (1^2; 1^2)(-2) & & A & & \\ (1; 1)(-2) & \rightarrow & A(-1) & \rightarrow & \tilde{\mathcal{O}}_{2,3,n} \rightarrow 0. \\ (1; 1)(-3) & & A(-2) & & \end{array}$$

By minimality, the maps from $(1^2; 1^2)(-2)$ and $(1; 1)(-2)$ to $A(-2)$ are 0, so we see that $\tilde{\mathcal{O}}_{2,3,n}/M \cong \mathcal{O}_{3,3,n}(-2)$. The first few terms of the comparison map of the resolutions of $\tilde{\mathcal{O}}_{2,3,n}$ and $\mathcal{O}_{3,3,n}(-2)$ is given by

$$\begin{array}{ccccccc} (3, 1; 1^4)(-4) & & (2, 1; 1^3)(-3) & & (1^2; 1^2)(-2) & & A \\ (2, 1^2; 2, 1^2)(-4) & & (2; 1^2)(-3) & & (1; 1)(-2) & & A(-1) \\ (3; 1^3)(-4) & & (1^3; 2, 1)(-3) & \rightarrow & (1; 1)(-3) & \rightarrow & A(-2) \\ (2, 1^2; 2^2)(-5) & \rightarrow & (1^3; 2, 1)(-4) & \rightarrow & & & \\ (2, 1^2; 2^2)(-5) & & (1^2; 2)(-4) & & & & \\ (3; 1^3)(-5) & & (2; 1^2)(-4) & & & & \\ (2, 1; 2, 1)(-5) & & & & & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (3; 1^3)(-5) & & (1^2; 2)(-4) & & (1; 1)(-3) & \rightarrow & A(-2) \\ (2, 1; 2, 1)(-5) & \rightarrow & (2; 1^2)(-4) & \rightarrow & & & \\ (1^3; 3)(-5) & & & & & & \end{array}$$

The maps $(1^2; 2)(-4) \rightarrow (1; 1)(-3)$ and $(2; 1^2)(-4) \rightarrow (1; 1)(-3)$ in the resolution of $\tilde{\mathcal{O}}_{2,3,n}$ are the Koszul relations on the linear equations $(1; 1)(-3)$. This implies that the vertical maps between the terms of type $(1^2; 2)(-4)$, $(2; 1^2)(-4)$, $(3; 1^3)(-5)$, and $(2, 1; 2, 1)(-5)$ are isomorphisms, and the result follows by a mapping cone construction. $\square$

Theorem 3.6. *The defining equations for $\mathcal{K}_{1,3,n}$ are*

$$(1^3; 1^3)(-3) \oplus (1^3; 2, 1)(-4) \oplus (1^3; 2, 1)(-5) \oplus (1^3; 3)(-6).$$

The projective dimension of $\mathcal{K}_{1,3,n}$ is $3n - 11$ and its regularity is 5.

Remark 3.7. Using (1.2), this proves [OS, Conjecture 3.6], which says that there are $\binom{n-3}{3}$ generators in degree 3, $2\binom{n-2}{3}$ generators in degrees 4 and 5 each, and $\binom{n-1}{3}$ generators in degree 6. All of these equations may be interpreted as 3×3 minors of the reduced Kalman matrix (1.1). We thank Giorgio Ottaviani for bringing this to our attention. $\square$

Proof. The proof is similar to that of Theorem 3.3. The presentation for $\tilde{\mathcal{O}}_{1,3,n}$ is

$$\begin{array}{ccccc} (1^3; 1^3)(-3) & & A & & \\ (1^2; 1^2)(-3) & \rightarrow & A(-1) & \rightarrow & \tilde{\mathcal{O}}_{1,3,n} \rightarrow 0. \\ (1; 1)(-3) & & A(-2) & & \end{array}$$

The map $(1^3; 1^3)(-3) \rightarrow A(-1) \oplus A(-2)$ is 0 since there are no nonzero such G -equivariant maps. Also, the maps from $(1^2; 1^2)(-3)$ and $(1; 1)(-3)$ to $A(-1) \oplus A(-2)$ are nonzero. If not, then they give generators for the ideal of $\mathcal{O}_{1,3,n}$. In particular, if we pick an ordered basis for V which first has a basis for L followed by a basis for W , then these equations correspond to the 2×2 minors and the 1×1 minors of the bottom-left block submatrix, respectively, and we can find matrices in $\mathcal{K}_{1,3,n}$ for which these equations do not vanish.

Hence from Proposition 3.5, $\tilde{\mathcal{O}}_{1,3,n}/\mathcal{O}_{1,3,n} \cong M(-1)$. The first few terms of the comparison maps between the free resolutions of $\tilde{\mathcal{O}}_{1,3,n}$ and $M(-1)$ are

$$\begin{array}{ccccc} (2, 1^2; 1^4)(-4) & & (1^3; 1^3)(-3) & \longrightarrow & A \\ (2, 1; 1^3)(-4) & \longrightarrow & (1^2; 1^2)(-3) & \longrightarrow & A(-1) \\ (2; 1^2)(-4) & & (1; 1)(-3) & & A(-2) \\ \downarrow & & \downarrow & & \downarrow \\ (2, 1; 1^3)(-4) & & (1^2; 1^2)(-3) & \longrightarrow & A(-1) \\ (2; 1^2)(-4) & \longrightarrow & (1; 1)(-3) & \longrightarrow & A(-2) \\ (1^3; 2, 1)(-4) & & & & \\ (1^3; 2, 1)(-5) & & & & \\ (1^3; 3)(-6) & & & & \end{array}$$

The vertical maps between the terms $(2, 1; 1^3)(-4)$ and $(2; 1^2)(-4)$ are isomorphisms. To see this, it is enough to show that the maps $(2, 1; 1^3)(-4) \rightarrow (1^2; 1^2)(-3)$ and $(2; 1^2)(-4) \rightarrow (1; 1)(-3)$ in the resolution of $\tilde{\mathcal{O}}_{1,3,n}$ are nonzero, but this is the content of Corollary 2.4. Now the result follows by a mapping cone construction. $\square$

3.3. Equations for $s = d - 1$. In this section, we assume that K has characteristic 0 and find the equations for $\mathcal{O}_{d-1,d,n}$. We can also do this in arbitrary characteristic when $d = 3$ since, in this case, the next result is implied by Proposition 2.5.

Proposition 3.8. *When $\text{char } K = 0$, the first few terms of the minimal A -free resolution $\mathbf{F}_\bullet$ of $\tilde{\mathcal{O}}_{d-1,d,n}$ are*

$$\begin{aligned}\mathbf{F}_0 &= \bigoplus_{j=0}^{d-1} A(-j), \\ \mathbf{F}_1 &= (1^2; 1^2)(-2) \oplus \bigoplus_{j=2}^d (1; 1)(-j), \\ \mathbf{F}_2 &= (1^3; 2, 1)(-4) \oplus \bigoplus_{j=3}^{d+1} (2; 1^2)(-j) \oplus \bigoplus_{j=4}^{d+1} (1^2; 2)(-j).\end{aligned}$$

Proof. In this case,

$$\bigwedge^q \xi = \bigoplus_{a=0}^{d-1} \bigwedge^a \mathcal{R} \otimes S^a \mathcal{Q}^* \otimes \bigwedge^{q-a} (\mathcal{R} \otimes W).$$

So we have to calculate the cohomology of sheaves of the form $S^a \mathcal{Q}^* \otimes S_\lambda \mathcal{R}$.

By Borel–Weil–Bott (Theorem 1.5), the sheaf $S^a \mathcal{Q}^* \otimes S_\lambda \mathcal{R}$ has cohomology in degree at most $\ell(\lambda)$, and such a term appears in $\bigwedge^{|\lambda|} \xi$. By Theorem 1.3, this term can only contribute to $\mathbf{F}_i$ with $i = 0, 1, 2$ if $|\lambda| - 2 \leq \ell(\lambda)$. So the only possibilities for λ with $|\lambda| = q$ are (1^q) , $(2, 1^{q-2})$, $(3, 1^{q-3})$, or $(2, 2, 1^{q-4})$. We will consider each of these four cases individually. Recall that $\rho = (d-1, d-2, \dots, 1, 0)$.

Consider a sequence $(-a, 1^q)$. Adding ρ , we get $(d-1-a, d-1, d-2, \dots, d-q, d-q-2, \dots, 1, 0)$. So in order to have nonzero cohomology, we need $a = q$, where $0 \leq a \leq d-1$. We get $H^0 = K$.

Now consider a sequence $(-a, 2, 1^{q-2})$. Adding ρ , we get $(d-1-a, d, d-2, \dots, d-q+1, d-q-1, d-q-2, \dots, 1, 0)$. To get nonzero cohomology, we need $a = 0$ and $q = 2$, or $a \geq 1$ and $q = a+1$. In the first case, we get $H^1 = \bigwedge^2 L$, and in the second case, we get $H^a = L$. The first case only comes from the sheaf $S^2 \mathcal{R} \otimes \bigwedge^2 W$. The second case only comes from $\bigwedge^a \mathcal{R} \otimes S^a \mathcal{Q}^* \otimes \mathcal{R} \otimes W$.

Now consider a sequence $(-a, 3, 1^{q-3})$. Adding ρ , we get $(d-1-a, d+1, d-2, d-3, \dots, d-q+2, d-q, \dots, 1, 0)$. So we need $a \geq 1$ and $q = a+2$. In this case, $H^a = S^2 L$. This can only come from the sheaf $S_{3,1^{a-1}} \mathcal{R} \otimes S^a \mathcal{Q}^* \otimes \bigwedge^2 W \subset \bigwedge^a \mathcal{R} \otimes S^a \mathcal{Q}^* \otimes \bigwedge^2 (\mathcal{R} \otimes W)$.

Finally consider a sequence $(-a, 2, 2, 1^{q-4})$. Adding ρ , we get $(d-1-a, d, d-1, d-3, \dots, d-q+2, d-q, \dots, 1, 0)$. So either $a = 1$, which gives $H^2 = \bigwedge^{q-1} L$ or $a \geq 2$ and $q = a+2$, which gives $H^a = \bigwedge^2 L$. If the first case contributes to $\mathbf{F}_2$, then $q = 4$. This comes from the sheaf $S_{2,2} \mathcal{R} \otimes \mathcal{Q}^* \otimes S_{2,1} W \subset \mathcal{R} \otimes \mathcal{Q}^* \otimes \bigwedge^3 (\mathcal{R} \otimes W)$. The second case comes from the sheaf $S_{2,2,1^{a-2}} \mathcal{R} \otimes S^a \mathcal{Q}^* \otimes S^2 W \subset \bigwedge^a \mathcal{R} \otimes S^a \mathcal{Q}^* \otimes \bigwedge^2 (\mathcal{R} \otimes W)$. $\square$

Theorem 3.9. *Assume either that $\text{char } K = 0$ or $d = 3$. Then the equations for $\mathcal{O}_{d-1,d,n}$ are*

$$\bigwedge^2 L \otimes \bigwedge^2 W \otimes A(-2), \quad \bigwedge^2 L \otimes S^2 W \otimes A(-3).$$

The interpretation of these equations is just as in Remark 3.7.

Proof. Using arguments similar to before, the presentation for $\tilde{\mathcal{O}}_{d-1,d,n}/\mathcal{O}_{d-1,d,n}$ is

$$\bigoplus_{j=2}^d (1; 1)(-j) \rightarrow \bigoplus_{j=1}^{d-1} A(-j).$$

The maps in the above are of the form $(1; 1)(-j-1) \rightarrow A(-j)$ for $j = 1, \dots, d-1$. So in some choice of basis, the cokernel is $\bigoplus_{j=1}^{d-1} \mathcal{O}_{d,d,n}(-j)$, which is resolved by a direct sum of Koszul complexes. The next term in the Koszul complex is $\bigoplus_{j=3}^{d+1} [(1^2; 2)(-j) \oplus (2; 1^2)(-j)]$. Let $\mathbf{F}_\bullet$ be the minimal free resolution of $\tilde{\mathcal{O}}_{d-1,d,n}$. Using arguments similar to before, all terms in $\mathbf{F}_2$ of the form $(1^2; 2)(-j)$ and $(2; 1^2)(-j)$ have a nonzero map to $(1; 1)(-j+1)$ in $\mathbf{F}_1$. Hence the maps from these terms to the corresponding terms of the Koszul complex of $\tilde{\mathcal{O}}_{d-1,d,n}/\mathcal{O}_{d-1,d,n}$ are nonzero, and we finish the proof by a mapping cone construction. $\square$

3.4. Conjecture 3.1 when $n = d + 1$. In this section, we sketch a proof of Conjecture 3.1 in the case when $\text{char } K = 0$ and $n = d + 1$. Since the details are fairly involved and because this result is not very substantial, we will just mention the important points and offer them as evidence for the validity of Conjecture 3.1.

Proposition 3.10. *When $\text{char } K = 0$, the terms of the minimal free resolution $\mathbf{F}_\bullet$ of $\tilde{\mathcal{O}}_{s,d,d+1}$ are*

$$\mathbf{F}_i = \bigoplus_{\lambda \subseteq (s-i) \times (d-s)} \bigwedge^i L \otimes S^i W \otimes A(-i(d-s+1) - |\lambda|),$$

where $\lambda \subseteq (s-i) \times (d-s)$ means $\ell(\lambda) \leq s-i$ and $\lambda_1 \leq d-s$, and the empty partition is allowed.

In particular, the generators in $\mathbf{F}_0$ can be written as

$$\bigoplus_{\lambda \subseteq s \times (d-s)} H^{|\lambda|}(\mathbf{Gr}(s, L); \mathbf{S}_\lambda \mathcal{R} \otimes \mathbf{S}_{\lambda'} \mathcal{Q}^*) \otimes A(-|\lambda|).$$

So we see that $\tilde{\mathcal{O}}_{s,d,d+1}$ is generated by $\bigoplus_{\lambda \subseteq s \times (d-s)} A(-|\lambda|)$. Let C_s be the submodule of $\tilde{\mathcal{O}}_{s,d,d+1}$ generated by $\bigoplus_{\lambda \subseteq (s-1) \times (d-s)} A(-|\lambda|)$ (this is unambiguous by the above remark). Also define $C_{d+1} = 0$. Note that $C_1 = \mathcal{O}_{1,d,d+1}$ and $C_d = \mathcal{O}_{d,d,d+1}$.

Proposition 3.11. *Suppose $\text{char } K = 0$. For $s = 1, \dots, d$, there are short exact sequences*

$$0 \rightarrow C_s \rightarrow \tilde{\mathcal{O}}_{s,d,d+1} \rightarrow C_{s+1}(-s) \rightarrow 0.$$

Hence Conjecture 3.1 is true in the case $n = d + 1$.

Proof. We claim that the first $s - 1$ terms of the minimal free resolution $\mathbf{F}_\bullet^s$ of C_s are

$$\mathbf{F}_i^s = \bigoplus_{\lambda \subseteq (s-1) \times (d-s)} \bigwedge^i L \otimes S^i W \otimes A(-i - |\lambda|) \quad (0 \leq i \leq s-1).$$

This hypothesis is just strong enough to allow one to prove the result and the claim by descending induction on s . The case $s = d$ is clear since $C_d = \mathcal{O}_{d,d,d+1}$ and is resolved by a Koszul complex. $\square$

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