QUASILINEAR ELLIPTIC EQUATIONS VIA PERTURBATION METHOD

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ABSTRACT. We present a new approach to studying a class of quasilinear problems including the so-called Modified Nonlinear Schrödinger Equations (MNLS). We show that solutions of quasilinear equations can be obtained as limits of 4-Laplacian perturbations.

1. INTRODUCTION

This paper is concerned with the existence and multiplicity of solutions for quasilinear equations of the form

$$\begin{cases} \sum_{i,j=1}^N D_j(a_{ij}(x, u)D_i u) - \frac{1}{2} \sum_{i,j=1}^N D_s a_{ij}(x, u)D_i u D_j u + f(x, u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $D_i = \frac{\partial}{\partial x_i}$ and $D_s a_{ij}(x, s) = \frac{\partial}{\partial s} a_{ij}(x, s)$. This includes the following special class of equations (with $a_{ij}(x, u) = (1 + u^2)\delta_{ij}$):

$$(1) \quad \Delta u + \frac{1}{2}u\Delta(u^2) + f(x, u) = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

which is referred to as the so-called Modified Nonlinear Schrödinger Equation (MNLS). Here $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, and f is a continuous function in $\bar{\Omega} \times \mathbf{R}$. This has been involved in models of mathematical physics, e.g., [3, 4, 5, 9, 10] and has received considerable attention in mathematical analysis in the last ten years.

In this paper we propose a new approach which works for the general form of the quasilinear equations. For the sake of simplicity of notation and of a clearer presentation of ideas we state and prove results for (MNLS) (1) in the first two sections, and mention and sketch generalizations in section 3. The weak form of the equation (1) is

$$(2) \quad \int_{\Omega} [(1 + u^2)\nabla u \nabla \phi + u|\nabla u|^2 \phi - f(x, u)\phi] dx = 0, \quad \forall \phi \in C_0^\infty(\Omega),$$

which is formally the variational formulation of the following functional:

$$(3) \quad I_0(u) = \frac{1}{2} \int_{\Omega} (1 + u^2)|\nabla u|^2 dx - \int_{\Omega} F(x, u) dx.$$

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Here $F(x, s) = \int_0^s f(x, t)dt$, the primitive of f . We may define the derivative of I_0 at u in the direction of $\phi \in C_0^\infty(\Omega)$ as follows:

$$\langle I'_0(u), \phi \rangle = \lim_{t \rightarrow 0+} \frac{1}{t} (I_0(u+t\phi) - I_0(u)) = \int_{\Omega} ((1+u^2)\nabla u \nabla \phi + u|\nabla u|^2 \phi - f(x, u)\phi) dx.$$

We call u a critical point of I_0 if $u \in W_0^{1,2}(\Omega)$, $\int_{\Omega} u^2 |\nabla u|^2 dx < \infty$ and $\langle I'_0(u), \phi \rangle = 0$ for all $\phi \in C_0^\infty(\Omega)$. That is, u is a weak solution of (1).

For the nonlinearity we make the following assumptions.

(f₁) $\lim_{s \rightarrow 0} \frac{f(x, s)}{s} = 0$ uniformly in x .

(f₂) There exist $C_1 > 0$, $4 < q < \frac{4N}{N-2}$ such that for $x \in \Omega$ and $s \in \mathbf{R}$,

$$|f(x, s)| \leq C_1(1 + |s|^{q-1}).$$

(f₃) There exist $M > 0$ and $p > 4$ such that for all $x \in \Omega$, $|s| \geq M$, $\frac{1}{p}sf(x, s) \geq F(x, s) > 0$.

(f₄) $f(x, s)$ is odd in s .

In general there is no suitable space in which I_0 enjoys both smoothness and compactness. There have been several ideas and approaches used in recent years to overcome the difficulties such as by minimizations [13, 15], the Nehari method [12], and change of variables [7, 11]. In this paper we propose a new approach. We consider a perturbed functional

$$(4) \quad \begin{aligned} I_\mu(u) &= \frac{1}{4}\mu \int_{\Omega} |\nabla u|^4 dx + I_0(u) \\ &= \frac{1}{4}\mu \int_{\Omega} |\nabla u|^4 dx + \frac{1}{2} \int_{\Omega} (1+u^2) |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx, \end{aligned}$$

where $\mu \in (0, 1]$ is a parameter. Then it is easy to see that I_μ is a C^1 -functional on $W_0^{1,4}(\Omega)$. For all $\phi \in W_0^{1,4}(\Omega)$,

$$(5) \quad \langle I'_\mu(u), \phi \rangle = \mu \int_{\Omega} |\nabla u|^2 \nabla u \nabla \phi dx + \int_{\Omega} ((1+u^2)\nabla u \nabla \phi + u|\nabla u|^2 \phi - f(x, u)\phi) dx.$$

The idea is to obtain existence of critical points of I_μ for $\mu > 0$ small and to establish suitable estimates for the critical points as $\mu \rightarrow 0$ so that we may pass to the limit to get solutions of the original problem. This turns out to be effective for dealing with multiple solutions of quasilinear equations of general forms to which the idea of change of variables does not apply. Here are the main results of our paper for equation (1); see section 3 for results for more general quasilinear equations.

Theorem 1.1. *Let $\mu_n \rightarrow 0$. Let $\{u_n\} \subset W_0^{1,4}(\Omega)$ be a sequence of critical points of I_{μ_n} satisfying $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \leq C$ for some C independent of n . Then up to a subsequence as $n \rightarrow \infty$, $u_n \rightarrow u$ in $W_0^{1,2}(\Omega)$, $u_n \nabla u_n \rightarrow u \nabla u$ in $L^2(\Omega)$, $\mu_n \int_{\Omega} |\nabla u_n|^4 dx \rightarrow 0$, and $I_{\mu_n}(u_n) \rightarrow I_0(u)$ and u is a critical point of I_0 .*

Using Theorem 1.1 we have the following existence result.

Theorem 1.2. *Assume (f₁), (f₂), (f₃). Then*

(i) I_μ has a positive and a negative critical point u_μ and v_μ and as $\mu \rightarrow 0$, u_μ (resp., v_μ) converges to a positive (resp., negative) solution of equation (1).

(ii) If in addition (f₄) holds, then I_μ has a sequence of critical points $u_{\mu,n}$ such that as $\mu \rightarrow 0$, $u_{\mu,n}$ converges to a solution u_n of equation (1) with $I_0(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Notation. We denote by $\|\cdot\|$ the norm of $W_0^{1,4}(\Omega)$ and by $|\cdot|_s$ the norm of $L^s(\Omega)$ ($1 \leq s < +\infty$).

2. PROOF OF THE MAIN RESULTS

Lemma 2.1. *For $\mu > 0$ fixed, the functional I_μ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence for I_μ , i.e., $I_\mu(u_n) \rightarrow c$ and $I'_\mu(u_n) \rightarrow 0$. We prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,4}(\Omega)$. First we have

$$(6) \quad \begin{aligned} I_\mu(u_n) - \frac{1}{p} \langle I'_\mu(u_n), u_n \rangle \\ = \left(\frac{1}{4} - \frac{1}{p}\right) \mu \int_\Omega |\nabla u_n|^4 dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega |\nabla u_n|^2 dx \\ + \left(\frac{1}{2} - \frac{2}{p}\right) \int_\Omega u_n^2 |\nabla u_n|^2 dx + \int_\Omega \left(\frac{1}{p} u_n f(x, u_n) - F(x, u_n)\right) dx. \end{aligned}$$

By (f₃), $\left(\frac{1}{4} - \frac{1}{p}\right) \mu \int_\Omega |\nabla u_n|^4 dx \leq c + o(\|u_n\|)$. This implies that $\{u_n\}$ is bounded in $W_0^{1,4}(\Omega)$. Up to a subsequence we may assume that $u_n \rightharpoonup u$ in $W_0^{1,4}(\Omega)$ and $u_n \rightarrow u$ in $L^s(\Omega)$ for $1 \leq s < \frac{4N}{N-2}$. In (5) choosing $\phi = u_n - u_m$ we have

$$(7) \quad \begin{aligned} o(1) \|u_n - u_m\| \\ = \langle I'_\mu(u_n) - I'_\mu(u_m), u_n - u_m \rangle \\ = \mu \int_\Omega (|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m) (\nabla u_n - \nabla u_m) dx + \int_\Omega |\nabla u_n - \nabla u_m|^2 dx \\ + \int_\Omega (u_n^2 \nabla u_n - u_m^2 \nabla u_m) (\nabla u_n - \nabla u_m) dx \\ + \int_\Omega (u_n |\nabla u_n|^2 - u_m |\nabla u_m|^2) (u_n - u_m) dx \\ - \int_\Omega (f(x, u_n) - f(x, u_m)) (u_n - u_m) dx. \end{aligned}$$

We may estimate the terms involved as follows:

$$(8) \quad \begin{aligned} & \int_\Omega (u_n^2 \nabla u_n - u_m^2 \nabla u_m) (\nabla u_n - \nabla u_m) dx \\ &= \int_\Omega u_n^2 |\nabla u_n - \nabla u_m|^2 dx + \int_\Omega (u_n^2 - u_m^2) \nabla u_m (\nabla u_n - \nabla u_m) dx \\ &\geq -|u_n - u_m|_4 (|u_n|_4 + |u_m|_4) \|u_m\| (\|u_n\| + \|u_m\|) \rightarrow 0, \end{aligned}$$

$$(9) \quad \begin{aligned} & \left| \int_\Omega (u_n |\nabla u_n|^2 - u_m |\nabla u_m|^2) (u_n - u_m) dx \right| \\ &\leq (|u_n|_4 \|u_n\|^2 + |u_m|_4 \|u_m\|^2) |u_n - u_m|_4 \rightarrow 0, \end{aligned}$$

$$(10) \quad \begin{aligned} & \left| \int_\Omega (f(x, u_n) - f(x, u_m)) (u_n - u_m) dx \right| \\ &\leq C \int_\Omega (1 + |u_n|^{q-1} + |u_m|^{q-1}) |u_n - u_m| dx \\ &\leq C(1 + |u_n|_q^{q-1} + |u_m|_q^{q-1}) |u_n - u_m|_q \rightarrow 0, \end{aligned}$$

$$(11) \quad \mu \int_\Omega (|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m) (\nabla u_n - \nabla u_m) dx \geq c\mu \int_\Omega |\nabla u_n - \nabla u_m|^4 dx.$$

Returning to (7) we have

$$c\mu \int_\Omega |\nabla u_n - \nabla u_m|^4 dx \leq o(1) \|u_n - u_m\| + o(1),$$

which implies that $\|u_n - u_m\| \rightarrow 0$, i.e., $u_n \rightarrow u$ in $W_0^{1,4}(\Omega)$. $\square$

Next we consider the limiting behavior of critical points of I_μ as $\mu \rightarrow 0$ and prove Theorem 1.1.

Proof of Theorem 1.1. Similar to the proof of Lemma 2.1, by using $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \leq C$ we obtain

$$(12) \quad \begin{aligned} C &\geq I_{\mu_n}(u_n) - \frac{1}{p} \langle I'_{\mu_n}(u_n), u_n \rangle \\ &= \left(\frac{1}{4} - \frac{1}{p}\right) \mu_n \int_{\Omega} |\nabla u_n|^4 dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} |\nabla u_n|^2 dx + \left(\frac{1}{2} - \frac{2}{p}\right) \int_{\Omega} u_n^2 |\nabla u_n|^2 dx \\ &\quad + \int_{\Omega} \left(\frac{1}{p} u_n f(x, u_n) - F(x, u_n)\right) dx. \end{aligned}$$

Thus by (f₃),

$$(13) \quad \mu_n \int_{\Omega} |\nabla u_n|^4 dx + \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} u_n^2 |\nabla u_n|^2 dx \leq C$$

for some C independent of n . Then up to a subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, $u_n \nabla u_n \rightharpoonup u \nabla u$ in $L^2(\Omega)$ and $u_n(x) \rightarrow u(x)$ a.e. $x \in \Omega$. Note that u_n satisfies the equation

$$(14) \quad \mu_n \int_{\Omega} |\nabla u_n|^2 \nabla u_n \nabla \phi dx + \int_{\Omega} ((1 + u_n^2) \nabla u_n \nabla \phi + u_n |\nabla u_n|^2 \phi - f(x, u_n) \phi) dx = 0,$$

for all $\phi \in W_0^{1,4}(\Omega)$. Since $(\int_{\Omega} |u_n|^{\frac{4N}{N-2}} dx)^{\frac{N-2}{N}} \leq c \int_{\Omega} u_n^2 |Du_n|^2 dx \leq c$, by Moser's iteration we have

$$(15) \quad |u_n|_{L^\infty(\Omega)} \leq c; \quad \text{hence } |u|_{L^\infty(\Omega)} \leq c$$

for some constant c independent of n . To show that u is a critical point of I_0 we use some arguments in [6, 14] (see more references therein). In (14) we choose $\phi = \psi \exp(-u_n)$, where $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Substituting ϕ into (14) we have

$$(16) \quad \begin{aligned} 0 &= \mu_n \int_{\Omega} |\nabla u_n|^2 \nabla u_n (\nabla \psi \exp(-u_n) - \psi \nabla u_n \exp(-u_n)) dx \\ &\quad + \int_{\Omega} (1 + u_n^2) \nabla u_n (\nabla \psi \exp(-u_n) - \psi \nabla u_n \exp(-u_n)) dx \\ &\quad + \int_{\Omega} u_n |\nabla u_n|^2 \psi \exp(-u_n) dx - \int_{\Omega} f(x, u_n) \psi \exp(-u_n) dx \\ &\leq \mu_n \int_{\Omega} |\nabla u_n|^2 \nabla u_n \nabla \psi \exp(-u_n) dx + \int_{\Omega} (1 + u_n^2) \nabla u_n \nabla \psi \exp(-u_n) dx \\ &\quad - \int_{\Omega} (1 + u_n^2 - u_n) |\nabla u_n|^2 \psi \exp(-u_n) dx - \int_{\Omega} f(x, u_n) \psi \exp(-u_n) dx. \end{aligned}$$

Note that $1 + u_n^2 - u_n \geq 0$. By Fatou's lemma, the weak convergence of u_n and the fact that $\mu_n \int_{\Omega} |\nabla u_n|^4 dx$ is bounded, we have

$$(17) \quad \begin{aligned} 0 &\leq \int_{\Omega} (1 + u^2) \nabla u \nabla \psi \exp(-u) dx - \int_{\Omega} (1 + u^2 - u) |\nabla u|^2 \psi \exp(-u) dx \\ &\quad - \int_{\Omega} f(x, u) \psi \exp(-u) dx \\ &= \int_{\Omega} (1 + u^2) \nabla u \nabla (\psi \exp(-u)) dx + \int_{\Omega} u |\nabla u|^2 \psi \exp(-u) dx \\ &\quad - \int_{\Omega} f(x, u) \psi \exp(-u) dx. \end{aligned}$$

Let $\chi \geq 0$, $\chi \in C_0^\infty(\Omega)$. We may choose a sequence of nonnegative functions $\psi_n \in C_0^\infty(\Omega)$ such that $\psi_n \rightarrow \chi \exp u$ in $W_0^{1,2}(\Omega)$, $\psi_n \rightarrow \chi \exp u$ a.e. $x \in \Omega$, and ψ_n is uniformly bounded in $L^\infty(\Omega)$. Then by approximations in (17) we may obtain for all $\chi \geq 0$, $\chi \in C_0^\infty(\Omega)$,

$$(18) \quad \int_{\Omega} (1 + u^2) \nabla u \nabla \chi dx + \int_{\Omega} u |\nabla u|^2 \chi dx - \int_{\Omega} f(x, u) \chi dx \geq 0.$$

Similarly we may obtain an opposite inequality. Thus we have for all $\chi \in C_0^\infty(\Omega)$,

$$(19) \quad \int_{\Omega} (1 + u^2) \nabla u \nabla \chi dx + \int_{\Omega} u |\nabla u|^2 \chi dx - \int_{\Omega} f(x, u) \chi dx = 0.$$

That is, u is a critical point of I_0 and a solution of equation (1). By doing approximations again we may have u in the place of χ of (19):

$$(20) \quad \int_{\Omega} (1 + 2u^2) |\nabla u|^2 dx - \int_{\Omega} f(x, u) u dx = 0.$$

Setting $\phi = u_n$ in (14) we have

$$(21) \quad \mu_n \int_{\Omega} |\nabla u_n|^4 dx + \int_{\Omega} (1 + 2u_n^2) |\nabla u_n|^2 dx - \int_{\Omega} f(x, u_n) u_n dx = 0.$$

Using $\int_{\Omega} f(x, u_n) u_n dx \rightarrow \int_{\Omega} f(x, u) u dx$, (20), (21) and lower semi-continuity we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^2 dx &\rightarrow \int_{\Omega} |\nabla u|^2 dx, \quad \int_{\Omega} u_n^2 |\nabla u_n|^2 dx \rightarrow \int_{\Omega} u^2 |\nabla u|^2 dx, \\ \mu_n \int_{\Omega} |\nabla u_n|^4 dx &\rightarrow 0. \end{aligned}$$

In particular, we have $u_n \rightarrow u$ in $W_0^{1,2}(\Omega)$, $u_n \nabla u_n \rightarrow u \nabla u$ in $L^2(\Omega)$, and $I_{\mu_n}(u_n) \rightarrow I_0(u)$. $\square$

Next we apply the Mountain Pass Theorem to obtain existence of critical points of I_{μ} . For $\rho > 0$ denote

$$(22) \quad \Sigma_{\rho} = \{u \in W_0^{1,4}(\Omega) \mid \int_{\Omega} (1 + u^2) |\nabla u|^2 dx \leq \rho^2\}.$$

Consider a functional

$$I_{\mu}^{+}(u) = \frac{1}{4} \mu \int_{\Omega} |\nabla u|^4 dx + \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 dx - \int_{\Omega} F(x, u^{+}) dx.$$

Here and in the following we denote $u^{+} = \max\{u, 0\}$ and $u^{-} = \max\{-u, 0\}$. The functional I_{μ} satisfies the (PS) condition. Similarly we may verify that I_{μ}^{+} satisfies the (PS) condition. By (f_1) and (f_2) , for any $\epsilon > 0$ there exists $C_{\epsilon} > 0$ such that $F(x, u) \leq \epsilon u^2 + C_{\epsilon} |u|^q$. Then for ϵ, ρ small,

$$\begin{aligned} \int_{\Omega} F(x, u^{+}) dx &\leq \int_{\Omega} (\epsilon u^2 + C_{\epsilon} |u|^q) dx \\ &\leq \frac{1}{8} \int_{\Omega} |\nabla u|^2 dx + C_{\epsilon} \left(\int_{\Omega} u^2 |\nabla u|^2 dx \right)^{q/4} \\ (23) \quad &\leq \frac{1}{8} \rho^2 + C_{\epsilon} \rho^{q/2} \\ &\leq \frac{1}{4} \rho^2. \end{aligned}$$

Thus for small $\rho > 0$ we have for $u \in \partial \Sigma_{\rho}$,

$$I_{\mu}^{+}(u) \geq \frac{1}{2} \int_{\Omega} (1 + u^2) |\nabla u|^2 dx - \int_{\Omega} F(x, u^{+}) dx \geq \frac{1}{4} \rho^2.$$

Choose $\phi \in C_0^{\infty}(\Omega)$, $\phi(x) \geq 0$ in Ω , and $T > 0$. Define a path $h : [0, 1] \rightarrow W_0^{1,4}(\Omega)$ by $h(t) = tT\phi$. When T is large enough we have $I_{\mu}^{+}(h(1)) < 0$, $\int_{\Omega} (1 + h^2(1)) |\nabla h(1)|^2 dx > \rho^2$ and $\sup_{t \in [0, 1]} I_{\mu}^{+}(h(t)) \leq m$ for some m independent of $\mu \in (0, 1]$. Define

$$c_{\mu} = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I_{\mu}^{+}(\gamma(t)),$$

where

$$\Gamma = \{\gamma \mid \gamma \in C([0, 1], W_0^{1,4}(\Omega)), \gamma(0) = 0, \gamma(1) = T\phi\}.$$

From the Mountain Pass Theorem we have that $c_{\mu} \geq \frac{1}{4} \rho^2$ is a critical value of I_{μ}^{+} . Let u_{μ} be a critical point corresponding to c_{μ} . We have $u_{\mu} \geq 0$. Thus u_{μ} is a positive critical point of I_{μ} . In summary we have

Proposition 2.2. *There exist positive constants ρ and m independent of μ such that I_μ has a positive critical point u_μ satisfying $\frac{1}{4}\rho^2 \leq I_\mu(u_\mu) \leq m$.*

Proof of Part (i) of Theorem 1.2. For a positive solution the proofs follow from Proposition 2.2 and Theorem 1.1. A similar argument gives a negative solution. $\square$

Next we turn to the symmetric case for which an unbounded sequence of critical values will be constructed. First we have the following.

Proposition 2.3. *Assume $(f_{1,2,3,4})$. Then for all $\mu \in (0, 1]$ fixed, I_μ has a sequence of critical points $u_{\mu,j}$ such that there exist $\beta_j > \alpha_j$ both of which are independent of $\mu > 0$, $\alpha_j \rightarrow \infty$ as $j \rightarrow \infty$ and $I_\mu(u_{\mu,j}) \in [\alpha_j, \beta_j]$ for all $\mu > 0$.*

Proof. Basically we apply the symmetric Mountain Pass Theorem due to Ambrosetti and Rabinowitz (see [2, 16]). In order to get uniform lower bounds of minimax critical values of I_μ in terms of $\mu > 0$ we make some necessary changes. For completeness, we give the proof in detail.

Let $\lambda_i, i = 1, 2, \dots$, be the i -th eigenvalue of $-\Delta$ with Dirichlet boundary condition and φ_i the eigenfunction corresponding to λ_i . By regularity we know that $\phi_i \in W_0^{1,4}(\Omega)$. Let $E_n = \text{span}\{\phi_1, \dots, \phi_n\}$. We write $W_0^{1,4}(\Omega) = E_j \oplus X_j$ for $j = 1, 2, \dots$, where X_j is orthogonal to E_j in $W_0^{1,2}(\Omega)$. Define

$$G_n = \{H \in C(\Sigma_{R_n} \cap E_n, W_0^{1,4}(\Omega)) : H \text{ is odd and } H = \text{id on } \partial\Sigma_{R_n} \cap E_n\},$$

where $R_n > 0$ is chosen such that $I_\mu(u) < 0$ for $u \in E_n \setminus \Sigma_{R_n}$. Due to (f_3) , we may choose R_n independent of $\mu > 0$. Moreover, we assume $R_n > \rho_n$; see the following (24) for the definition of ρ_n . Set

$$\begin{aligned} \Gamma_j : &= \{H(\Sigma_{R_n} \cap E_n \setminus Y) : H \in G_n, n \geq j, Y = -Y \subset \Sigma_{R_n} \cap E_n \text{ open,} \\ &\text{and } \gamma(Y) \leq n - j\}, \end{aligned}$$

where $\gamma(\cdot)$ is the genus. Define

$$c_j(\mu) = \inf_{B \in \Gamma_j} \sup_{u \in B} I_\mu(u), \quad j = 1, 2, \dots$$

We claim that $c_j(\mu), j = 1, 2, \dots$, are critical values of I_μ and there exist $\alpha_j < \beta_j$ such that $c_j(\mu) \in [\alpha_j, \beta_j]$ and $\alpha_j \rightarrow \infty$ as $j \rightarrow \infty$.

Since I_μ is increasing in μ , we have $c_j(\mu) \leq c_j(1) := \beta_j, j = 1, 2, \dots$. We estimate the lower bound for $c_j(\mu)$. By the following Lemma 2.4, we have an intersection property: If $\rho < R_n$ for all $n \geq j$, then

$$\text{for } B \in \Gamma_j, \text{ we have } B \cap \partial\Sigma_\rho \cap X_{j-1} \neq \emptyset.$$

Hence

$$c_j(\mu) \geq \inf_{u \in \partial\Sigma_\rho \cap X_{j-1}} I_\mu(u) \geq \inf_{u \in \partial\Sigma_\rho \cap X_{j-1}} I_0(u).$$

We may choose $\rho = \rho_j > 0$ such that $\inf_{\partial\Sigma_{\rho_j} \cap X_{j-1}} I_\mu(u) \geq \alpha_j := \frac{1}{8}\rho_j^2 \rightarrow \infty$ as $j \rightarrow \infty$. In fact by (f_1) and (f_2) , for any $\epsilon > 0$ there exists $C = C_\epsilon > 0$ such that $F(x, u) \leq \epsilon u^2 + C|u|^q$. Then for ϵ small and $u \in \partial\Sigma_\rho \cap X_{j-1}$,

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}\rho^2 - \epsilon|u|_2^2 - C|u|_q^q \\ &\geq \frac{1}{4}\rho^2 - C|u|_{2^*}^{aq}|u|_2^{(1-a)q} \\ &\geq \frac{1}{4}\rho^2 - C\lambda_j^{-\frac{(1-a)q}{2}}\rho^{\frac{aq}{2}+(1-a)q} \\ &= \rho^2\left(\frac{1}{4} - C\lambda_j^{-\frac{(1-a)q}{2}}\rho^{\frac{aq}{2}+(1-a)q-2}\right). \end{aligned}$$

Here $a \in (0, 1)$ satisfies $\frac{1}{q} = \frac{a}{2 \cdot 2^*} + \frac{1-a}{2}$. As $\frac{aq}{2} + (1-a)q > \frac{q}{2} > 2$ we may choose $\rho = \rho_j$ such that

$$(24) \quad C\lambda_j^{-\frac{(1-a)q}{2}} \rho_j^{\frac{aq}{2} + (1-a)q-2} = \frac{1}{8}.$$

Then $I_\mu(u) \geq \frac{1}{8}\rho_j^2$, and we have $\alpha_j = \frac{1}{8}\rho_j^2 \rightarrow \infty$ as $j \rightarrow \infty$.

Now we show that $c_j(\mu), j = 1, 2, \dots$, are critical values of I_μ . If $\varphi : W_0^{1,4}(\Omega) \rightarrow W_0^{1,4}(\Omega)$ is odd, and $\varphi = id$ on $\partial\Sigma_{R_n} \cap E_n$ for all $n \geq j$, then $\varphi : \Gamma_j \rightarrow \Gamma_j$. Now $c_j(\mu) \geq \alpha_j > 0$; hence $c_j(\mu)$ is a critical value of I_μ . Otherwise there exist $\epsilon > 0$ and $\varphi : W_0^{1,4}(\Omega) \rightarrow W_0^{1,4}(\Omega)$ such that φ is odd, $\varphi(I_\mu^{c_j(\mu)+\epsilon}) \subset I_\mu^{c_j(\mu)-\epsilon}$ and $\varphi = id$ in $I_\mu^{c_j(\mu)-\epsilon}$; in particular, $\varphi = id$ on $\partial\Sigma_{R_n} \cap E_n$ for all n . By the definition of $c_j(\mu)$ there exists $B \in \Gamma_j$ such that $\sup_{u \in B} I_\mu(u) \leq c_j(\mu) + \epsilon$. Now $A = \varphi(B) \in \Gamma_j$ and $\sup_{u \in A} I_\mu(u) \leq c_j(\mu) - \epsilon$, which is a contradiction. $\square$

Lemma 2.4. *For $B \in \Gamma_j$, it follows that $B \cap \partial\Sigma_\rho \cap X_{j-1} \neq \emptyset$ provided $\rho < R_n$ for all $n \geq j$.*

Proof. Set $B = H(\Sigma_{R_n} \cap E_n \setminus Y)$ for some $n \geq j$ and $\gamma(Y) \leq n - j$. Let $\hat{O} = \{x \in \Sigma_{R_n} \cap E_n \mid H(x) \in \Sigma_\rho\}$. Since H is odd, $0 \in \hat{O}$. Let O be the connected component of $\hat{O}$ containing 0. Then O is a bounded symmetric neighborhood of 0 in E_n . By Borsuk's theorem, $\gamma(\partial O) = n$. Since $H(u) = u$ for $u \in \partial\Sigma_{R_n} \cap E_n$, $\int_\Omega (1 + H^2(u)) |DH(u)|^2 dx = R_n^2 > \rho^2$. Thus we have $H(u) \in \partial\Sigma_\rho$ for $u \in \partial O$. Set $W = \{u \in \Sigma_{R_n} \cap E_n \mid H(u) \in \partial\Sigma_\rho\}$. Then $\partial O \subset W$ and $\gamma(W) = n$. Also $\gamma(W \setminus Y) \geq n - (n - j) = j > j - 1$. Then we have $\gamma(H(W \setminus Y)) > j - 1$. Since $\text{codim} X_{j-1} = j - 1$, $H(W \setminus Y) \cap X_{j-1} \neq \emptyset$. As $H(W \setminus Y) \subset B \cap \partial\Sigma_\rho$, the proof is finished. $\square$

Proof of Part (ii) of Theorem 1.2. For the existence of infinitely many solutions we proceed by using Proposition 2.3 and Theorem 1.1 and passing to the limit for $\mu \rightarrow 0$. As $\alpha_j \leq c_j(\mu) \leq \beta_j$ for all μ we may obtain critical point u_j of I_0 with critical values in $[\alpha_j, \beta_j]$ and $I_0(u_j) \geq \alpha_j \rightarrow \infty$. $\square$

Remark 2.1. We note that the uniform lower bounds on the minimax values of I_μ involved always hold as long as there is $a_0 > 0$ such that $I_\mu(u) \geq \int_\Omega (\frac{a_0}{2}(1 + u^2)|\nabla u|^2 - F(x, u))dx$ for $u \in W_0^{1,4}(\Omega)$. This is important for general quasilinear problems considered in the next section.

3. GENERAL QUASILINEAR PROBLEMS

In this section we show that our new approach works for more general quasilinear equations. As the proofs are similar we will be sketchy from place to place. Consider

$$(25) \quad D_j(a_{ij}(x, u)D_i u) - \frac{1}{2}D_s a_{ij}(x, u)D_i u D_j u + f(x, u) = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

Here we use the conventional notation for summations. The functions a_{ij} satisfy the following conditions.

(A1) $a_{ij} \in C^1(\overline{\Omega} \times \mathbf{R}, \mathbf{R})$, $a_{ij} = a_{ji}$, there exist constants $a_1 \geq a_0 > 0$ such that

$$a_0(1 + s^2)|\xi|^2 \leq a_{ij}(x, s)\xi_i \xi_j \leq a_1(1 + s^2)|\xi|^2, \quad \forall x \in \overline{\Omega}, s \in \mathbf{R}, \xi \in \mathbf{R}^N.$$

(A2) $0 \leq D_s a_{ij}(x, s)s\xi_i \xi_j \leq 2a_{ij}(x, s)\xi_i \xi_j$ for $\xi \in \mathbf{R}^N, s \in \mathbf{R}$.

(A3) $a_{ij}(x, s)$ is odd in s .

The variational functional now is

$$I_\mu(u) = \frac{1}{4}\mu \int_\Omega |\nabla u|^4 dx + \frac{1}{2} \int_\Omega a_{ij}(x, u) D_i u D_j u dx - \int_\Omega F(x, u) dx.$$

Lemma 3.1. *Assume (A1), (A2), (f₁), (f₂), (f₃). For $\mu > 0$ fixed, the functional I_μ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence for I_μ , i.e., $I_\mu(u_n) \rightarrow c$ and $I'_\mu(u_n) \rightarrow 0$. We prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,4}(\Omega)$. First we have

$$\begin{aligned} (26) \quad & I_\mu(u_n) - \frac{1}{p} \langle I'_\mu(u_n), u_n \rangle \\ &= \left(\frac{1}{4} - \frac{1}{p}\right) \mu \int_\Omega |\nabla u_n|^4 dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega a_{ij}(x, u_n) D_i u_n D_j u_n dx \\ &\quad - \frac{1}{2p} \int_\Omega D_s a_{ij}(x, u_n) u_n D_i u_n D_j u_n dx + \int_\Omega \left(\frac{1}{p} u_n f(x, u_n) - F(x, u_n)\right) dx. \end{aligned}$$

By (f₄) and (A1) – (A2), $\left(\frac{1}{4} - \frac{1}{p}\right) \mu \int_\Omega |\nabla u_n|^4 dx \leq c + o(\|u_n\|)$. This implies that $\{u_n\}$ is bounded in $W_0^{1,4}(\Omega)$. Up to a subsequence we may assume $u_n \rightharpoonup u$ in $W_0^{1,4}(\Omega)$ and $u_n \rightarrow u$ in $L^s(\Omega)$ for $1 \leq s < \frac{4N}{N-2}$. In (5) choosing $\phi = u_n - u_m$ we have

$$\begin{aligned} (27) \quad & o(1) \|u_n - u_m\| \\ &= \langle I'_\mu(u_n) - I'_\mu(u_m), u_n - u_m \rangle \\ &= \mu \int_\Omega (|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m) (\nabla u_n - \nabla u_m) dx \\ &\quad + \int_\Omega (a_{ij}(x, u_n) D_i u_n - a_{ij}(x, u_m) D_i u_m) (D_j u_n - D_j u_m) dx \\ &\quad + \int_\Omega \frac{1}{2} (D_s a_{ij}(x, u_n) D_i u_n D_j u_n - D_s a_{ij}(x, u_m) D_i u_m D_j u_m) (u_n - u_m) dx \\ &\quad - \int_\Omega (f(x, u_n) - f(x, u_m)) (u_n - u_m) dx. \end{aligned}$$

Using (A1)–(A2) we show that the limits of all terms except the first are nonnegative and we obtain

$$c\mu \int_\Omega |\nabla u_n - \nabla u_m|^4 dx \leq o(1) \|u_n - u_m\| + o(1),$$

which implies $\|u_n - u_m\| \rightarrow 0$, i.e., $u_n \rightarrow u$ in $W_0^{1,4}(\Omega)$. $\square$

Next we show that Theorem 1.1 still holds in this general situation. We assume (A1), (A2), (f₁), (f₂), (f₃).

Proof. Similar to the proof of the above lemma, by using $I'_{\mu_n}(u_n) = 0$ and $I_{\mu_n}(u_n) \leq C$,

$$\mu_n \int_\Omega |\nabla u_n|^4 dx + \int_\Omega a_{ij}(x, u_n) D_i u_n D_j u_n dx \leq C$$

for some C independent of n . Then by (A1) up to a subsequence we have $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, $u_n \nabla u_n \rightharpoonup u \nabla u$ in $L^2(\Omega)$ and $u_n(x) \rightarrow u(x)$ a.e. $x \in \Omega$. Note that u_n satisfies, for all $\phi \in W_0^{1,4}(\Omega)$,

$$\begin{aligned} (28) \quad & \mu_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \phi dx + \int_\Omega a_{ij}(x, u_n) D_i u_n D_j \phi dx \\ &+ \int_\Omega \frac{1}{2} D_s a_{ij}(x, u_n) D_i u_n D_j u_n \phi dx - \int_\Omega f(x, u_n) \phi dx = 0. \end{aligned}$$

Since $(\int_{\Omega} |u_n|^{\frac{4N}{N-2}} dx)^{\frac{N-2}{N}} \leq c \int_{\Omega} a_{ij}(x, u_n) D_i u_n D_j u_n dx \leq c$, by Morser's iteration we have

$$(29) \quad |u_n|_{L^\infty(\Omega)} \leq c; \quad \text{hence } |u|_{L^\infty(\Omega)} \leq c$$

for some c independent of n . We use the same argument as before to show that u is a critical point of I_0 . In (28) we choose $\phi = \psi \exp(-Mu_n)$, where $\psi \geq 0, \psi \in C_0^\infty(\Omega)$, and $M > 0$ is a constant. Substituting ϕ into (28) we have

$$(30) \quad \begin{aligned} 0 &= \mu_n \int_{\Omega} |\nabla u_n|^2 \nabla u_n (\nabla \psi \exp(-Mu_n) - M \psi \nabla u_n \exp(-Mu_n)) dx \\ &\quad + \int_{\Omega} a_{ij}(x, u_n) D_i u_n (D_j \psi \exp(-Mu_n) - M \psi D_j u_n \exp(-Mu_n)) dx \\ &\quad + \int_{\Omega} \frac{1}{2} D_s a_{ij}(x, u_n) D_i u_n D_j u_n \psi \exp(-Mu_n) dx \\ &\quad - \int_{\Omega} f(x, u_n) \psi \exp(-Mu_n) dx \\ &\leq \mu_n \int_{\Omega} |\nabla u_n|^2 \nabla u_n \nabla \psi \exp(-Mu_n) dx \\ &\quad + \int_{\Omega} a_{ij}(x, u_n) D_i u_n D_j \psi \exp(-Mu_n) dx \\ &\quad - \int_{\Omega} (M a_{ij}(x, u_n) - \frac{1}{2} D_s a_{ij}(x, u_n)) D_i u_n D_j u_n \psi \exp(-Mu_n) dx \\ &\quad - \int_{\Omega} f(x, u_n) \psi \exp(-Mu_n) dx. \end{aligned}$$

Note that $M a_{ij}(x, u_n) - \frac{1}{2} D_s a_{ij}(x, u_n)$ is positive for M large enough. By Fatou's lemma, the weak convergence of u_n and the fact that $\mu_n \int_{\Omega} |\nabla u_n|^4 dx$ is bounded, we have

$$(31) \quad \begin{aligned} 0 &\leq \int_{\Omega} a_{ij}(x, u) D_i u D_j \psi \exp(-Mu) dx \\ &\quad - \int_{\Omega} (M a_{ij}(x, u) - \frac{1}{2} D_s a_{ij}(x, u)) D_i u D_j u \psi \exp(-Mu) dx \\ &\quad - \int_{\Omega} f(x, u) \psi \exp(-Mu) dx \\ &= \int_{\Omega} a_{ij}(x, u) D_i u D_j (\psi \exp(-Mu)) dx \\ &\quad + \frac{1}{2} \int_{\Omega} D_s a_{ij}(x, u) D_i u D_j u \psi \exp(-Mu) dx - \int_{\Omega} f(x, u) \psi \exp(-Mu) dx. \end{aligned}$$

From here the same arguments used before give for all $\chi \in C_0^\infty(\Omega)$,

$$(32) \quad \int_{\Omega} a_{ij}(x, u) D_i u D_j \chi dx + \frac{1}{2} \int_{\Omega} D_s a_{ij}(x, u) D_i u D_j u \chi dx - \int_{\Omega} f(x, u) \chi dx = 0.$$

That is, u is a critical point of I_0 and a solution of equation (25). By doing approximations again we may have u in the place of χ of (32):

$$(33) \quad \int_{\Omega} a_{ij}(x, u) D_i u D_j u dx + \frac{1}{2} \int_{\Omega} D_s a_{ij}(x, u) u D_i u D_j u dx - \int_{\Omega} f(x, u) u dx = 0.$$

Setting $\phi = u_n$ in (28) we have

$$(34) \quad \begin{aligned} \mu_n \int_{\Omega} |\nabla u_n|^4 dx + \int_{\Omega} a_{ij}(x, u_n) D_i u_n D_j u_n dx + \frac{1}{2} \int_{\Omega} D_s a_{ij}(x, u_n) u_n D_i u_n D_j u_n dx \\ - \int_{\Omega} f(x, u_n) u_n dx = 0. \end{aligned}$$

Using $\int_{\Omega} f(x, u_n) u_n dx \rightarrow \int_{\Omega} f(x, u) u dx$, (33), (34) and lower semi-continuity we obtain

$$\mu_n \int_{\Omega} |\nabla u_n|^4 dx \rightarrow 0, \quad \int_{\Omega} a_{ij}(x, u_n) D_i u_n D_j u_n dx \rightarrow \int_{\Omega} a_{ij}(x, u) D_i u D_j u dx,$$

which implies that $u_n \rightarrow u$ in $W_0^{1,2}(\Omega)$, $u_n \nabla u_n \rightarrow u \nabla u$ in $L^2(\Omega)$, and $I_{\mu_n}(u_n) \rightarrow I_0(u)$. $\square$

Using this convergence theorem we can obtain the following existence result for equation (25).

Theorem 3.2. *Assume (A1), (A2), $(f_1), (f_2), (f_3)$. Then*

- (i) I_μ has a positive and a negative critical point u_μ and v_μ and as $\mu \rightarrow 0$, u_μ (resp., v_μ) converges to a positive (resp., negative) solution of equation (25).
- (ii) If in addition $(A_3), (f_4)$ hold, then I_μ has a sequence of critical points $u_{\mu,n}$ such that as $\mu \rightarrow 0$, $u_{\mu,n}$ converges to a solution u_n of equation (25) with $I_0(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

The proof is similar to the proof of Theorem 1.2 with a few modifications. First, by using the Mountain Pass Theorem we obtain a positive critical point of I_μ , and the convergence theorem gives a positive critical point of I_0 . Second, by the symmetric Mountain Pass Theorem we obtain a sequence of unbounded critical values of I_μ with uniform lower and upper bounds (see Remark 2.1). Then the convergence theorem gives solutions for I_0 . The details are omitted.

We close the paper by pointing out that to our knowledge there have not been any multiplicity results for general quasilinear equations such as (25) and our results above are the first attempt in this direction, though for the special form of equations (1), multiplicity results could be derived by using the change of variable arguments from [7, 11] (we refer to recent work [1, 8, 17] for more references on using the arguments of changing variables for various problems).

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