ON $K3$ SURFACES WHICH DOMINATE KUMMER SURFACES

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ABSTRACT. We study isogeny relations between $K3$ surfaces and Kummer surfaces. Specifically, we prove a Torelli-type theorem for the existence of rational maps from $K3$ surfaces to Kummer surfaces, and a Kummer sandwich theorem for $K3$ surfaces with Shioda-Inose structure.

1. INTRODUCTION

In the present paper we study rational maps between $K3$ surfaces in terms of their periods. Let X be a complex algebraic $K3$ surface and T_X be the transcendental lattice of X , which is endowed with a natural Hodge structure. For a natural number $n > 0$ let $T_X(n)$ be the lattice obtained by multiplying the quadratic form on T_X by n . In [7] Shafarevich posed the following question.

Problem 1.1 ([7] Question 1.1). Let X and Y be complex algebraic $K3$ surfaces. Is it true that there exists a dominant rational map $X \dashrightarrow Y$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_Y(n) \otimes \mathbb{Q}$ for some natural number n ?

Shafarevich’s question is a variation of a Torelli-type problem. It proposes to consider the $\mathbb{Q}$ -Hodge structures $T_X \otimes \mathbb{Q}$ (up to scaling) for the existence of rational maps. A recent paper of Chen [1] implies that the answer is in general negative. On the other hand, the problem has been solved affirmatively in certain cases: for $K3$ surfaces X with Picard number $\rho(X) = 20$ (“singular $K3$ surfaces”) by Inose and Shioda [9], [2] already before [7]; for $K3$ surfaces X with $\rho(X) = 19$ by Nikulin-Shafarevich [7]; Nikulin [7] studied rational maps obtained as compositions of double coverings. When both X and Y are Kummer surfaces, Problem 1.1 is obviously true by the corresponding property of Abelian surfaces. The first purpose of this paper is to answer Problem 1.1 affirmatively when the target Y is a Kummer surface.

Theorem 1.2. *Let X and Y be complex algebraic $K3$ surfaces. Assume that Y is dominated by some Kummer surface; e.g., Y admits a Shioda-Inose structure or Y itself is a Kummer surface. Then there exists a dominant rational map $X \dashrightarrow Y$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_Y(n) \otimes \mathbb{Q}$ for some natural number n .*

In order to produce a desired map $X \dashrightarrow Y$, we will compose the following three types of rational maps: (1) double coverings, (2) rational maps between Kummer surfaces induced by isogenies of Abelian surfaces, and (3) multiplication maps from

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elliptic $K3$ surfaces to the associated Jacobian fibrations. The first two have been also used in [9], [2], and [7]. A new ingredient of this paper is a systematic use of the third type of rational maps. In the course of the proof, we shall characterize those $K3$ surfaces which dominate Kummer surfaces by their Hodge structures.

The approach of Inose and Shioda for Problem 1.1 was to use a Kummer sandwich theorem, which roughly says that a singular $K3$ surface is two-isogenous to a Kummer surface. Recently the Kummer sandwich theorem has been extended to a larger class of $K3$ surfaces by Shioda [8] and has found some arithmetic applications. The $K3$ surfaces studied in [8] are characterized by the existence of Shioda-Inose correspondences with products of elliptic curves. The second purpose of this paper is to prove a Kummer sandwich theorem for all complex algebraic $K3$ surfaces with Shioda-Inose structure (Theorem 2.5). It is independent of Theorem 1.2 and shows a more precise isogeny relation between Kummer surfaces and $K3$ surfaces with Shioda-Inose structure.

Throughout this paper, the varieties are assumed to be complex algebraic. The transcendental lattice of an algebraic surface X will be denoted by T_X . By U we denote the rank 2 even indefinite unimodular lattice. By E_8 we denote the rank 8 even negative-definite unimodular lattice. For a lattice $L = (L, (\cdot, \cdot)_L)$ and a natural number n , we denote by $L(n)$ the scaled lattice $(L, n(\cdot, \cdot)_L)$.

2. KUMMER SANDWICH THEOREM

Let X be an algebraic $K3$ surface. Recall that a *Nikulin involution* of X is an involution $\iota : X \rightarrow X$ which acts trivially on $H^{2,0}(X)$. A Nikulin involution of X canonically corresponds to a double covering $X \dashrightarrow Y$ to another $K3$ surface Y . Indeed, if we have a double covering $\pi : X \dashrightarrow Y$, then the covering transformation of π is a Nikulin involution of X . Conversely, for a Nikulin involution ι of X , the minimal resolution $Y = \widetilde{X/\langle \iota \rangle}$ of the quotient surface is a $K3$ surface ([6]), and we have the rational quotient map $\pi : X \dashrightarrow Y$ of degree 2. The transcendental lattices T_X and T_Y are related by the chain of inclusions

$$(2.1) \quad 2T_Y \subseteq \pi_* T_X = T_X(2) \subseteq T_Y,$$

which preserves the quadratic forms and the Hodge structures.

Nikulin [6], [7] and Morrison [5] developed the lattice-theoretic aspect of Nikulin involution. Let us denote

$$(2.2) \quad \Lambda_0 := E_8(2) \oplus U^3,$$

$$(2.3) \quad \Lambda_1 := \frac{1}{2}E_8(2) \oplus U^3.$$

We regard Λ_0 as a submodule of Λ_1 in a natural way. Then Λ_1 is the dual lattice of Λ_0 . The following proposition reduces the construction of a Nikulin involution to a purely arithmetic problem.

Proposition 2.1 ([7] Section 2.1 and Lemma 2.2.4). *Let X be an algebraic $K3$ surface. Suppose that one is given a primitive embedding $T_X \subset \Lambda_0$ of lattices. Then there exists a Nikulin involution $\iota : X \rightarrow X$ such that if we denote $Y = \widetilde{X/\langle \iota \rangle}$, then T_Y is Hodge isometric to the lattice*

$$(2.4) \quad T := (T_X \otimes \mathbb{Q} \cap \Lambda_1)(2),$$

where the Hodge structure of T is induced from T_X . Conversely, if one has a rational map $X \dashrightarrow Y$ of degree 2 to a K3 surface Y , then there exists a primitive embedding $T_X \subset \Lambda_0$ such that T_Y is Hodge isometric to the lattice T defined by (2.4).

The Shioda-Inose structure is a special kind of Nikulin involution.

Definition 2.2 ([5]). An algebraic K3 surface X admits a *Shioda-Inose structure* if there exists a Kummer surface $Y = \text{Km}A$ and a rational map $\pi : X \dashrightarrow Y$ of degree 2 such that π_* induces a Hodge isometry $T_X(2) \simeq T_Y$.

There is a lattice-theoretic characterization of K3 surfaces admitting Shioda-Inose structures due to Morrison.

Theorem 2.3 ([5], Theorem 6.3). *An algebraic K3 surface X admits a Shioda-Inose structure if and only if there exists a primitive embedding $T_X \hookrightarrow U^3$ of lattices.*

Shioda [8], extending the work of Inose [2], proved a Kummer sandwich theorem for elliptic K3 surfaces with section and with two II^* -fibers over an arbitrary algebraically closed field of characteristic $\neq 2, 3$. When the ground field is $\mathbb{C}$, one can characterize the K3 surfaces studied in [8] by the existence of Shioda-Inose structures such that the corresponding Abelian surfaces are products of elliptic curves. Here we shall derive in a transcendental way a Kummer sandwich theorem for all complex algebraic K3 surfaces with Shioda-Inose structure. We denote the Dynkin diagram of E_8 by

We identify the $\mathbb{Z}$ -modules underlying E_8 and $E_8(2)$ in a natural way and regard the above set $\{v_i\}_{i=1}^8$ as a basis of $E_8(2)$. Then we have $(v_i, v_i) = -4$ for $i = 1, \dots, 8$, $(v_i, v_j) = 2$ if v_i and v_j are connected by an edge, and $(v_i, v_j) = 0$ otherwise. Let $\{e_i, f_i\}_{i=1}^3$ be the standard basis of U^3 . We have $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. For $i = 1, 2, 3$, we define the vectors $l_i, m_i \in \Lambda_0 = E_8(2) \oplus U^3$ by

$$\begin{aligned}
 l_1 &= -v_5 + v_7 + 2(e_1 + f_1), \\
 m_1 &= -v_4, \\
 l_2 &= v_1 + v_8 + 2(e_2 + f_2), \\
 m_2 &= v_2, \\
 l_3 &= v_7 + v_8 + 2(e_1 + e_2 + e_3 + f_3), \\
 m_3 &= v_6,
 \end{aligned}$$

and put $L := \langle l_1, m_1, l_2, m_2, l_3, m_3 \rangle$.

Lemma 2.4. *The sublattice $L \subset \Lambda_0$ has the following properties:*

- (1) $L \simeq U(2)^3$.
- (2) $L \subset 2\Lambda_1$.
- (3) L is a primitive sublattice of Λ_0 .

Proof. We can extend the set $\{l_i, m_i\}_{i=1}^3$ to a $\mathbb{Z}$ -basis of Λ_0 by adding the set of vectors $\{v_3, v_5, e_1, f_1, e_2, f_2, e_3, f_3\}$. Thus L is primitive in Λ_0 . The assertion (2) is obvious, and the assertion (1) is proved by direct calculations. $\square$

Theorem 2.5. *Let X be an algebraic $K3$ surface admitting a Shioda-Inose structure $X \dashrightarrow Y = \mathrm{Km}A$. Then there exists a Nikulin involution ι on Y such that the minimal resolution of the quotient surface $Y/\langle\iota\rangle$ is isomorphic to X . In particular, one has the following sequence of rational maps of degree 2:*

$$(2.5) \quad \mathrm{Km}A \dashrightarrow X \dashrightarrow \mathrm{Km}A.$$

Proof. By the definitions, we have the Hodge isometries

$$T_Y \simeq T_A(2), \quad T_X \simeq T_A.$$

Since T_A is embedded into $H^2(A, \mathbb{Z}) \simeq U^3$ primitively, there exists a primitive embedding $\varphi : T_Y \hookrightarrow U(2)^3$. By composing φ with an isometry $U(2)^3 \simeq L$, we obtain a primitive embedding $\psi : T_Y \hookrightarrow \Lambda_0$ such that $\psi(T_Y) \subset 2\Lambda_1$. We have

$$\psi(T_Y) \otimes \mathbb{Q} \cap \Lambda_1 = \frac{1}{2}\psi(T_Y).$$

By Proposition 2.1, there exists a Nikulin involution $\iota : Y \rightarrow Y$ such that for the minimal resolution Z of $Y/\langle\iota\rangle$ the transcendental lattice T_Z is Hodge isometric to

$$\frac{1}{2}T_Y(2) \simeq \frac{1}{2}T_A(4) \simeq T_A \simeq T_X.$$

Since a Hodge isometry $T_Z \simeq T_X$ can be extended to a Hodge isometry $H^2(Z, \mathbb{Z}) \simeq H^2(X, \mathbb{Z})$ (cf. [5], Corollary 2.10), we have $Z \simeq X$ by the Torelli theorem. $\square$

The rational quotient map $\pi : \mathrm{Km}A \dashrightarrow X$ constructed in Theorem 2.5 induces a Hodge isometry $\pi^* : T_X(2) \rightarrow T_{\mathrm{Km}A}$. Thus a $K3$ surface X with Shioda-Inose structure can be defined not only as a double cover of a Kummer surface $\mathrm{Km}A$ but also as a double quotient of $\mathrm{Km}A$, which exhibits an isogeny relation between X and $\mathrm{Km}A$. Unfortunately, as we rely on the Torelli theorem, our Kummer sandwich theorem is not explicit as in [2], [8], and our argument works only over $\mathbb{C}$.

$K3$ surfaces with Shioda-Inose structure are particular double covers of Kummer surfaces. Now, is it true in general that a double cover X of a Kummer surface $\mathrm{Km}A$ admits a double covering $\mathrm{Km}A \dashrightarrow X$ of the opposite direction, as do isogenies of elliptic curves? Here is a negative example.

Example 2.6. Let A be an Abelian surface with $T_A \simeq U \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ and X be the $K3$ surface with T_X Hodge isometric to $2T_A$. Then there exists a rational map $X \dashrightarrow \mathrm{Km}A$ of degree 2, but there does *not* exist a rational map $\mathrm{Km}A \dashrightarrow X$ of degree 2.

Proof. The existence of a double covering $X \dashrightarrow \mathrm{Km}A$ follows from Mehran's criterion for double covers of Kummer surfaces ([4], Theorem 3.1). Suppose that we have a rational map $\mathrm{Km}A \dashrightarrow X$ of degree 2. By Proposition 2.1 there exists a primitive embedding $T_A(2) \hookrightarrow \Lambda_0$ such that

$$(2.6) \quad T_A(2) \otimes \mathbb{Q} \cap \Lambda_1 = T_A(2).$$

Via this embedding, we regard $T_A(2)$ as a primitive sublattice of Λ_0 . Let $\pi : T_A(2) \rightarrow U^3$ be the orthogonal projection, which is injective by the condition (2.6). Let M be the lattice $\pi(T_A(2))$ and N be the primitive closure of M in U^3 . By the condition (2.6) again, the Abelian group N/M has no 2-component. For an

even lattice L , let $L^\vee$ be the dual lattice of L , $D_L = L^\vee/L$ be the discriminant group of L , and $(D_L)_2$ be the 2-component of D_L . We see from the inclusions $M \subset N \subset N^\vee \subset M^\vee$ that

$$(2.7) \quad (D_M)_2 \simeq (D_N)_2 \simeq (D_{N^\perp \cap U^3})_2.$$

The second isomorphism follows from the fact that N is a primitive sublattice of the unimodular lattice U^3 . In particular, the length of $(D_M)_2$ is less than or equal to 2. On the other hand, we have $(v, w) \in 2\mathbb{Z}$ for every $v, w \in M$. Thus we have $\frac{1}{2}M \subset M^\vee$, which is absurd. $\square$

Remark 2.7. It follows from [7], Theorem 1.3, that for X and A as in Example 2.6, there nevertheless exists a rational map $\text{Km}A \dashrightarrow X$ of degree 2^μ for some $\mu > 1$.

3. RATIONAL MAPS TO KUMMER SURFACES

In this section we study rational maps from K3 surfaces to Kummer surfaces in general. We shall use the following.

Proposition 3.1 ([3], Section 4). *Let X and Y be algebraic K3 surfaces with $\text{rk}(T_X) = \text{rk}(T_Y) \leq 9$ such that there exists an embedding $T_X \rightarrow T_Y$ of lattices preserving the periods. Then there exists a sequence $X_1 = X, X_2, \dots, X_n = Y$ of K3 surfaces such that X_{i+1} is isomorphic to the surface underlying the Jacobian fibration of an elliptic fibration $\pi_i : X_i \rightarrow \mathbb{P}^1$. In particular, for a line bundle $L \in \text{Pic}(X)$ we have a rational map $X_i \dashrightarrow X_{i+1}$ defined by $x \mapsto \mathcal{O}_F(dx) \otimes L^{-1}$, where F is the π_i -fiber containing $x \in X_i$ and $d = (L.F)$.*

We shall characterize K3 surfaces X dominating Kummer surfaces by the lattices T_X .

Proposition 3.2. *For an algebraic K3 surface X the following conditions are equivalent.*

- (i) *There exists a dominant rational map $X \dashrightarrow \text{Km}A$ to some Kummer surface $\text{Km}A$.*
- (ii) *There exists an embedding $T_X \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of quadratic spaces.*
- (iii) *There exists an embedding $T_X \hookrightarrow U^3$ of lattices.*

Proof. (i) $\Rightarrow$ (ii): A rational map $f : X \dashrightarrow \text{Km}A$ of finite degree d induces a Hodge isometry

$$f_* : T_X(d) \otimes \mathbb{Q} \xrightarrow{\sim} T_{\text{Km}A} \otimes \mathbb{Q} \simeq T_A(2) \otimes \mathbb{Q}.$$

Then the quadratic space $T_X \otimes \mathbb{Q}$ is isometric to $T_A(2d) \otimes \mathbb{Q}$ and thus is embedded into $H^2(A, \mathbb{Q})(2d) \simeq U^3(2d) \otimes \mathbb{Q}$. By the property $U^3(2d) \otimes \mathbb{Q} \simeq U^3 \otimes \mathbb{Q}$ of the lattice U , we obtain an embedding $T_X \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q}$ of quadratic spaces.

(ii) $\Rightarrow$ (iii): Recall that an even lattice of rank r can be embedded (primitively) into U^r . In particular, we may assume that $\text{rk}(T_X) = 4$ or 5 . When $\text{rk}(T_X) = 4$, the condition (ii) is equivalent to the existence of an embedding $U \otimes \mathbb{Q} \rightarrow T_X \otimes \mathbb{Q}$ by Witt's theorem for $(T_X \otimes \mathbb{Q})^\perp \cap U^3 \otimes \mathbb{Q}$. Thus we have an isotropic vector in T_X . Let T be a maximal even overlattice of T_X . A primitive isotropic vector $v \in T$ induces an embedding $U \rightarrow T$ because $(v, T) = \mathbb{Z}$. Hence $T \simeq U \oplus L$ for some rank 2 lattice L so that T can be embedded into U^3 . When $\text{rk}(T_X) = 5$, as in the case

of $\text{rk}(T_X) = 4$, the condition (ii) is equivalent to the existence of a rank 2 totally isotropic sublattice of T_X . Then every maximal even overlattice of T_X is of the form $T = U^2 \oplus L$, $\text{rk}(L) = 1$, and thus can be embedded into U^3 .

(iii) $\Rightarrow$ (i): We fix an embedding $T_X \subset U^3$. Let T be the primitive closure of T_X in U^3 and endow T with the Hodge structure induced from T_X . We regard T as a primitive sublattice of $U^3 \oplus E_8^2$. By the surjectivity of the period map, there exists a $K3$ surface Y with T_Y Hodge isometric to T . We have an embedding $T_X \hookrightarrow T_Y$ of finite index which preserves the periods. It follows from Proposition 3.1 that there exists a dominant rational map $X \dashrightarrow Y$. Since the lattice T_Y can be embedded primitively into U^3 , the $K3$ surface Y admits a Shioda-Inose structure $Y \dashrightarrow \text{Km}A$ by Theorem 2.3. $\square$

Proposition 3.2 is analogous to Theorem 2.3: replacing Shioda-Inose structures by general rational maps corresponds to replacing primitive embeddings of lattices by embeddings of rational quadratic spaces.

An Abelian surface A is a product of two elliptic curves if and only if T_A can be embedded primitively into U^2 . Hence by a similar argument as in the above proof we have the following variant of Proposition 3.2.

Proposition 3.3. *For an algebraic $K3$ surface X the following conditions are equivalent.*

- (i) *There exists a dominant rational map $X \dashrightarrow \text{Km}A$ to some Kummer surface $\text{Km}A$, where A is a product of two elliptic curves.*
- (ii) *There exists an embedding $T_X \otimes \mathbb{Q} \hookrightarrow U^2 \otimes \mathbb{Q}$ of quadratic spaces.*
- (iii) *There exists an embedding $T_X \hookrightarrow U^2$ of lattices.*

By using Proposition 3.2 we deduce the next theorem, from which Theorem 1.2 follows immediately.

Theorem 3.4. *Let X be an algebraic $K3$ surface and $\text{Km}A$ be an algebraic Kummer surface. Then there exists a dominant rational map $X \dashrightarrow \text{Km}A$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_A(n) \otimes \mathbb{Q}$ for some natural number n .*

Proof. It suffices to prove the “if” part. Assume the existence of a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_A(n) \otimes \mathbb{Q}$. As $T_A \otimes \mathbb{Q}$ is embedded into $U^3 \otimes \mathbb{Q}$, by Proposition 3.2 we can find a Kummer surface $\text{Km}B$ and a finite rational map $X \dashrightarrow \text{Km}B$. Since we have a Hodge isometry $T_B(m) \otimes \mathbb{Q} \simeq T_A \otimes \mathbb{Q}$ for some natural number m , the Abelian surface B is isogenous to the Abelian surface A . Thus there exists a dominant rational map $\text{Km}B \dashrightarrow \text{Km}A$. $\square$

Corollary 3.5. *Let X be an algebraic $K3$ surface and A be an Abelian surface. If we have a dominant rational map $A \dashrightarrow X$, then there exists a dominant rational map $X \dashrightarrow \text{Km}A$.*

Corollary 3.6. *Let X and Y be algebraic $K3$ surfaces dominated by some Kummer surfaces. Then there exists a dominant rational map $X \dashrightarrow Y$ if and only if there exists a dominant rational map $Y \dashrightarrow X$.*

Thus, as in Inose’s paper [2], we are able to define a notion of isogeny for those $K3$ surfaces dominated by Kummer surfaces by the existence of a rational map.

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