COMPLETELY REGULAR PROPER REFLECTION OF LOCALES OVER A GIVEN LOCALE

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ABSTRACT. Let X be a completely regular locale. We present a construction which shows that every locale $f : Y \rightarrow X$ over X has a completely regular proper reflection in the slice category Loc/X and the reflection map is a dense embedding if and only if Y is completely regular.

Let X be a locale. We know that the category $Loc(Sh(X))$ of internal locales in the sheaves topos $Sh(X)$ is equivalent to the slice category Loc/X of locales over X (see [1], [2]). Also a localic map $f : Y \rightarrow X$ is proper if and only if it corresponds to a compact internal locale in the topos $Sh(X)$ (see [8]). By a classical result given by B. Banaschewski and C. J. Mulvey in [3], we know that the category of compact completely regular locales (compact regular locales) is reflective in the category of locales. So the following question is interesting:

Dose there exist a compact completely regular reflection for every internal locale in $Sh(X)$?

Suppose $f : Y \rightarrow X$ is a locale over X . Let $g : Z \rightarrow X$ be a proper map with Z completely regular. We call $g : Z \rightarrow X$ a completely regular proper reflection of $f : Y \rightarrow X$ if there exists a localic map $h : Y \rightarrow Z$ such that $f = gh$,

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

and $h : Y \rightarrow Z$ is universal among all those localic maps from $f : Y \rightarrow X$ to a completely regular proper map over X . Now we consider a similar question in the slice category Loc/X as follows.

Does there exist a completely regular proper reflection for every locale $f : Y \rightarrow X$ over X ? If so, can we give an explicit description of the reflection?

Suppose $f : Y \rightarrow X$ is a proper surjection and Y is regular. Then X is regular by proposition 2 in [9]. Hence if every locale over X has a completely regular proper reflection, then we take a surjection $g : Y \rightarrow X$ such that its completely regular proper reflection $p : Z \rightarrow X$ will still be surjective. This requires X to be regular,

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which shows that for a non-regular locale X the answer to the above question is negative.

Lemma 1. *If every locale over X has a completely regular proper reflection (regular proper reflection), then X is regular.*

For X being the terminal locale 2, $Sh(X) \cong Set$, then it is clear that the compact completely regular reflection for any locale Y is just the Stone–Čech compactification of Y . In this paper we will show that if X is completely regular, then every locale $f : Y \rightarrow X$ over X has a completely regular proper reflection $P_f : P_Y \rightarrow X$ in the slice category Loc/X . Moreover, the reflection map $\varepsilon_f : Y \rightarrow P_Y$ is a dense embedding if and only if Y is completely regular. In this case, $P_f : P_Y \rightarrow X$ becomes a maximal proper reflection of $f : Y \rightarrow X$ like that in the classical case.

Recall that a localic map $f : X \rightarrow Y$ is said to be proper whenever f is closed and the right adjoint $f_* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ of the frame morphism $f^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ preserves directed joins. This is a point-free version of the classical notion of what are called perfect maps in the category of topological spaces. Properness of $f : X \rightarrow Y$ has the following characterizations (see [6], [7], [8]):

- (i) f is proper;
- (ii) f is stably closed; i.e., its pullback along any morphism with codomain Y is closed;
- (iii) the internal locale in the topos $Sh(Y)$ corresponding to f is compact;
- (iv) $id_Z \times f : Z \times X \rightarrow Z \times Y$ is closed for every locales Z .

If X is a compact locale. Then the projection $p_Y : X \times Y \rightarrow Y$ is closed for any Y (see [5]). Thus we have $id_Z \times p_Y = p_{Z \times Y} : Z \times X \times Y \rightarrow Z \times Y$ is closed for all locale Z . Hence we have the following result.

Lemma 2. *Suppose X is a compact locale. Then the projection $X \times Y \rightarrow Y$ is proper for any locale Y .*

Let X be a completely regular locale and let $f : Y \rightarrow X$ be a localic map. Consider the diagonal $\langle \beta, f \rangle : Y \rightarrow \beta Y \times X$, where βY is the compact completely regular reflection of Y . Write $P_Y = \uparrow \{(u, x) \mid \beta^*(u) \wedge f^*(x) = 0\}$, the closure of the image of Y under $\langle \beta, f \rangle$, and $P_f : P_Y \rightarrow X$ the composite of the closed inclusion $P_Y \hookrightarrow \beta Y \times X$ and the projection $\beta Y \times X \rightarrow X$. By Lemma 2, we know that $P_f : P_Y \rightarrow X$ is proper and P_Y is completely regular since both X and βY are completely regular. Write $\varepsilon_f : Y \rightarrow P_Y$ to be the co-restriction of $\langle \beta, f \rangle$. Then we have a commutative triangle of localic maps

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon_f} & P_Y \\ & \searrow f & \swarrow P_f \\ & X & \end{array}$$

We shall show $P_f : P_Y \rightarrow X$ to be the completely regular proper reflection of $f : Y \rightarrow X$ in the slice category Loc/X . We note that the above construction can be applied to any locale X so that it becomes a functor $P : Loc/X \rightarrow Loc/X$.

Proposition 1. *$P : Loc/X \rightarrow Loc/X$ is a functor such that ε becomes a natural transformation from the identity to P .*

Proof. Let $h : Y \rightarrow Z$ be a localic map such that the following triangle commutes:

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

Write $\bar{h} : \beta Y \rightarrow \beta Z$ to be the extension of h ; i.e., the following square commutes:

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ \beta_Y \downarrow & & \downarrow \beta_Z \\ \beta Y & \xrightarrow{\bar{h}} & \beta Z \end{array}$$

Denote $0_{P_Y} = \{(u, x) \mid \beta_Y^*(u) \wedge f^*(x) = 0\}$ the least element of P_Y , and $0_{P_Z} = \{(v, x) \mid \beta_Z^*(v) \wedge g^*(x) = 0\}$ the least element of P_Z respectively. Consider the product $\bar{h} \times id_X : \beta Y \times X \rightarrow \beta Z \times X$. For $(v, x) \in 0_{P_Z}$, we have $\beta_Y^* \bar{h}^*(v) \wedge f^*(x) = h^* \beta_Z^*(v) \wedge h^* g^*(x) = h^*(\beta_Z^*(v) \wedge g^*(x)) = 0$, i.e. $(\bar{h} \times id_X)^*(0_{P_Z}) \leq 0_{P_Y}$. Now define $P(h) : P_Y \rightarrow P_Z$ for which the corresponding frame morphism is defined by $P(h)^*(I) = (\bar{h} \times id_X)^*(I) \vee 0_{P_Y}$. Then the following triangle commutes:

$$\begin{array}{ccc} P_Y & \xrightarrow{P(h)} & P_Z \\ & \searrow P_f & \swarrow P_g \\ & X & \end{array}$$

P clearly preserves composites. Hence $P : Loc/X \rightarrow Loc/X$ is a functor. To show that $\varepsilon : id \rightarrow P$ is a natural transformation, it suffices to check the commutativity of the following square. But it is straightforward.

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon_f} & P_Y \\ h \downarrow & & \downarrow P(h) \\ Z & \xrightarrow{\varepsilon_g} & P_Z \end{array}$$

□

Lemma 3. *If $f : Y \rightarrow X$ is proper with Y completely regular, then $\varepsilon_f : Y \rightarrow P_Y$ is an isomorphism.*

Proof. By the fact already proved in [8] (or [9]), the diagonal $\langle \beta, f \rangle : Y \rightarrow \beta Y \times X$ is proper. Hence $\langle \beta, f \rangle$ is a closed inclusion since $\beta : Y \rightarrow \beta Y$ is an inclusion (see [3]). This shows that the co-restriction $\varepsilon_f : Y \rightarrow P_Y$ is an isomorphism. □

Theorem 1. *Let X be a completely regular locale, and let $f : Y \rightarrow X$ be a localic map. Then the localic map $\varepsilon_f : Y \rightarrow P_Y$ is universal among all localic maps from $f : Y \rightarrow X$ to a completely regular proper map over X ; i.e., $P_f : P_Y \rightarrow X$ is the completely regular proper reflection of $f : Y \rightarrow X$ in the slice category Loc/X . Moreover, the reflection map $\varepsilon_f : Y \rightarrow P_Y$ is a dense embedding if and only if Y is completely regular.*

Proof. Suppose $g : Z \rightarrow X$ is a proper map with Z a completely regular locale. Let $h : Y \rightarrow Z$ be a localic map such that the following triangle commutes:

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

Then we have a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon_f} & P_Y \\ h \downarrow & & \downarrow P(h) \\ Z & \xrightarrow{\varepsilon_g} & P_Z \end{array}$$

By Lemma 3, $\varepsilon_g : Z \rightarrow P_Z$ is an isomorphism. Hence h has a factorization $h = \varepsilon_g^{-1}P(h)\varepsilon_f$. If $h = r\varepsilon_f$ is another factorization of h , then the equalizer of $\varepsilon_g^{-1}P(h)$ and r is a closed sublocale of P_Y since Z is completely regular, hence must be Hausdorff. But the equalizer contains Y and so must be P_Y itself. Hence $r = \varepsilon_g^{-1}P(h)$. If Y is completely regular, then $\varepsilon_f : Y \rightarrow P_Y$ is an embedding since $\beta : Y \rightarrow \beta Y$ is an embedding. Conversely if $\varepsilon_f : Y \rightarrow P_Y$ is an embedding, then Y is completely regular since P_Y is completely regular. $\square$

It is interesting that we can apply the above reflection construction to the category of topological spaces and thus get a completely regular perfect reflection for every topological space over a given completely regular space.

Let X be a completely regular space, and let $f : Y \rightarrow X$ be a continuous map. Write P_Y for the closure of the image of Y under the diagonal $\langle \beta, f \rangle : Y \rightarrow \beta Y \times X$, where βY is the Stone–Čech compactification of Y . Then the restriction $P_f : P_Y \rightarrow X$ of the projection is a perfect map and P_Y is completely regular. Denote by $\varepsilon_Y : Y \rightarrow P_Y$ the co-restriction of the diagonal $\langle \beta, f \rangle$. Assuming the axiom of choice, we have the following result.

Corollary 1. $\varepsilon_Y : Y \rightarrow P_Y$ is universal among all continuous maps from the topological space $f : Y \rightarrow X$ over X to a completely regular perfect map over X ; i.e., $P_f : P_Y \rightarrow X$ is the completely regular perfect reflection of $f : Y \rightarrow X$ in the slice category Top/X .

Now we consider a more general case when the given locale X is not completely regular. In this case we will show that for a special class of locales over X , i.e. those locales $f : Y \rightarrow X$ over X such that the closure $\overline{f(Y)}$ of the image $f(Y)$ of Y under f is completely regular, it indeed has a completely regular proper reflection in Loc/X .

Lemma 4. Let $p : Y \rightarrow X$ be a proper map. If the image $p(Y)$ of Y under p is contained in a sublocale Y_j of Y , then the co-restriction map $\bar{p} : Y \rightarrow Y_j$ is proper.

Proof. $\bar{p}$ is clearly closed. Suppose $\{y_s \mid s \in S\} \subset Y_j$ is directed. Then $\bar{p}^*(\bigvee^j y_s) = \bar{p}^*(j(\bigvee y_s)) = p^*(\bigvee y_s) = \bigvee p^*(y_s) = \bigvee \bar{p}^*(y_s)$, where $\bigvee^j$ represents the join in Y_j . Hence $\bar{p}^*$ preserves directed joins. $\square$

Let X be a locale. Suppose $f : Y \rightarrow X$ is a locale over X such that the closure $\overline{f(Y)}$ of the image $f(Y)$ of Y under f is completely regular. We have a factorization $Y \xrightarrow{f} X = Y \xrightarrow{\bar{f}} \overline{f(Y)} \xrightarrow{f'} X$. Similar to the case for X being completely regular, we have a diagonal map $\langle \beta, \bar{f} \rangle : Y \rightarrow \beta Y \times \overline{f(Y)}$. Write $\hat{P}_Y = \uparrow \{(u, x) \mid \beta^*(u) \wedge \bar{f}^*(x) = 0\}$, the closure of the image of Y under $\langle \beta, \bar{f} \rangle$, and $\hat{P}_f : \hat{P}_Y \rightarrow X$ the composite of the closed inclusion $\hat{P}_Y \hookrightarrow \beta Y \times \overline{f(Y)}$, the projection $\beta Y \times \overline{f(Y)} \rightarrow \overline{f(Y)}$ and the inclusion $\overline{f(Y)} \hookrightarrow X$. Then $\hat{P}_f : \hat{P}_Y \rightarrow X$ is proper and $\hat{P}_Y$ is completely regular since both $\overline{f(Y)}$ and βY are completely regular. Denote by $\varepsilon : Y \rightarrow \hat{P}_Y$ the co-restriction of $\langle \beta, \bar{f} \rangle$. We have a commutative triangle

$$\begin{array}{ccc} Y & \xrightarrow{\varepsilon} & \hat{P}_Y \\ & \searrow f & \swarrow \hat{P}_f \\ & X & \end{array}$$

Theorem 2. $\hat{P}_f : \hat{P}_Y \rightarrow X$ is a completely regular proper reflection of $f : Y \rightarrow X$ in the slice category Loc/X and $\varepsilon : Y \rightarrow \hat{P}_Y$ is the reflection map. Moreover, the reflection map $\varepsilon : Y \rightarrow P_Y$ is a dense embedding if and only if Y is completely regular.

Proof. Suppose $g : Z \rightarrow X$ is a proper map with Z completely regular. Let $h : Y \rightarrow Z$ be a localic map such that $f = gh$. Write for $\overline{h(Y)}$ the closure of the image $h(Y)$ of Y under h . Then $h : Y \rightarrow Z$ has a factorization $Y \xrightarrow{\bar{h}} \overline{h(Y)} \xrightarrow{h'} Z$. We have $g(\overline{h(Y)}) = \overline{g(h(Y))} = \overline{f(Y)}$ since g is closed. Note that the composite $\overline{h(Y)} \hookrightarrow Z \xrightarrow{g} X$ is proper, so its co-restriction $s : \overline{h(Y)} \rightarrow \overline{h(Y)}$ is proper by Lemma 4. Moreover we have $f's\bar{h} = gh'\bar{h} = gh = f'\bar{f}$; hence $s\bar{h} = \bar{f}$ since f' is monic. By Theorem 1, the co-restriction $\bar{h}$ can be uniquely factored through $\varepsilon : Y \rightarrow \hat{P}_Y$ by a map $r : \hat{P}_Y \rightarrow \overline{h(Y)}$. Thus $h : Y \rightarrow Z$ can be factored through $\varepsilon : Y \rightarrow \hat{P}_Y$ as $Y \xrightarrow{\varepsilon} \hat{P}_Y \xrightarrow{h} Z = Y \xrightarrow{\bar{h}} \overline{h(Y)} \hookrightarrow Z = Y \xrightarrow{\varepsilon} \hat{P}_Y \xrightarrow{r} \overline{h(Y)} \hookrightarrow Z$. The factorization through ε is essentially unique since $\varepsilon : Y \rightarrow \hat{P}_Y$ is dense and Z is completely regular. $\square$

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