CURVES OF GENUS g WHOSE CANONICAL MODEL LIES ON A SURFACE OF DEGREE $g + 1$

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ABSTRACT. Let C be a non-hyperelliptic curve of genus g . We prove that if the minimal degree of a surface containing the canonical model of C in $\mathbb{P}_k^{g-1}$ is $g + 1$, then either $g \geq 9$ and C carries exactly one g_4^1 or $7 \leq g \leq 15$ and C is birationally isomorphic to a plane septic curve with at most double points as singularities.

1. INTRODUCTION AND NOTATION

Let k be an algebraically closed field. Let C be a curve (i.e. a smooth, connected, projective scheme of dimension 1) and denote by $g \geq 2$ its genus. The canonical system on C gives rise to a natural morphism $\phi: C \rightarrow \mathbb{P}_k^{g-1}$. The morphism ϕ is either the double cover of a rational normal curve or it is an isomorphism. In the former case C is said to be *hyperelliptic*; in the latter case $C_{can} := \phi(C)$ is called the *canonical model* of C .

The study of the intrinsic properties of a non-hyperelliptic curve C is strictly related to the description of the embedding $C_{can} \subseteq \mathbb{P}_k^{g-1}$.

We are interested in characterising curves C such that C_{can} lies on a surface $S \subseteq \mathbb{P}_k^{g-1}$ (i.e. an integral, projective scheme of dimension 2) for which $\deg(S)$ is fixed. Due to the geometric characterisation of the degree, we know that $\deg(S) \geq g - 2$. It is thus natural to ask first what happens for low values of $\deg(S)$, say $g - 2, g - 1, g$, and so on.

The first result along such lines is the following well-known Babbage–Enriques–Noether–Petri Theorem (see e.g. [A–C–G–H], p. 131, when the characteristic of k is 0 and [SD] otherwise).

Theorem A. *Let C be a non-hyperelliptic curve of genus $g \geq 3$. Then $C_{can} \subseteq \mathbb{P}_k^{g-1}$ is contained in an integral surface $S \subseteq \mathbb{P}_k^{g-1}$ of degree $g - 2$ if and only if exactly one of the following two conditions holds.*

- i) C is trigonal (i.e. there exists a map $C \rightarrow \mathbb{P}_k^1$ which is finite of degree 3): in this case S is a rational normal scroll.
- ii) $g = 5$ and C is isomorphic to a plane quintic curve: in this case S is a Veronese surface.

A second result is the following theorem (see [B–C–N] and the references therein).

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Theorem B. *Let C be a non-hyperelliptic curve of genus $g \geq 5$. Then $C_{can} \subseteq \mathbb{P}_k^{g-1}$ is contained in an integral surface $S \subseteq \mathbb{P}_k^{g-1}$ of degree $g-1$ if and only if one of the three following conditions holds.*

i) C is bielliptic (i.e. there exists a double cover $C \rightarrow E$ onto an elliptic curve): in this case S is a cone on an elliptic normal curve contained in a hyperplane of $\mathbb{P}_k^{g-1}$ with vertex not on C_{can} .

ii) $g \leq 10$ and C is birationally isomorphic to a plane sextic carrying $10-g$, possibly infinitely near, double points as singularities: in this case S is the anticanonical image of the blowup of $\mathbb{P}_k^2$ at such points.

iii) $g = 9$ and C is isomorphic to a smooth complete intersection inside $\mathbb{P}_k^3$ of an integral quadric Q with an integral quartic surface: in this case S is the anticanonical image of Q .

Both of the above results also rest on the classification of surfaces of degree $g-2$ and $g-1$ in $\mathbb{P}_k^{g-1}$. Such an approach is not optimal as $\deg(S)$ increases. In [Cs] we succeed to prove Theorem C below, via an analysis of the linear system cut out on S by the quadrics through C_{can} .

Theorem C. *Let C be a non-hyperelliptic curve of genus g . If the minimal degree of a surface $S \subseteq \mathbb{P}_k^{g-1}$ containing C_{can} is g , then $7 \leq g \leq 12$ and S is a conic bundle over $\mathbb{P}_k^1$ whose fibres cut out on C a base-point-free g_4^1 . The curve C does not carry any other base-point-free g_4^1 .*

The aim of the present paper is to examine the next case, namely the case of curves C such that C_{can} lies on a surface $S \subseteq \mathbb{P}_k^{g-1}$ with $\deg(S) = g+1$. We restrict our study to the case of a field k of characteristic 0. A refinement of the techniques used for proving Theorem C above allow us to obtain the following.

Theorem D. *Let C be a non-hyperelliptic curve of genus g over an algebraically closed field k of characteristic 0. If the minimal degree of a surface $S \subseteq \mathbb{P}_k^{g-1}$ containing the canonical model of C is $g+1$, then exactly one of the two following conditions holds.*

i) $7 \leq g \leq 15$ and C is birationally isomorphic to a plane septic D with exactly $15-g$ possibly infinitely near double points as singularities: in this case C_{can} is contained in the surface $\widehat{S} \subseteq \mathbb{P}_k^{g-1}$ of degree $g+1$ obtained as the birational image of $\mathbb{P}_k^2$ via the map defined by the linear system of quartics through the double points of the plane model D (such an $\widehat{S}$ will be called a weak Bordiga surface).

ii) $g \geq 9$ and C is endowed with a unique g_4^1 : in this case S is a conic bundle onto $\mathbb{P}_k^1$ whose fibres cut out on the canonical model of C the aforementioned base-point-free g_4^1 . If $g \geq 16$ such a g_4^1 is composed with an involution on a curve of genus 2.

Very recently the aforementioned Theorems B, C, D have been generalised in [Co], where the author proves that if the canonical model of a curve C of genus g lies on a surface of degree $g+\kappa < 2g-3$ with $\kappa \geq -1$ and $(g, \kappa) \neq (10, -1)$, then either $g \leq 3\kappa+12$ or κ is odd and C is the double cover of a curve of genus $(\kappa+3)/2$. Moreover, also a careful characterisation of such curves is given therein, under the extra hypothesis that S is smooth.

The proof of Theorem D is quite long and has been divided into some steps from section 3 to section 6. In section 2 we recall some facts regarding tetragonal curves which are used in the proof. At the end of section 6 we list some remarks and comments.

Notation. We work over an algebraically closed field k of characteristic 0.

If V is a vector space, then we denote by $\mathbb{P}(V)$ the corresponding projective space. In particular, we set $\mathbb{P}_k^n := \mathbb{P}(k^{n+1})$.

Let $X \subseteq \mathbb{P}_k^n$ be a closed subscheme and let $\mathcal{L}$ be a sheaf on X . If $q \in H^0(X, \mathcal{L})$ we will denote by $D_0(q) \subseteq X$ its zero-scheme.

For other definitions, results and notation we always refer to [Ha].

2. COVERS OF DEGREE 4

We start this paper recalling some standard facts about covers of degree 4. If C is a curve, then a morphism $\varrho: C \rightarrow \mathbb{P}_k^1$ is a cover of degree 4 if it is finite, flat and $\mathrm{rk}(\varrho_* \mathcal{O}_C) = 4$. There exists a natural exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^1} \xrightarrow{\varrho^\#} \varrho_* \mathcal{O}_C \longrightarrow \check{\mathcal{E}} \longrightarrow 0,$$

where $\check{\mathcal{E}}$ is a locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -sheaf of rank 3 called the *Tschirnhausen module* of ϱ .

Since every locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -sheaf splits, we can fix a decomposition $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$. The sequence above and Serre duality yields $g + 3 = \alpha_1 + \alpha_2 + \alpha_3$.

Theorem 2.1. *Let C be a curve of genus g and $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4. There exists an embedding $i: C \hookrightarrow \mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^1$, a locally free $\mathcal{O}_{\mathbb{P}_k^1}$ -sheaf $\mathcal{F} := \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$ such that $\sum_{j=1}^2 \beta_j = g + 3$ and a morphism $\delta: \pi^* \mathcal{F} \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}}(2)$ such that $C = D_0(\delta)$ and $\varrho = \pi|_{D_0(\delta)}$. Moreover*

$$(2.1.1) \quad \omega_C \cong i^*(\mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-2)).$$

Conversely let $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$ and $\mathcal{F} \cong \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$ be such that $\sum_{i=1}^3 \alpha_i = \sum_{j=1}^2 \beta_j$, define $\mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^1$ and let $\delta: \pi^ \mathcal{F} \rightarrow \mathcal{O}_{\mathbb{P}}(2)$. If $\varrho = \pi|_{D_0(\delta)}$ is a cover, then it has degree 4 and the Tschirnhausen module of ϱ is $\check{\mathcal{E}}$.*

Proof. See [C–E], Theorem 2.1 and section 4. □

The following Bertini-type theorem will be used.

Theorem 2.2. *Let $\mathcal{E} := \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$ and $\mathcal{F} := \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$ be such that $\sum_{i=1}^3 \alpha_i = \sum_{j=1}^2 \beta_j$. Define $\mathbb{P} := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^1$ and*

$$\mathcal{U} := \{\delta \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \check{\mathcal{F}}) \mid D_0(\delta) \text{ is smooth and } \pi|_{D_0(a,b)} \text{ is a cover of degree 4}\}.$$

If $S^2 \mathcal{E} \otimes \check{\mathcal{F}}$ is globally generated, then $\mathcal{U} \neq \emptyset$ and is open.

Proof. See [C–E], Theorem 4.5. □

Let C be a curve, let $\varrho: C \rightarrow \mathbb{P}_k^1$ be a cover of degree 4 and fix decompositions $\mathcal{E} \cong \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_k^1}(\alpha_i)$, $\alpha_1 \leq \alpha_2 \leq \alpha_3$, $\mathcal{F} \cong \bigoplus_{j=1}^2 \mathcal{O}_{\mathbb{P}_k^1}(\beta_j)$, $\beta_1 \leq \beta_2$, of the associated sheaves.

The triple $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ is called the *scrollar invariants* of ϱ . The number $\tau := \beta_1 - 4$ is called the *tetragonality* of ϱ .

Remark 2.3. We list below some results about the tetragonality and the scollar invariants of a cover that we will use in the next sections. We are assuming that C is a curve of genus $g \geq 4$, endowed with a cover $\varrho: C \rightarrow \mathbb{P}_k^1$ of degree 4 with invariants $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and τ . Such invariants have already been studied in several places. Besides the aforementioned paper [C–E], we recall for the reader’s benefit [Ch] and the two papers [Sch1, Sch2] and [Br], where $e_1 = \alpha_3 - 2$, $e_2 = \alpha_2 - 2$, $e_3 = \alpha_1 - 2$, $b_1 = g - 5 - \tau$, $b_2 = \tau$.

The following assertions hold true.

- (1) $1 \leq \alpha_1 \leq g + 3/3$, $-2 \leq \tau \leq \min\{(g - 5)/2, 2(\alpha_1 - 2)\}$ (see [Cs], Proposition 2.5).
- (2) If C is non-hyperelliptic, then $\alpha_1 \geq 2$ and $\tau \geq -1$ (see Section 6.3 of [Sch1]).
- (3) If C is non-hyperelliptic and $\tau = -1$, then either C is trigonal or it is isomorphic to a smooth plane quintic. Moreover, $g \leq 6$ (see Section 6.3 of [Sch1]).
- (4) If C is non-hyperelliptic and $\tau < (g - 9)/3$, then ϱ is composed with an involution onto a curve of genus $\alpha_1 - 1$. Moreover, $\tau = 2(\alpha_1 - 2)$ (see [Sch2], Corollary 4.2).
- (5) If C is neither hyperelliptic nor trigonal nor isomorphic to a smooth plane quintic and $\varrho': C \rightarrow \mathbb{P}_k^1$ is a second cover of degree 4 with tetragonality τ' , then $\tau = \tau' = 0$. If C is not bielliptic, then $g \leq 9$ (Proposition 3.4 of [Cs] and the references therein).

The Picard group $\text{Pic}(\mathbb{P})$ of $\mathbb{P}$ is generated by the class ξ of the tautological sheaf $\mathcal{O}_{\mathbb{P}}(1)$ and by the class f of a fibre of π . Moreover, the Chern equation on $\mathbb{P}$ becomes

$$\xi^3 = (g + 3)\xi^2 f.$$

Since $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}_k^1}(\tau + 4) \oplus \mathcal{O}_{\mathbb{P}_k^1}(g - 1 - \tau)$, it follows that the section $\delta \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \tilde{\mathcal{F}})$ defining the cover ϱ corresponds to a pair

$$(a, b) \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(-\tau - 4)) \oplus H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^1}(\tau - g + 1)).$$

In particular, $C \subseteq \mathbb{P}$ is the complete intersection of the two conic bundles $A := D_0(a) \in |2\xi - (\tau + 4)f|$ and $B := D_0(b) \in |2\xi - (g - 1 - \tau)f|$.

Due to Remark 2.3 (2) one deduces $\alpha_1 \geq 2$; hence the linear system $|\xi - 2f|$ on $\mathbb{P}$ is globally generated and it defines a morphism $\phi: \mathbb{P} \rightarrow \Sigma := \text{im}(\phi) \subseteq \tilde{\mathbb{P}}_k^{g-1}$ onto a rational normal scroll. The restriction of ϕ to C is exactly the canonical embedding (see Formula (2.1.1)). Moreover, ϕ maps the two conic bundles A and B onto two embedded surfaces A_{can} and B_{can} whose degrees are

$$\begin{aligned} \deg(A_{can}) &= (2\xi - (\tau + 4)f)(\xi - 2f)^2 = 2g - 6 - \tau, \\ \deg(B_{can}) &= (2\xi - (g - 1 - \tau)f)(\xi - 2f)^2 = g - 1 + \tau. \end{aligned}$$

Since $\tau \leq (g - 5)/2$, it follows that $\deg(B_{can}) \leq \deg(A_{can})$.

Notice that the planes in $\mathbb{P}$ cutting the g_4^1 on C are mapped onto planes in Σ . In particular, the divisors of the g_4^1 are cut on C_{can} by suitable planes. Each such a divisor is the base locus of a suitable pencil of conics in those planes, namely the pencil generated by the sections of A_{can} and B_{can} .

The above discussion and Theorem 3.3 of [Cs] prove the following result.

Theorem 2.4. *Let C be a non-hyperelliptic curve of genus $g \geq 4$, $\varrho: C \rightarrow \mathbb{P}_k^1$ a cover of degree 4 with tetragonality τ .*

Then the g_4^1 on C corresponding to ϱ is cut out on C_{can} by a family of planes in $\mathbb{P}_k^{g-1}$ whose union Σ is a scroll over $\mathbb{P}_k^1$. Moreover, C_{can} is the complete intersection inside Σ of two (in general non-uniquely determined) conic bundles A_{can} and B_{can} of respective degrees $2g - 6 - \tau$ and $g - 1 + \tau$. If $\tau \geq 1$, then Σ is smooth.

Finally the minimal degree of an integral surface $S \subseteq \Sigma$, different from B_{can} and containing the canonical model C_{can} of C is $2g - 6 - \tau$.

3. A TECHNICAL LEMMA

From now on, C will denote a non-hyperelliptic curve of genus g . The canonical map $\phi: C \rightarrow \check{\mathbb{P}}_k^{g-1}$ is an embedding. We will assume that the minimal degree of a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ containing C_{can} is $g + h$.

Since S is necessarily non-degenerate in $\check{\mathbb{P}}_k^{g-1}$, it follows that $h \geq -2$. As explained in the Introduction, the cases $-2 \leq h \leq 0$ have been examined in several papers.

Thus we will assume $h \geq 1$, from now on. In particular, C is not trigonal (see Theorem A); thus C_{can} is an ideal-theoretic intersection of quadrics. The cones projecting C_{can} from one of its points have trivially degree $2g - 3$; thus we deduce that $h \leq g - 3$. In [C–H] it has been proved that equality actually occurs for C general in its moduli space and $g \geq 23$.

Each quadric through C_{can} cuts out on S , residually to C_{can} , a curve Γ such that $\deg(\Gamma) = 2 \deg(S) - \deg(C_{can}) = 2h + 2$. If $C_{can} \subseteq \Gamma$ (e.g. if $C_{can} \subseteq \text{Sing}(S)$), then $2h + 2 = \deg(\Gamma) \geq \deg(C_{can}) = 2g - 2$, a contradiction, since $h \leq g - 3$. Thus $C_{can} \not\subseteq \Gamma$ (in particular, $C_{can} \not\subseteq \text{Sing}(S)$).

Let us consider a minimal desingularisation $p: \tilde{S} \rightarrow S$ (see [Za]). Let $\tilde{C}$ be the strict transform of C_{can} in $\tilde{S}$ and denote by H the total transform of a general hyperplane in $\check{\mathbb{P}}_k^{g-1}$. The linear system $|2H - \tilde{C}|$ on $\tilde{S}$ has a fixed part F and a movable part $|M|$. In particular, we have the linear equivalence $2H \sim \tilde{C} + M + F$. We have $H \cdot M = \deg(\Gamma) = 2h + 2$, whence

$$(3.1) \quad 4h + 4 = 2H \cdot M = \tilde{C} \cdot M + M^2 + F \cdot M,$$

where $\tilde{C} \cdot M$, M^2 , $F \cdot M$ are all non-negative.

Lemma 3.2. *With the notation above,*

$$(3.2.1) \quad h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(M)) \geq h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(M)) \geq h + 2.$$

Proof. Since $2g - 2 = \tilde{C} \cdot H > M \cdot H = 2h + 2$, it follows that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(M - \tilde{C})) = 0$. Thus the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(M - \tilde{C}) \longrightarrow \mathcal{O}_{\tilde{S}}(M) \longrightarrow \mathcal{O}_{\tilde{C}}(M) \longrightarrow 0$$

and [C–H, Lemma 2.5 (i)] yield inequalities (3.2.1). $\square$

The following result is now immediate.

Proposition 3.3. *Let C be a non-hyperelliptic curve of genus g and let $g + h$ ($h \geq 1$) be the minimal degree of a surface $S \subseteq \check{\mathbb{P}}_k^{g-1}$ containing C_{can} . Denote by $p: \tilde{S} \rightarrow S$ a minimal resolution of singularities of S , $\tilde{C} \subseteq \tilde{S}$ the proper transform of C_{can} and $|M|$ the movable part of $|p^* \mathcal{O}_S(2) \otimes \mathcal{O}_{\tilde{S}}(-\tilde{C})|$.*

Then $|\mathcal{O}_{\tilde{C}}(M)|$ contains a base-point-free g_d^{h+1} with $2h + 3 \leq d \leq 4h + 4$.

Proof. Due to the above lemma, the existence of a base-point-free g_d^{h+1} contained in $|\mathcal{O}_{\tilde{C}}(M)|$ with $d \leq 4h + 4$ is clear. If $d \leq 2h + 2$, then such a g_d^{h+1} would be special, since $h \leq g - 3$. This last fact and inequalities (3.2.1) contradict Clifford's Theorem, since C is non-hyperelliptic and $2h + 2 \leq 2g - 4$. It follows that $d \geq 2h + 3$. $\square$

From now on, let $h = 1$: we are thus assuming that the minimal degree of a surface $S \subseteq \tilde{\mathbb{P}}_k^{g-1}$ containing C_{can} is $g + 1$. Thanks to Proposition 3.3, C carries at least a possibly non-complete g_d^2 with $5 \leq d \leq \tilde{C} \cdot M \leq 8$.

Lemma 3.4. *Let C be a non-hyperelliptic curve of genus g and let $g + 1$ be the minimal degree of a surface $S \subseteq \tilde{\mathbb{P}}_k^{g-1}$ containing C_{can} . Then $g \geq 7$ and C does not carry any g_d^2 with $d \leq 6$.*

Proof. If $g \leq 4$, then C is trigonal; thus the minimal degree of a surface containing C_{can} would be $g - 1$, thanks to Theorem A. It follows that $g \geq 5$. If $g \leq 6$, then C carries a g_4^1 . Its tetragonality is at most $\tau = 0$, thanks to Remark 2.3 (1). Theorem 2.4 yields the existence of a surface of degree at most $g - 1$ containing C_{can} , contradicting the hypothesis of Theorem D. Thus, necessarily, $g \geq 7$.

Consider the morphism $f: C \rightarrow \mathbb{P}_k^2$ associated to the g_d^2 . Its image $D \subseteq \mathbb{P}_k^2$ is a non-degenerate integral curve and $d = \deg(f) \deg(D)$. If $d = 5$, then $(\deg(f), \deg(D)) = (1, 5)$; i.e., f is necessarily birational. Thus $g \leq p_a(D) = 6$, a contradiction, since C has genus $g \geq 7$. Thus, the case $d = 5$ does not occur.

If $d = 6$, we have three possible cases for the pair $(\deg(f), \deg(D))$, namely $(1, 6)$, $(2, 3)$, $(3, 2)$. In the last case D is a conic; thus C would be trigonal and hence C_{can} would be contained on a surface of degree $g - 2$ by Theorem A.

In the second case D is a plane cubic; thus it is either rational or elliptic, whence C turns out to be either hyperelliptic or bielliptic. The former case is a priori excluded and the latter is not possible since, otherwise, C_{can} would be contained on a surface of degree $g - 1$ by Theorem B.

Thus f is birational and D is a plane sextic. Again by Theorem B we deduce that also this case cannot occur. It follows that also the case $d = 6$ does not occur. $\square$

Let C be a non-hyperelliptic curve of genus g and let $g + 1$ be the minimal degree of a surface $S \subseteq \tilde{\mathbb{P}}_k^{g-1}$ containing C_{can} . Due to Proposition 3.3 and Lemma 3.4, we know that it carries at least a base-point-free g_d^2 with $7 \leq d \leq 8$. Such a g_d^2 is the movable part of $|p^* \mathcal{O}_S(2) \otimes \mathcal{O}_{\tilde{S}}(-\tilde{C})|$. It is trivial to check that each g_d^2 on C with $d \leq 8$ is special. Thus, Clifford's Theorem yields that such a g_d^2 can be contained in a g_d^3 at most.

4. CURVES WITH A BASE-POINT-FREE g_7^2

Let C be endowed with a base-point-free g_7^2 . We can assume that such a g_7^2 is actually complete. If not C would be endowed with a g_7^3 . Fixing a point $p_0 \in C$ we thus would obtain, residually to p_0 , a g_6^2 on C , which is not possible (see Lemma 3.4). Thus such a g_7^2 is actually complete and it defines a morphism $f: C \rightarrow \mathbb{P}_k^2$ which is birational onto its image $D \subseteq \mathbb{P}_k^2$, which is a non-degenerate integral curve of degree 7.

Lemma 4.1. *The curve D carries at most a single triple point.*

Proof. Since C cannot be trigonal (due to the hypothesis and Theorem A), it follows that D carries at most triple points. Let p_1 and p_2 be triple points on D . If p_2 is

not infinitely near to p_1 , then C would be endowed with two distinct g_4^1 . Due to Remark 2.3 (5), each of them would have tetragonality $\tau = 0$. Thus Theorem 2.4 would yield that C_{can} would lie on a surface of degree $g - 1$, again a contradiction.

It follows that p_2 is necessarily infinitely near to p_1 ; hence the curve D has a single tangent at p_1 . Let (x, y) be affine coordinates such that $p_1 = (0, 0)$ and the tangent line at p_1 is $\{y = 0\}$. Thus the equation of D has necessarily the form

$$(4.1.1) \quad f(x, y) := y^3 + f_4(x, y) + f_5(x, y) + f_6(x, y) + f_7(x, y),$$

where $f_h := \sum_{i=0}^h a_i^{(h)} x^{h-i} y^i \in k[x, y]$ is a form of degree h . The strict transform of D after the blowup $y = xu$ has equation

$$(4.1.2) \quad u^3 + x f_4(1, u) + x^2 f_5(1, u) + x^3 f_6(1, u) + x^4 f_7(1, u),$$

so that p_2 is the point $(0, 0)$ in the (x, u) coordinates. If such a point is triple, then $a_0^{(4)} = a_1^{(4)} = a_0^{(5)} = 0$, whence we infer that

$$\begin{aligned} f_4(x, y) &= a_2^{(4)} x^2 y^2 + a_3^{(4)} x y^3 + a_4^{(4)} y^4, \\ f_5(x, y) &= a_1^{(5)} x^4 y + a_2^{(5)} x^3 y^2 + a_3^{(5)} x^2 y^3 + a_4^{(5)} x y^4 + a_5^{(5)} y^5. \end{aligned}$$

The lines through p_1 cut out a g_4^1 on C ; thus they define a cover $\varrho: C \rightarrow \mathbb{P}_k^1$ with scrollar invariants $(\alpha_1, \alpha_2, \alpha_3)$ and tetragonality τ . We have

$$h^0(C, \varrho^* \mathcal{O}_{\mathbb{P}_k^1}(2)) = h^0(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}(2)) + \sum_{i=1}^3 h^0(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}(2 - \alpha_i)).$$

The integral conic $\widehat{D}$ with equation $y - x^2$ intersects D at p_1 with multiplicity 6; hence it defines a divisor in $|\varrho^* \mathcal{O}_{\mathbb{P}_k^1}(2)|$ which cannot be cut out by the union of two lines through p_1 . It follows that $h^0(C, \varrho^* \mathcal{O}_{\mathbb{P}_k^1}(2)) \geq 4$; hence $\alpha_1 \leq 2$. We conclude that $\tau \leq 2(\alpha_1 - 2) \leq 0$. Arguing as in the previous case of two distinct triple points we have a contradiction. $\square$

Let $p_1, \dots, p_r$ be the singular (possibly infinitely near) points of D . We assume that $p_1 \in \mathbb{P}_k^2$, and we can assume that p_2 is in the blowup $\widetilde{S}_1$ of $\mathbb{P}_k^2$ at p_1 , that p_3 is in the blowup $\widetilde{S}_2$ of $\widetilde{S}_1$ at p_2 , and so on. We denote by $q: \widetilde{S} := \widetilde{S}_r \rightarrow \mathbb{P}_k^2$ the composition of such a sequence of blowups. Let $e_1, \dots, e_r$ be the total transforms in $\widetilde{S}$ of the exceptional divisors of $\widetilde{S}_1, \dots, \widetilde{S}_r$; hence $e_i^2 = -1$. Let ℓ be the pull-back of a general line of $\mathbb{P}_k^2$. Finally let $D_0 := D$ and $D_i \subseteq \widetilde{S}_i$ be the strict transform of D_{i-1} for $i = 1, \dots, r$. We start by recalling the following definition (see [De], Séminaire III, Définition 1).

Definition 4.2. With the above notation, the points $p_1, \dots, p_r$ are said to be in almost general position if the following conditions hold.

- i) No four among them are on the same line.
- ii) No seven among them are on the same conic.
- iii) For each $j = 1, \dots, r - 1$ the point $p_{j+1} \in \widetilde{S}_j$ does not lie on any proper transform $\widehat{e}_i$ of e_i , $i = 1, \dots, j$, such that $\widehat{e}_i^2 = -2$.

Lemma 4.3. *The curve D carries at most double points as singularities.*

Proof. Assume that D carries a triple point p_1 , the other singularities $p_2, \dots, p_r$ being double points due to Lemma 4.1 above. Since $g \geq 7$, it follows that $r \leq 6$. We first show that the points $p_1, \dots, p_r$ are in almost general position.

The lines through p_1 induce a base-point-free g_4^1 on C . If $g \leq 8$, due to Remark 2.3 (1) the tetragonality τ of such a g_4^1 would be at most 1, a contradiction thanks to Theorem 2.4; thus $g \geq 9$.

Since $g \geq 9$, we have $r \leq 6$. We claim that the points $p_1, \dots, p_r$ are in almost general position. Indeed, in our setup, condition ii) is automatically satisfied. Since the points $p_1, \dots, p_r$ are singular points of a plane septic, it follows that condition i) is trivially satisfied too. It remains to examine condition iii). Let p_i be a double point and let p_{i+1} be an infinitely near (necessarily double) point. We have $D_{i+1} = q_i^* D_i - 2e$, where $q_i: \tilde{S}_{i+1} \rightarrow \tilde{S}_i$ is the blowup map at p_i and e its exceptional divisor. In particular, $e \cdot D_{i+1} = 2$; thus the intersection of D_{i+1} and e at the double point p_{i+1} is transversal. Blowing up again, the new double point infinitely near to p_{i+1} , if any, cannot lie on the strict transform of e . It follows that condition iii) is satisfied for the points p_i that are not infinitely near to p_1 .

Now we fix our attention on the point p_1 . If the triple point p_1 has at least two tangent lines, then one can trivially repeat the above argument. The only non-trivial case that we have to examine is the case of a plane septic with a triple point with one tangent line. If this is the case, then the equation of D is of the form (4.1.1), in a suitable affine neighbourhood of $p_1 = (0, 0)$. After the blowup $y = xu$, the strict transform of D has an equation of the form (4.1.2). Again the point p_2 is $(0, 0)$ in the coordinates (x, u) . Since it is assumed to be a double point, we have $a_0^{(4)} = 0$. If $a_1^{(4)} \neq 0$, it follows that p_2 is a node; thus condition iii) above is trivially satisfied. If $a_1^{(4)} = 0$, then necessarily $a_0^{(5)} \neq 0$, since p_2 is a double point. It follows that p_2 is an ordinary cusp, whence again condition iii) is satisfied.

The strict transform $\tilde{D}$ of D on $\tilde{S}$ is in the linear system $|7\ell - 3e_1 - 2e_2 - \dots - 2e_r|$ and, by adjunction on $\tilde{S}$, the canonical system on $\tilde{D}$ is cut out by the linear system $|4\ell - 2e_1 - e_2 - \dots - e_r|$. Such a linear system contains the divisors splitting as the sum of a divisor in $|3\ell - e_1 - e_2 - \dots - e_r|$ plus a divisor in $|\ell - e_1|$. Both such linear systems are free from fixed components (for the first one, see [De], Séminaire III, Théorème 1); thus $|4\ell - 2e_1 - e_2 - \dots - e_r|$ is free from fixed components as well.

Let $A \in |4\ell - 2e_1 - e_2 - \dots - e_r|$. Since $\mathcal{O}_{\tilde{S}}(A - \tilde{D}) \cong \omega_{\tilde{S}}$ and $h^i(\tilde{S}, \omega_{\tilde{S}}) = h^{2-i}(\tilde{S}, \mathcal{O}_{\tilde{S}}) = h^{2-i}(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2})$ (see [Ha], Proposition V.3.4), it follows that the cohomology of

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(A - \tilde{D}) \longrightarrow \mathcal{O}_{\tilde{S}}(A) \longrightarrow \mathcal{O}_{\tilde{D}}(A) \longrightarrow 0$$

yields

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(4\ell - 2e_1 - e_2 - \dots - e_r)) = h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}(A)) = h^0(C, \omega_C) = g;$$

thus $|4\ell - 2e_1 - e_2 - \dots - e_r|$ induces a rational map $s: \tilde{S} \dashrightarrow \mathbb{P}_k^{g-1}$. Let $\hat{S}$ be the closure of the image of s . Clearly $C_{can} \subseteq \hat{S}$. Since the divisors in $|4\ell - 2e_1 - e_2 - \dots - e_r|$ cut out on D the canonical divisor, it follows that such divisors are mapped on curves via s , which are necessarily different from C_{can} . We conclude that $\hat{S}$ must be a surface of degree $\deg(\hat{S}) \leq (4\ell - 2e_1 - e_2 - \dots - e_r)^2 = g$, contradicting the hypothesis. It follows that D cannot carry triple points. $\square$

Up to now we have proved that the plane model D of C is a septic curve endowed with $r := 15 - g$ possibly infinitely near double points $p_1, \dots, p_r$. As above the canonical system on $\tilde{D}$ is cut out by the linear system $|4\ell - e_1 - \dots - e_r|$ and again

we have a natural rational map $s: \tilde{S} \dashrightarrow \hat{S} \subseteq \mathbb{P}_k^{g-1}$ whose image $\hat{S}$ contains C_{can} . We claim that s is birational.

To this purpose we examine the position of the points $p_1, \dots, p_r$. Obviously condition i) of Definition 4.2 is satisfied since $\deg(D) = 7$. Moreover, the same argument used before Lemma 4.3 shows that condition iii) holds as well. Assume that condition ii) is satisfied (which certainly occurs if $r \leq 6$). As in the proof of Lemma 4.3 we deduce that $|4\ell - e_1 - \dots - e_r|$ is free from fixed components, that $\hat{S}$ is a surface and that inequality

$$\deg(s) \deg(\hat{S}) \leq (4\ell - e_1 - \dots - e_r)^2 = g + 1$$

holds. Due to the hypothesis we deduce that s is birational and that the above inequality is actually an equality; thus $|4\ell - e_1 - \dots - e_r|$ is base-point-free.

Now assume that condition ii) does not hold (which implies $r = 7, 8$). Since $\deg(D) = 7$, at most seven among the points $p_1, \dots, p_r$ can actually lie on a conic. Assume that $|2\ell - e_1 - \dots - e_7|$ contains an element Γ . The linear system $|4\ell - e_1 - \dots - e_r|$ contains all the divisors splitting as the sum of Γ plus a divisor in either $|2\ell|$ (if $r = 7$) or $|2\ell - e_8|$ (if $r = 8$). Such a second linear system has no fixed components; thus the only possible fixed component of $|4\ell - e_1 - \dots - e_r|$ could be Γ . If this is the case the map s coincides with the map induced by either $|2\ell|$ (if $r = 7$) or $|2\ell - e_8|$ (if $r = 8$), which is birational. It follows that $\hat{S}$ is a surface and $\deg(\hat{S}) \leq 4 < g + 1$, a contradiction, whence we deduce that $|4\ell - e_1 - \dots - e_r|$ has no fixed components. Arguing as in the previous cases it turns out that $\hat{S}$ is a surface and $\deg(\hat{S}) = g + 1$. We can summarise the results of this section in the following.

Proposition 4.4. *Let C be a non-hyperelliptic curve of genus g and let $g + 1$ be the minimal degree of a surface $S \subseteq \mathbb{P}_k^{g-1}$ containing C_{can} .*

If C carries a base-point-free g_7^2 , then $7 \leq g \leq 15$; such a g_7^2 is complete and induces a birational morphism of C onto a plane septic D with exactly $15 - g$ (possibly infinitely near) double points. The linear system of quartics through the double points of D induces a birational map onto a surface $\hat{S} \subseteq \mathbb{P}_k^{g-1}$ of degree $g + 1$ and $C_{can} \subseteq \hat{S}$.

The above discussion and the construction of the classical Bordiga surface of degree 6 in $\mathbb{P}_k^4$ yield naturally to calling such a surface $\hat{S}$ a *weak Bordiga surface* (associated to $\{p_1, \dots, p_r\}$).

5. CURVES WITH A BASE-POINT-FREE g_8^2

With the notation of section 3, let $|M|$ be the movable part of $|p^* \mathcal{O}_S(2) \otimes \mathcal{O}_{\tilde{S}}(-\tilde{C})|$. We know that $|\mathcal{O}_{\tilde{C}}(M)|$ contains a base-point-free g_d^2 with $7 \leq d \leq 8$. The case when C is endowed with a g_7^2 has been analysed in the previous section. Thus we will assume in this section that $d = 8$.

We can assume that such a g_8^2 is actually complete since, otherwise, C would be endowed with a g_8^3 . Fixing a point $p_0 \in C$ we thus would obtain, residually to p_0 , a g_7^2 on C ; hence we would be in the previous case $d = 7$. Thus, from now on, we will assume that C does not carry any g_7^2 .

Recall that inequalities (3.2.1) give $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(M)) = 3$. Since $\tilde{C} \cdot M = 8$, it follows that $M^2 = F \cdot M = 0$. Thus $|M|$ is base-point-free; hence its general element $\overline{M}$ is

smooth due to Bertini's Theorem. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{S}} \longrightarrow \mathcal{O}_{\tilde{S}}(M) \longrightarrow \mathcal{O}_{\overline{M}}(M) \longrightarrow 0$$

gives $h^0(M, \mathcal{O}_{\overline{M}}(M)) \geq 2$. Since the restriction of $|M|$ to $\overline{M}$ is obviously effective and it has degree 0, it follows that $\overline{M}$ has at least two connected components. Since $\overline{M} \cdot H = 4$, it also follows that $\overline{M}$ has at most four connected components.

In what follows we will examine the possible configurations of the components of $\overline{M}$.

Lemma 5.1. *With the above notation, let M' be a component of $\overline{M}$. Then $M' \cdot \tilde{C} = 2M' \cdot H$.*

Proof. Due to the arbitrariness of the component M' , it suffices to prove that it is not possible to have $M' \cdot \tilde{C} > 2M' \cdot H$. Due to the hypothesis and Theorem A, we can exclude that the curve C is either trigonal or isomorphic to a plane quintic; thus C_{can} is an ideal-theoretic intersection of quadrics. Let $\Gamma' := p(M') \subseteq S$ and let Q be any quadric containing C_{can} . Then $M' \cdot H = \deg(\Gamma')$ and $\tilde{C} \cdot M' = \deg(C_{can} \cap \Gamma')$, whence

$$2M' \cdot H = 2 \deg(\Gamma') = \deg(Q \cap \Gamma') \geq \deg(C_{can} \cap \Gamma') = \tilde{C} \cdot M' > 2M' \cdot H,$$

a contradiction. $\square$

Lemma 5.2. *Let $\overline{M}_1$ be an integral component of the aforementioned general divisor $\overline{M}$. Then $\overline{M}_1 \cdot H \leq 2$.*

Proof. Since $\overline{M}$ is not connected, it follows that $\overline{M} = \overline{M}_1 + \overline{M}_2$, where $\overline{M}_2 \neq 0$ is effective and $\overline{M}_1 \cdot \overline{M}_2 = 0$. We have that $|M|$ has no fixed components; thus $\overline{M}_i^2 = \overline{M} \cdot \overline{M}_i \geq 0$. Since $\overline{M}^2 = \overline{M}_1^2 + \overline{M}_2^2 = 0$, the vanishing $\overline{M}_1^2 = \overline{M}_2^2 = 0$ follows.

Thus the subset of $|M|$ consisting of divisors containing $\overline{M}_1$ coincides with the sublinear system of $|M|$ of divisors through a fixed general point $P \in \overline{M}_1$. We conclude that

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\overline{M}_2)) = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(M)) - 1 \geq 2.$$

Since $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\overline{M}_2 - \tilde{C})) = 0$, the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(\overline{M}_2 - \tilde{C}) \longrightarrow \mathcal{O}_{\tilde{S}}(\overline{M}_2) \longrightarrow \mathcal{O}_{\tilde{C}}(\overline{M}_2) \longrightarrow 0$$

yields

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\overline{M}_2)) \geq h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\overline{M}_2)) \geq 2.$$

Due to Lemma 5.1 we have $\overline{M}_2 \cdot \tilde{C} = 2$. Thus C would be endowed with a pencil of divisor of degree 2, a contradiction since C is neither rational nor hyperelliptic. It follows that $\overline{M}_1 \cdot H \leq 2$. $\square$

Lemma 5.3. *With the above notation, C carries a g_4^1 which is cut out on C_{can} by a family of curves of degree 2 contained in S .*

Proof. As above, we write $\overline{M} = \overline{M}_1 + \overline{M}_2$, where $\overline{M}_1$ is integral and $\overline{M}_1 \cdot \overline{M}_2 = 0$. We know from Lemma 5.2 that $\overline{M}_1 \cdot H \leq 2$; i.e., the irreducible components of $\overline{\Gamma} := p(\overline{M}) \subseteq S$ are either lines or integral conics.

Assume that $\overline{M}_1$ is mapped onto an integral conic. As in the proof of Lemma 5.2, $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\overline{M}_2)) \geq 2$ and $\overline{M}_2 \cdot \tilde{C} = 4$; thus we obtain a g_4^1 which is cut out on C_{can} by the images of the elements of $|\mathcal{O}_{\tilde{C}}(\overline{M}_2)|$ which are curves on S of degree 2. The general curve of this linear system is either a conic or pair of skew lines.

Now assume that all the components of $\overline{M}$ are mapped via p on lines in S . In this case $\overline{M}$ has four components, say $\overline{M}_i$, $i = 1, \dots, 4$. Since $\overline{M}^2 = 0$ and $|M|$ has no fixed points or base components, again it follows that $\overline{M}_i^2 = 0$, $i = 1, \dots, 4$.

Let p_0 be a general point on $\tilde{S}$. There are many divisors in $|M|$ containing p_0 but all such divisors must have a unique common component through p_0 whose image via p is necessarily a line on S . Thus we can define naturally a rational map from $\tilde{S}$, hence from the ruled surface S , onto the general hyperplane section E of S . Notice that E has degree $g + 1$ and it can be assumed to be non-degenerate in a projective space of dimension $g - 2$. Due to Castelnuovo's bound (see [A-C-G-H], p.116) we have that $p_a(E) \leq 3$. Due to Lemma 5.1 such a rational map induces a morphism $\alpha: C \rightarrow E$ of degree 2. Since C is neither rational nor hyperelliptic, it follows that $p_a(C) \geq 2$.

If $p_a(E) = 3$, then E is smooth and the singular locus of S must have dimension 0. Moreover, $\alpha: C \rightarrow E$ is a double cover; thus $\omega_C = \alpha^* \mathcal{L}$ for a suitable $\mathcal{L} \in \text{Pic}(E)$ of degree $g - 1$. Due to the definition of α , there exists a canonical divisor K without double points which is cut out on C_{can} by $g - 1$ pairwise distinct lines $L_1, \dots, L_{g-1}$. Since the canonical system is cut out on C_{can} by the hyperplanes, there exists a hyperplane $H \subseteq \mathbb{P}_k^{g-1}$ such that $H \cdot C_{can} = K$ in $\text{Pic}(C)$. Notice that such an H intersects each line L_i at two points; thus it must contain it, whence $H \cap S = L_1 \cup \dots \cup L_{g-1} \cup D$, where D is a curve on S of degree 2. The irreducible components of D are smooth and rational.

Let D_0 be one of them. Since H cuts out on C_{can} a canonical divisor, necessarily $C_{can} \cap D_0 = \emptyset$. Since the singular locus of S has dimension 0, the rational map $\tilde{S} \dashrightarrow E$ defined above induces a well-defined morphism $D_0 \rightarrow E$. Such a map cannot be constant, since $C_{can} \cap D_0 = \emptyset$. This is absurd, since E is not unirational.

We conclude that $p_a(E) = 2$. It follows that there exists a map $\beta: E \rightarrow \mathbb{P}_k^1$ of degree 2; thus $\varrho := \beta \circ \alpha$ would induce a g_4^1 on $\tilde{C}$. Such a g_4^1 is cut out on C_{can} by a pair of lines contained in S . $\square$

Let $\varrho: C \rightarrow \mathbb{P}_k^1$ be the cover of degree 4 associated to the aforementioned g_4^1 . Recall that ϱ factors through an embedding $i: C \rightarrow \mathbb{P} = \mathbb{P}(\mathcal{E})$ followed by the natural projection $\pi: \mathbb{P} \rightarrow \mathbb{P}_k^1$. Moreover, the canonical system on C is cut out by the linear system $|\xi - 2f|$. Thus C_{can} is contained in the threefold Σ ruled in planes which is the image of $\mathbb{P}$ via $|\xi - 2f|$. The planes in Σ cut out on C_{can} the g_4^1 .

If any one of the curves in S cutting the g_4^1 were a pair of skew lines, then it could not cut out on C_{can} a divisor lying in a plane. It follows that such curves must be (possibly reducible) conics; hence they are also contained in the fibres of Σ . We thus conclude that $S \subseteq \Sigma$.

Let τ be the tetragonality of ϱ . Since $S \subseteq \Sigma$ and $\deg(S) = g + 1$, it follows from Theorem 2.4 again that either $g + 1 = g - 1 + \tau$, whence $\tau = 2$, or $g + 1 \geq 2g - 6 - \tau$, whence $\tau \geq g - 7$.

In the second case, since $\tau \leq (g - 5)/2$ (see Remark 2.3 (1)), we obtain $g \leq 9$, whence $\tau \leq 2$. The case $\tau < 2$ cannot occur since, otherwise, C_{can} would be contained in a surface of degree $g - 1 - \tau < g + 1$. We conclude that $\tau = 2$ in both cases. In particular, the invariant β is $(6, g - 3)$ with $g - 3 \geq 6$, i.e., $g \geq 9$. If $g \geq 16$, then $\tau < (g - 9)/3$; hence the g_4^1 is composed with an involution onto a curve of genus 2 (see Proposition 2.5). Finally, since $\tau = 2 > 0$, Remark 2.3 (5) guarantees the uniqueness of the g_4^1 on C .

We can summarise the results of this section in the following.

Proposition 5.4. *Let C be a non-hyperelliptic curve of genus g and let $g + 1$ be the minimal degree of a surface $S \subseteq \mathbb{P}_k^{g-1}$ containing C_{can} .*

If C does not carry any g_7^2 , then $g \geq 9$ and C is endowed with a unique g_4^1 with tetragonality $\tau = 2$. If $g \geq 16$ such a g_4^1 is composed with an involution on a curve of genus 2. The surface S is a conic bundle onto $\mathbb{P}_k^1$ whose fibres cut out on the canonical model of C a g_4^1 .

6. THEOREM D: ITS PROOF AND SOME CONSEQUENCES

We start this section by proving the Main Theorem stated in the introduction.

Proof of Theorem D. Let C be a curve of genus g whose canonical model lies on a surface of degree $g + 1$ but not on surfaces of lesser degree. In Propositions 3.3, 4.4 and 5.4 above we showed that either C is tetragonal with $g \geq 9$ (and tetragonality $\tau = 2$) or it is birationally isomorphic to a plane septic curve with at most double points as singularities and $7 \leq g \leq 15$. It remains to prove that a curve cannot fall simultaneously in the two aforementioned classes.

To this purpose it suffices to prove that the gonality of a plane septic curve of genus $g \geq 9$ and with at most double points as singularities cannot be 4. This is well-known when the double points are nodes or ordinary cusps (see [C–K1] and [C–K2]). Since infinitely near points can occur in our setup we need to make use of some more refined results proved in [O–S]. Indeed, Proposition 2 of [O–S] can be easily applied with $d = 7$, $\nu = 2$, $\delta = 15 - g \leq 6$ in order to show that the gonality of the plane septic curve we are interested in is exactly 5 and not 4. $\square$

We dedicate the rest of this section to a corollary and some remarks.

Corollary 6.1. *Let C be a non-hyperelliptic curve of genus g . If C_{can} is contained in a surface in $\mathbb{P}_k^{g-1}$ of degree $g + 1$, then C is either trigonal or tetragonal or pentagonal or isomorphic to a plane septic curve. If C is pentagonal, then $7 \leq g \leq 14$.*

Proof. Let $S \subseteq \check{\mathbb{P}}_k^{g-1}$ be a surface of minimal degree containing C_{can} .

We know that $\deg(S) \leq g + 1$. If $d = g - 2$, then C is trigonal (Theorem A). If $\deg(S) = g - 1$, g , then C is either tetragonal or isomorphic to a plane sextic, whence it is pentagonal and $g = 10$ (Theorems B and C). If $\deg(S) = g + 1$, then C is either tetragonal (Theorem D ii)) or it is isomorphic to a plane septic (Theorem D i) with $g = 15$) or it is birationally isomorphic to a plane septic with at least a double point; then it is pentagonal (Theorem D i) with $7 \leq g \leq 14$).

If C is isomorphic to a plane septic, then it cannot carry any g_d^1 with $d \leq 5$ (see [A–C–G–H], Appendix A, Ex. 18). $\square$

Let C be a curve of genus g whose canonical model lies on a surface S of degree $g + 1$. Let us now examine the possible values for the scollar invariant α .

Remark 6.2. When the g_4^1 is composed with an involution onto a curve of genus 2 (e.g. if $g \geq 16$); then $\alpha = (3, \alpha_2, g - \alpha_2)$ (due to Remark 2.3 (4)). Moreover, $3 + \alpha_2 \geq g - \alpha_2$ (see [Ch], Proposition 3.3); thus we finally obtain that $\alpha_2 \geq (g - 3)/2$.

In particular, we have the following four possible cases for the scrollar invariant α :
(6.2.1)

$$\left(3, \frac{g-3}{2}, \frac{g+3}{2}\right), \quad \left(3, \frac{g-2}{2}, \frac{g+2}{2}\right), \quad \left(3, \frac{g-1}{2}, \frac{g+1}{2}\right), \quad \left(3, \frac{g}{2}, \frac{g}{2}\right).$$

Obviously only two of them are possible for a fixed value of g , according with its parity. Moreover, such cases actually occur (see [B–C–F] for the details).

Remark 6.3. The picture when $9 \leq g \leq 15$ is richer since many other cases are possible for the scrollar invariant α . Recall that in any case we have $\beta = (6, g - 3)$. Taking into account Remark 2.3 (1) and that $\alpha_1 + \alpha_2 \geq \alpha_3$ (see [Ch], Proposition 3.3), then α is one of the following:

$$\begin{aligned} g = 9 & \quad (4, 4, 4), (3, 4, 5), (3, 3, 6); \\ g = 10 & \quad (4, 4, 5), (3, 5, 5), (3, 4, 6); \\ g = 11 & \quad (4, 5, 5), (4, 4, 6), (3, 5, 6), (3, 4, 7); \\ g = 12 & \quad (5, 5, 5), (4, 5, 6), (4, 4, 7), (3, 6, 6), (3, 5, 7); \\ g = 13 & \quad (5, 5, 6), (4, 6, 6), (4, 5, 7), (4, 4, 8), (3, 6, 7), (3, 5, 8); \\ g = 14 & \quad (5, 6, 6), (5, 5, 7), (4, 6, 7), (4, 5, 8), (3, 7, 7), (3, 6, 8); \\ g = 15 & \quad (6, 6, 6), (5, 6, 7), (5, 5, 8), (4, 7, 7), (4, 6, 8), (4, 5, 9), (3, 7, 8), (3, 6, 9). \end{aligned}$$

On the one hand it is not difficult to show (see e.g. [Br], Proposition 3.1) that the cases

$$(4, 4, 7), \quad (4, 4, 8), \quad (5, 5, 7), \quad (4, 5, 8), \quad (5, 5, 8), \quad (4, 7, 7), \quad (4, 5, 9)$$

never occur, since C is smooth. On the other hand, Theorem 2.2 guarantees that the cases

$$(4, 4, 4), \quad (4, 4, 5), \quad (4, 5, 5), \quad (4, 4, 6), \quad (5, 5, 5), \quad (5, 5, 6), \quad (6, 6, 6)$$

actually occur. Finally, taking double covers of hyperelliptic curves of genus 2 branched along suitable divisors of degree $2g - 6$ it is possible to show that also all the cases with $\alpha_1 = 3$ occur (see [B–C–F] for the details).

It could be, perhaps, interesting to deal with the remaining cases, namely

$$(4, 5, 6), \quad (4, 6, 6), \quad (4, 5, 7), \quad (5, 6, 6), \quad (4, 6, 7), \quad (5, 6, 7), \quad (4, 6, 8).$$

Remark 6.4. Since the Brill–Noether number $\varrho(7, 2, 7)$ is 1, it follows that each curve C of genus $g = 7$ is endowed with a morphism $f: C \rightarrow D \subseteq \mathbb{P}_k^2$ onto a non-degenerate integral curve D , where $7 = \deg(f) \deg(D)$. It follows that f is actually a birational morphism; thus D is necessarily a singular septic curve. By adjunction the canonical system on C is cut out on D by the quartic adjoint curves in $\mathbb{P}_k^2$.

An argument similar to the one used in Section 4 then shows that the canonical model C_{can} of each such curve is always contained in a surface of degree at most 8. Moreover, we are now able to define a stratification

$$\bigcup_{\ell=5}^8 \mathfrak{D}_7(\ell)$$

of the non-hyperelliptic part of the moduli space $\mathfrak{M}_7$: $\mathfrak{D}_7(\ell)$ is the locus of points corresponding to curves C such that the minimal degree of a surface containing their canonical models C_{can} is ℓ .

More precisely we have that the points in $\mathfrak{D}_7(5)$ correspond to trigonal curves (see Theorem A: this is usually denoted by $\mathfrak{M}_{7,3}^1$ in the literature). The points in $\mathfrak{D}_7(6)$ correspond to curves which are either bielliptic or birationally isomorphic to a plane

sextic curve with three (possibly infinitely near) double points (see Theorem B). The points in $\mathfrak{D}_7(7)$ correspond to tetragonal curves with scollar invariants $\alpha = (3, 3, 4)$ and $\beta = (5, 5)$. Finally we have the largest stratum $\mathfrak{D}_7(8)$ whose points correspond to curves which are isomorphic to a plane septic with eight (possibly infinitely near) double points.

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