ON THE NORM CLOSURE PROBLEM FOR COMPLEX SYMMETRIC OPERATORS

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ABSTRACT. We prove that the set of all complex symmetric operators on a separable, infinite-dimensional Hilbert space is not norm closed.

In [2, Sect. 3], it is asked whether the set of all complex symmetric operators on a separable, infinite-dimensional Hilbert space is norm closed. We answer this question in the negative. Let $S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots)$ denote the unilateral shift on $\ell^2(\mathbb{N})$ and let $\cong$ denote unitary equivalence. Note that

$$T_n = \frac{n}{n+1}S \oplus \left(\bigoplus_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{j}{j+1}S \right) \oplus \left(\bigoplus_{j=1}^{\infty} \frac{j}{j+1}S^* \right) \cong \bigoplus_{j=1}^{\infty} \frac{j}{j+1}(S \oplus S^*)$$

is complex symmetric by [1, Ex. 5]. On the other hand, T_n converges in norm to

$$T = S \oplus \left(\bigoplus_{j=1}^{\infty} \frac{j}{j+1}S \right) \oplus \left(\bigoplus_{j=1}^{\infty} \frac{j}{j+1}S^* \right) \cong S \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1}(S \oplus S^*).$$

Since $\|S^k(1, 0, 0, \dots)\| = 1$, there is an x so that $\|T^k x\| = 1$ for $k \geq 0$. However,

$$T^* = S^* \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1}(S^* \oplus S) = S^* \oplus (\text{a strict contraction})$$

possesses no such vector since $(S^*)^k$ tends strongly to zero. This precludes the existence of a conjugation C (i.e., an isometric, conjugate-linear involution) such that $T = CT^*C$. Thus T is not complex symmetric. $\square$

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