SIMULTANEOUS LINEARIZATION FOR COMMUTING QUASIPERIODICALLY FORCED CIRCLE DIFFEOMORPHISMS

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(Communicated by Yingfei Yi)

ABSTRACT. For commuting smooth quasiperiodically forced circle diffeomorphisms, we show that if the base frequencies and the fibred rotation numbers jointly satisfy some simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞ neighborhood of the corresponding rotations, then they are simultaneously C^∞ -linearizable.

1. INTRODUCTION

In this paper, we study the homeomorphisms of the torus which are isotopic to the identity and of the form

$$(1.1) \quad \begin{aligned} (\alpha, f) : \mathbb{T}^d \times \mathbb{T} &\rightarrow \mathbb{T}^d \times \mathbb{T} \\ (\varphi, \theta) &\mapsto (\varphi + \alpha, f(\varphi, \theta)), \end{aligned}$$

where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, the fibre maps $f_\varphi := f(\varphi, \cdot)$ are all orientation-preserving circle homeomorphisms and $\alpha = (\alpha_1, \dots, \alpha_d)$ where $(1, \alpha_1, \alpha_2, \dots, \alpha_d)$ are rational and independent. Such homeomorphisms are often called *quasiperiodically forced* (or *qpf* for short) *circle homeomorphisms*. If $f \in C^r(\mathbb{T}^{d+1}, \mathbb{R})$ ($1 \leq r \leq \infty, \omega$), we call (α, f) a C^r -*qpf* circle diffeomorphism.

Skew-product transformations such as this have been widely studied as a source of example of interesting dynamics and for modeling fundamental phenomena in mathematical physics. One well-known example is the Harper map, which appears in the study of quasiperiodic crystals, and the corresponding Schrödinger operators [7], also in the rich theory of quasiperiodic $SL(2, \mathbb{R})$ cocycles and their associated projective actions [1, 2]. Another one is the *qpf* Arnold circle maps, which is used as a simple model for oscillators forced with two or more incommensurate frequencies [4].

Considering the system (1.1) with $d = 1$, Jäger and Stark [12] gave a Poincaré-like classification: (i) if (α, f) is ρ -bounded, then either there exists a (p, q) -invariant strip and $\rho_f(\alpha, f), \alpha$ and 1 are rationally dependent or (α, f) is *semi-conjugate* to the irrational torus translation $(\varphi, \theta) \mapsto (\varphi + \alpha, \theta + \rho_f(\alpha, f))$; (ii) if (α, f) is ρ -unbounded, then neither of these alternatives can occur and the map is always topologically transitive. Further, Jäger completed this classification [11], proving

Received by the editors November 30, 2010 and, in revised form, July 10, 2011.

2010 *Mathematics Subject Classification.* Primary 37C15; Secondary 37C05.

This work was supported by NNSF of China (Grant 10531050), NNSF of China (Grant 11031003), and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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that (α, f) is *semi-conjugate* to the irrational rotation if and only if it is a ρ -bounded motion.

The next natural question is the regularity problem: when can a qpf circle diffeomorphism be linearized? We say that a C^r - qpf circle diffeomorphism (α, f) is C^r -linearizable if there exists a homeomorphism $H \in C^r(\mathbb{T}^{d+1}, \mathbb{R}^{d+1})$ and a rotation of $\mathbb{T}^{d+1}$ denoted by (α, R_ρ) , where $(\alpha, R_\rho)(\varphi, \theta) = (\varphi + \alpha, \theta + \rho)$, such that $H^{-1} \circ (\alpha, f) \circ H = (\alpha, R_\rho)$. A pair (α, f) and (β, g) is *simultaneously linearizable* if the same conjugating homeomorphism linearizes both (α, f) and (β, g) . For a single system, linearization results with Diophantine conditions are easy to verify [8]. In this paper, we are concerned with the simultaneous linearization for commuting qpf circle diffeomorphisms.

As in the one-dimensional commuting circle diffeomorphisms case, Moser [14] proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞ neighborhood of the corresponding rotations, then they are simultaneously C^∞ linearizable. In terms of small divisors, this result presented a new and striking phenomenon: suppose m is the number of commuting diffeomorphisms and that the rotation numbers of some or of all the diffeomorphisms may not be Diophantine, but still, the m diffeomorphisms are smoothly linearizable due to the absence of simultaneous resonances. Precisely, Moser showed that the problem cannot be reduced to that of a single diffeomorphism with a Diophantine frequency.

Moser's result is a perturbative result; that is, the neighborhood of the rotation depends on the constant appearing in the simultaneous Diophantine condition, as usual in classical KAM theorems. Recently, Fayad and Khanin [6] proved that the related global theory of linearization for commuting circle diffeomorphisms is also available. Their paper used the renormalization method that mostly relied on the idea used by Herman [8] and Yoccoz [15] proving the global linearization theory of circle diffeomorphisms.

Recently, Damjanovic, Fayad, and Krikorian [3] also proved the local simultaneous reducibility for commuting $SL(2, \mathbb{R})$ cocycles above circular rotations. In fact, they proved more: using ideas developed in proving reducibility results with Liouvillean frequency [5], they proved the global simultaneous reducibility results. If we consider the projective flows of the cocycles, this system is a special case of qpf circle diffeomorphisms. Unfortunately, most methods developed from quasi-periodic $SL(2, \mathbb{R})$ cocycles cannot simply be carried over to our systems, since they strongly depend on the linear structure.

We should prove the result following the steps of Moser [14]. Our result is perturbative. For $d \geq 2$, the result is optimal. But for $d = 1$, the global linearization for commuting qpf circle diffeomorphisms remains a challenging open problem.

2. PRELIMINARIES AND MAIN RESULT

2.1. The fibred rotation number. Suppose (α, f) is a qpf circle homeomorphism. Given any lift of (α, f) ,

$$\begin{aligned} (\alpha, F) : \mathbb{T}^d \times \mathbb{R} &\rightarrow \mathbb{T}^d \times \mathbb{R} \\ (\varphi, \theta) &\mapsto (\varphi + \alpha, F_\varphi(\theta)) \end{aligned}$$

and $n \in \mathbb{N}$, let

$$F_\varphi^n := F_{\varphi+(n-1)\alpha} \circ \cdots \circ F_\varphi.$$

Then the limit

$$(2.1) \quad \rho(F) = \lim_{n \rightarrow \infty} \frac{F_\varphi^n(\theta) - \theta}{n}$$

exists and does not depend on $(\varphi, \theta) \in \mathbb{T}^d \times \mathbb{R}$. Furthermore, the convergence in (2.1) is uniform [9]. Then we call $\rho(f) = \rho(F) \bmod 1$ the fibred rotation number of (α, f) , and denote it by $\rho_f(\alpha, f)$.

2.2. Diophantine conditions. For $x \in \mathbb{R}$, we use the notation

$$\|x\| = \inf_{k \in \mathbb{Z}} |x - k|.$$

We say that $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ is a Diophantine vector if for every $k = (k_1, \dots, k_d) \in \mathbb{N}^d \setminus \{0\}$, we have

$$\|\langle k, \alpha \rangle\| \geq \frac{\gamma}{|k|^\tau},$$

where $|k| = |k_1| + \dots + |k_d|$, $\gamma > 0$, $\tau > 0$, and we denote this set by $DC(\gamma, \tau)$.

Let $\alpha \in \mathbb{R}^d$, $\rho \in \mathbb{T}$. We say that ρ is Diophantine with respect to α if there exist $K > 0$, $\tau_0 > 0$, such that

$$(2.2) \quad \|\langle k, \alpha \rangle + l\rho\| \geq \frac{K}{(|k| + |l|)^{\tau_0}}, \quad \text{for } l \neq 0, (k, l) \in \mathbb{Z}^d,$$

denoted by $DC_\alpha(K, \tau_0)$.

Definition 2.1. A pair of rational independent vectors $(\alpha, \beta) \in \mathbb{R}^{2d}$ is said to be a simultaneous Diophantine vector pair if there exist $\gamma > 0$, $\tau > 0$ such that for every $k \in \mathbb{N}^d \setminus \{0\}$, the following holds:

$$(2.3) \quad \max\{\|\langle k, \alpha \rangle\|, \|\langle k, \beta \rangle\|\} \geq \frac{\gamma}{|k|^\tau},$$

where $|k| = |k_1| + \dots + |k_d|$. We denote it by $(\alpha, \beta) \in SDC(\gamma, \tau)$.

2.3. Main result. Now we can give our main theorem:

Theorem 2.2. *Let (α_k, f_k) ($k = 1, 2$) be two commuting quasiperiodically forced C^∞ circle diffeomorphisms with fibred rotation numbers ρ_k respectively. We denote $\xi_k = (\alpha_k, \rho_k) \in \mathbb{R}^d \times \mathbb{T}$. If $\xi = (\xi_1, \xi_2) \in SDC(\gamma, \tau)$ for some $\gamma > 0, \tau > 0$. Then there exists $\varepsilon = \varepsilon(\gamma, \tau) > 0$, such that if $\|(\alpha_k, f_k) - (\alpha_k, R_{\rho_k})\|_{C^\infty} < \varepsilon$ ($k = 1, 2$), we can find $H \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R}^{d+1})$ that conjugates $(\alpha_1, f_1), (\alpha_2, f_2)$ to rotations simultaneously.*

Remark 2.3. To alleviate the notation, we will restrict to the case of two *qpf* circle diffeomorphisms, but the result of Theorem 2.2 holds for an arbitrary finite number of commuting *qpf* circle diffeomorphisms.

3. A BASIC LEMMA

For a single quasiperiodically forced C^∞ circle diffeomorphism, the following holds:

Theorem 3.1 ([8]). *Let (α, f) be a quasiperiodically forced C^∞ circle diffeomorphism with fibred rotation number ρ_f . Suppose that $\alpha \in DC(\gamma, \tau)$ and $\rho_f \in DC_\alpha(K, \tau_0)$ for some $\gamma > 0, \tau > 0, K > 0, \tau_0 > 0$. Then there exists $\varepsilon = \varepsilon(\gamma, \tau, K, \tau_0) > 0$, such that if $\|(\alpha, f) - (\alpha, R_{\rho_f})\|_{C^\infty} < \varepsilon$, we can find $H \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R}^{d+1})$ that conjugates (α, f) to rotation.*

Remark 3.2. The simultaneous Diophantine condition on two fibred rotation numbers with respect to the base frequencies is weaker than the Diophantine condition on one rotation number or some linear combination of the rotation numbers of individual maps. That is to say, the result of Theorem 2.2 cannot be obtained from existing local results as in Theorem 3.1.

Outline of the proof of Theorem 2.2. We use a kind of KAM iteration to prove the linearization of commuting qpf circle diffeomorphisms.

For the sake of simplicity, we denote $\tilde{f}_k := \Pi_2((\alpha_k, f_k) - (\alpha_k, R_{\rho_k}))$, where

$$\Pi_1 : \mathbb{T}^d \times C^\infty(\mathbb{T}^{d+1}, \mathbb{T}) \rightarrow \mathbb{T}^d, \quad \Pi_2 : \mathbb{T}^d \times C^\infty(\mathbb{T}^{d+1}, \mathbb{T}) \rightarrow C^\infty(\mathbb{T}^{d+1}, \mathbb{T})$$

is the coordinate projection and $\rho_k = \rho_f(\alpha_k, f_k)$, $k = 1, 2$.

The idea of the proof is to produce a diffeomorphism $H \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R}^{d+1})$ such that after conjugating both qpf circle diffeomorphisms (α_k, f_k) , $k = 1, 2$ by H , the new maps

$$G_k := H^{-1} \circ (\alpha_k, f_k) \circ H = (\alpha_k, R_{\rho_k} + \tilde{g}_k)$$

commute and are much closer to rotations than $F_k = (\alpha_k, f_k)$.

To get H , it is standard to consider the linearization of $H \mapsto H^{-1} \circ (\alpha_k, f_k) \circ H$ at $H = id$. Then we can easily get the cohomological equation $h \circ (\alpha_k, R_{\rho_k}) - h = \tilde{f}_k$, where $h = \Pi_2(H - id)$. If there exists a tame inverse to this cohomological equation, then $H := id + (0, h)$ makes the new maps G_k much closer to rotations. So after n steps, we get $F_k^{(n)} := \mathcal{H}^{-(n)} \circ (\alpha_k, f_k) \circ \mathcal{H}^{(n)}$, which converges to rotation of the torus. Here the H_n are closer and closer to id with a super-exponential speed, so we can prove that $\mathcal{H}^{(n)} = H_1 \circ H_2 \circ \cdots \circ H_n$ converges to some $\mathcal{H}$ as $n \rightarrow \infty$.

Precisely, at each step of the iterations, the goal is to solve the two equations simultaneously, i.e. with the same h :

$$(3.1) \quad \begin{aligned} h(\varphi + \alpha_1, \theta + \rho_1) - h(\varphi, \theta) &= \tilde{f}_1(\varphi, \theta), \\ h(\varphi + \alpha_2, \theta + \rho_2) - h(\varphi, \theta) &= \tilde{f}_2(\varphi, \theta). \end{aligned}$$

If such an h exists, then the following holds:

$$(3.2) \quad \tilde{f}_1(\varphi + \alpha_2, \theta + \rho_2) - \tilde{f}_1(\varphi, \theta) = \tilde{f}_2(\varphi + \alpha_1, \theta + \rho_1) - \tilde{f}_2(\varphi, \theta).$$

There is one problem that will not appear in the linearization of a single system. In order to have simultaneous linearization, the condition (3.2) is a necessary one, but it may not be satisfied in most situations. However, due to the commutativity of qpf circle homeomorphisms F_1, F_2 , i.e.

$$\tilde{f}_1 \circ F_2 - \tilde{f}_1 = \tilde{f}_2 \circ F_1 - \tilde{f}_2,$$

it is easy to see that their linearizations almost commute, i.e.

$$(3.3) \quad \tilde{f}_1 \circ (\alpha_2, R_{\rho_2}) - \tilde{f}_1 \sim \tilde{f}_2 \circ (\alpha_1, R_{\rho_1}) - \tilde{f}_2.$$

The difference of the right side and the left side of (3.3) is of the quadratic order with respect to $\tilde{f}_1, \tilde{f}_2$. What we want to do is to approximate $\tilde{f}_1, \tilde{f}_2$ by maps which satisfy (3.2) with error quadratic w.r.t. $\tilde{f}_1, \tilde{f}_2$. Meanwhile, we have to show that for such maps the equations (3.1) have simultaneous solutions. By using the approach of Moser [14], it is shown that such an approximation exists if $(\alpha_1, \rho_f(\alpha_1, f_1))$ and $(\alpha_2, \rho_f(\alpha_2, f_2))$ satisfy the Diophantine condition simultaneously.

We introduce the semi-norm

$$|h|_r = \sup_{x \in \mathbb{T}^{d+1}} \max_{\eta \in \mathbb{N}^{d+1}, |\eta|=r} \left| \frac{\partial^r h}{\partial x^\eta} \right|,$$

the C^r -norm

$$\|h\|_r = \max_{x \in \mathbb{T}^{d+1}} |h(x)| + |h|_r,$$

and the C^∞ -norm

$$\|h\|_\infty = \sup_{r \in \mathbb{N}} \|h\|_r.$$

Let $C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}) := \{[f] = 0 : f \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R})\}$, where $[f]$ means the mean value of f on $\mathbb{T}^{d+1}$.

Let

$$L_k h := h(\varphi + \alpha_k, \theta + \rho_k) - h(\varphi, \theta), \quad k = 1, 2.$$

It is obvious that

$$\|L_k h\|_r \leq 2\|h\|_r.$$

Since $F_1 \circ F_2 = F_2 \circ F_1$, we get

$$(3.4) \quad \tilde{f}_1 \circ F_2 - \tilde{f}_1 = \tilde{f}_2 \circ F_1 - \tilde{f}_2,$$

i.e.,

$$\begin{aligned} (3.5) \quad L_2 \tilde{f}_1 - L_1 \tilde{f}_2 &= \tilde{f}_1(\varphi + \alpha_2, \theta + \rho_2) - \tilde{f}_1(\varphi, \theta) - \tilde{f}_2(\varphi + \alpha_1, \theta + \rho_1) + \tilde{f}_2(\varphi, \theta) \\ &= \tilde{f}_1(\varphi + \alpha_2, \theta + \rho_2) - \tilde{f}_1(\varphi + \alpha_2, \theta + \rho_2 + \tilde{f}_2(\varphi, \theta)) \\ &\quad + \tilde{f}_2(\varphi + \alpha_1, \theta + \rho_1 + \tilde{f}_1(\varphi, \theta)) - \tilde{f}_2(\varphi + \alpha_1, \theta + \rho_1) \\ &= - \int_0^1 \frac{\partial \tilde{f}_1}{\partial \theta}(\varphi + \alpha_2, \theta + \rho_2 + t \tilde{f}_2(\varphi, \theta)) \tilde{f}_2(\varphi, \theta) dt \\ &\quad + \int_0^1 \frac{\partial \tilde{f}_2}{\partial \theta}(\varphi + \alpha_1, \theta + \rho_1 + t \tilde{f}_1(\varphi, \theta)) \tilde{f}_1(\varphi, \theta) dt. \end{aligned}$$

Define $A : C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}) \rightarrow C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}) \times C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R})$ by

$$A(f) := (L_1 f, L_2 f),$$

where

$$(3.6) \quad |A(f)|_r := \max\{|L_1 f|_r, |L_2 f|_r\} \leq 2|f|_r.$$

Let

$$B(f_1, f_2) := L_1 f_2 - L_2 f_1.$$

By (3.5), we have the following estimate for B :

$$(3.7) \quad |B(f_1, f_2)|_0 \leq |f_1|_0 |f_2|_1 + |f_2|_0 |f_1|_1.$$

For $k \in \{1, 2\}$, let L_k^* be the formal adjoint of L_k defined by

$$L_k^* h := h(\varphi - \alpha_k, \theta - \rho_k) - h(\varphi, \theta), \quad k = 1, 2.$$

We can easily get

$$(3.8) \quad |L_k^* h|_r \leq 2|h|_r.$$

Let $A^* : C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}) \times C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}) \rightarrow C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R})$ be the operator defined as

$$A^*(f_1, f_2) := \sum_{k=1}^2 L_k^* f_k.$$

Then we have

$$(3.9) \quad |A^* f|_r \leq 4|f|_r.$$

Let

$$B^* : h \mapsto (-L_2^* h, L_1^* h).$$

Then

$$(3.10) \quad |B^* h|_r \leq 2|h|_r.$$

Since $L_1^* L_2 = L_2 L_1^*$ and $L_2^* L_1 = L_1 L_2^*$, it follows that

$$(3.11) \quad (AA^* + B^* B)(f_1, f_2) = (A^* A f_1, A^* A f_2).$$

We denote $M = A^* A$, and we have the following lemma:

Lemma 3.3. *Let $\xi_k = (\alpha_k, \rho_f(\alpha_k, f_k)) \in \mathbb{R}^d \times \mathbb{T}$, $k = 1, 2$. The operator $M = A^* A : C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}) \rightarrow C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R})$ is a bijection if and only if $\xi = (\xi_1, \xi_2) \in SDC(\gamma, \tau)$ for some $\gamma > 0, \tau > 0$. Moreover, the following inequality holds:*

$$|M^{-1} g|_r \leq C_1(\gamma, \tau) |g|_{r+\sigma},$$

where $C_1(\gamma, \tau) = \frac{4}{(2\pi)^\sigma \gamma^2}$, and $\sigma = 2\tau + d + 2$.

Proof. $\forall g \in C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R})$, we want to find a map $f \in C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R})$, such that $Mf = g$. Using the Fourier expansion, we get

$$f(\varphi, \theta) = \sum_{|k|+|l| \neq 0} f_l^k e^{2\pi i(\langle k, \varphi \rangle + l\theta)},$$

where

$$f_l^k = \int_{\mathbb{T}} \int_{\mathbb{T}^d} f(\varphi, \theta) e^{-2\pi i(\langle k, \varphi \rangle + l\theta)} d\varphi d\theta.$$

By the definition of A and A^* , we have

$$\begin{aligned} Mf &= A^* A f = L_1^* L_1 f + L_2^* L_2 f \\ &= 4f(\varphi, \theta) - f(\varphi + \alpha_1, \theta + \rho_1) - f(\varphi - \alpha_1, \theta - \rho_1) \\ &\quad - f(\varphi + \alpha_2, \theta + \rho_2) - f(\varphi - \alpha_2, \theta - \rho_2) \\ &= \sum_{|k|+|l| \neq 0} \mu_l^k f_l^k e^{2\pi i(\langle k, \varphi \rangle + l\theta)}, \end{aligned}$$

where

$$\begin{aligned} \mu_l^k &= 4 - e^{2\pi i(\langle k, \alpha_1 \rangle + l\rho_1)} - e^{-2\pi i(\langle k, \alpha_1 \rangle + l\rho_1)} - e^{2\pi i(\langle k, \alpha_2 \rangle + l\rho_2)} - e^{-2\pi i(\langle k, \alpha_2 \rangle + l\rho_2)} \\ &= 4[\sin^2 \pi(\langle k, \alpha_1 \rangle + l\rho_1) + \sin^2 \pi(\langle k, \alpha_2 \rangle + l\rho_2)]. \end{aligned}$$

By comparing the Fourier coefficients of each term, we obtain

$$f_l^k = \frac{g_l^k}{\mu_l^k}.$$

The operator M^{-1} is bounded in the C^∞ -topology if and only if $\frac{1}{\mu_l^k}$ grows at most like a power of $|k| + |l|$, which is expressed exactly by Definition 2.1. On the other hand, if $\xi = (\xi_1, \xi_2) \in SDC(\gamma, \tau)$, then we have

$$\frac{1}{\mu_l^k} \leq \frac{1}{\|\langle k, \alpha_1 \rangle + l\rho_1\|^2 + \|\langle k, \alpha_2 \rangle + l\rho_2\|^2} \leq \frac{(|k| + |l|)^{2\tau}}{\gamma^2} \quad \text{for } |k| + |l| \neq 0.$$

Since $|g_l^k| \leq \frac{|g|_r}{(2\pi(|k| + |l|))^r}$, we obtain

$$\begin{aligned} |M^{-1}g|_r &\leq \sum_{|k| + |l| \neq 0} \frac{1}{\mu_l^k} (2\pi(|k| + |l|))^r |g_l^k| \\ &\leq \sum_{|k| + |l| \neq 0} \frac{1}{\mu_l^k} (2\pi(|k| + |l|))^r \frac{|g|_{r+\sigma}}{(2\pi(|k| + |l|))^{r+\sigma}} \\ &\leq \frac{1}{(2\pi)^\sigma \gamma^2} \sum_{|k| + |l| \neq 0} (|k| + |l|)^{2\tau + d - \sigma} |g|_{r+\sigma} \\ &\leq \frac{4}{(2\pi)^\sigma \gamma^2} |g|_{r+\sigma} \end{aligned}$$

if $\sigma = 2\tau + d + 2$.

The proof is completed. $\square$

In the following, we will determine an approximating solution of $Av = f$ for $f \in C_0^\infty(\mathbb{T}^{d+1}, \mathbb{R}^2)$ as

$$v = A^* M^{-1} f.$$

For $f = (f_1, f_2)$, we define $M^{-1}f := (M^{-1}f_1, M^{-1}f_2)$. Then (3.11) implies that

$$\begin{aligned} Av - f &= AA^* M^{-1} f - f \\ &= MM^{-1} f - B^* BM^{-1} f - f \\ &= -B^* BM^{-1} f, \end{aligned}$$

since all operators commute. If $Bf = 0$, then $Av = f$; that is to say, v is the desired solution.

Define $f'' = B^* BM^{-1} f$. Then

$$Av + f'' = f.$$

Moreover,

$$|f''|_r = |Av - f|_r = |B^* BM^{-1} f|_r \leq 2|BM^{-1} f|_r.$$

4. ITERATIVE STEP

For a map $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$, we define $T_N f$ to be the part of the Fourier series of f up to the order $N \in \mathbb{N}$, and denote $S_N f := f - T_N f$. Then the following estimates hold:

Proposition 4.1 ([13]). *If $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$, then for any $N, r, r' \in \mathbb{N}$, we have*

$$(4.1) \quad |T_N f|_{r'} \leq (2\pi)^{r' - r} |f|_r N^{r' - r + d + 1} \quad \text{for } r' > r;$$

$$(4.2) \quad |S_N f|_r \leq (2\pi)^{r - r'} |f|_{r'} N^{r - r' + d + 1} \quad \text{for } r' > r + d + 1.$$

$\square$

Proposition 4.2 (Generalized Faà di Bruno's Formula [10]). *Given $g \in C^r(\mathbb{R}^m)$, $h = (h_1, \dots, h_m) \in C^r(\mathbb{R}^n, \mathbb{R}^m)$, where $h_i = \Pi_i h$ with $r \geq 1$, it follows that*

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(g \circ h) = \sum_{|\sigma|=1}^r \alpha! \frac{\partial^{|\sigma|} g \circ h}{\partial x^\sigma} \sum_{E_\sigma} \sum_{j=1}^m \prod_{A_\alpha} \frac{1}{c_{j\alpha_j}!} \left(\frac{1}{\alpha_j!} \frac{\partial^{|\alpha_j|} h_j}{\partial x^{\alpha_j}} \right),$$

where E_σ and A_α are given by

$$E_\sigma = \{(c_{1\alpha_1}, \dots, c_{m\alpha_m}) : c_{j\alpha_j} \in \mathbb{N}, 1 \leq |\alpha_j| \leq r, \sum_{|\alpha_j|=1}^r c_{j\alpha_j} = \sigma, 1 \leq j \leq m\},$$

$$r = |\alpha|, \quad A_\alpha = \{(\alpha_1, \dots, \alpha_m) : 1 \leq |\alpha_j| \leq r, 1 \leq j \leq m, \sum_{j=1}^m \sum_{|\alpha_j|=1}^r c_{j\alpha_j} \cdot \alpha_j = \alpha\}.$$

□

Lemma 4.1 (Iterative lemma). *Let $F_\nu := (\alpha_\nu, f_\nu)$, $(\nu = 1, 2)$ be two commuting quasiperiodically forced C^∞ circle diffeomorphisms with fibred rotation numbers ρ_ν respectively. Let $\xi = (\xi_1, \xi_2) \in \text{SDC}(\gamma, \tau)$ for some $\gamma > 0, \tau > 0$, where $\xi_\gamma = (\alpha_\gamma, \rho_f(\alpha_\gamma, f_\gamma)) \in \mathbb{R}^d \times \mathbb{T}$. Then there exists $\varepsilon = \varepsilon(\gamma, \tau) > 0$, such that if $\|(\alpha_\nu, f_\nu) - (\alpha_\nu, R_{\rho_\nu})\|_{C^\infty} < \varepsilon$ ($\nu = 1, 2$), then we can find $H \in C^\infty(\mathbb{T}^{d+1}, \mathbb{R}^{d+1})$ that conjugates F_1, F_2 to*

$$G_1(\varphi, \theta) = (\varphi + \alpha_1, \theta + \rho_1 + \tilde{g}_1(\varphi, \theta)),$$

$$G_2(\varphi, \theta) = (\varphi + \alpha_2, \theta + \rho_2 + \tilde{g}_2(\varphi, \theta))$$

simultaneously. Moreover,

$$|\tilde{g}|_r \leq c_r N^{\sigma+d+2} (1 + |\tilde{f}|_r),$$

$$|\Pi_2(H - id)|_r \leq C_3 N^{\sigma+d+2} |\tilde{f}|_r,$$

$$|\tilde{g}|_0 \leq C_4 \{N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1 + |\tilde{f}|_s N^{-s+d+2}\},$$

where $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$, c_r, C_3, C_4 are all constants that only depend on γ, τ, r and $\sigma = 2\tau + d + 2$, $s > d + 2$.

Proof. Define $h = A^*v$, $v = M^{-1}((T_N \tilde{f}_1, T_N \tilde{f}_2) - ([\tilde{f}_1], [\tilde{f}_2]))$, where $\tilde{f}_\nu = \Pi_2(F_\nu - (\alpha_\nu, R_{\rho_\nu}))$.

Let

$$H(\varphi, \theta) = (\varphi, \theta + h(\varphi, \theta)),$$

$$G_\nu(\varphi, \theta) := H^{-1} \circ F_\nu \circ H(\varphi, \theta)$$

$$= (\varphi + \alpha_\nu, \theta + \rho_\nu + \tilde{g}_\nu(\varphi, \theta)).$$

By $H \circ G_\nu = F_\nu \circ H$, we obtain

$$\tilde{g}_\nu + h \circ G_\nu = h + \tilde{f}_\nu \circ H,$$

that is,

$$\begin{aligned} \tilde{g}_\nu(\varphi, \theta) &= h + \tilde{f}_\nu \circ H - h \circ G_\nu \\ &= \tilde{f}_\nu \circ H + h - h \circ (\alpha_\nu, R_{\rho_\nu}) + h \circ (\alpha_\nu, R_{\rho_\nu}) - h \circ G_\nu \\ &= -L_\nu h + h \circ (\alpha_\nu, R_{\rho_\nu}) - h \circ G_\nu + \tilde{f}_\nu \circ H. \end{aligned}$$

So we obtain the following relations:

$$(4.3) \quad \begin{cases} \tilde{g} = -Ah + T_N \tilde{f} + \Psi, \\ \Psi = \tilde{f} \circ H - \tilde{f} + h \circ (\alpha_\nu, R_{\rho_\nu}) - h \circ G_\nu + S_N \tilde{f}. \end{cases}$$

By inequality (3.9), Lemma 3.3 and Proposition 4.1, we have

$$(4.4) \quad \begin{aligned} |h|_r &= |A^* v|_r \leq 4|v|_r \\ &\leq 4C_1 |T_N \tilde{f}|_{r+\sigma} \\ &\leq 4(2\pi)^\sigma C_1 N^{\sigma+d+2} |\tilde{f}|_r \\ &\leq C_3 N^{\sigma+d+2} |\tilde{f}|_r, \end{aligned}$$

for $\sigma = 2\tau + d + 2$. If $|\tilde{f}|_1$ is small enough, we can assume that $|h|_1 < \frac{1}{4}$. From the mean value theorem,

$$\begin{aligned} |h \circ (\alpha_\nu, R_{\rho_\nu}) - h \circ G_\nu|_0 &\leq |h|_1 |\tilde{g}_\nu|_0 \leq \frac{1}{4} |\tilde{g}_\nu|_0, \\ |\tilde{f} \circ H - \tilde{f}|_0 &\leq |\tilde{f}|_1 |h|_0 \leq C_3 N^{\sigma+d+2} |\tilde{f}|_1 |\tilde{f}|_0. \end{aligned}$$

Therefore, we have

$$(4.5) \quad |\Psi|_0 \leq C_3 N^{\sigma+d+2} |\tilde{f}|_1 |\tilde{f}|_0 + \frac{1}{4} |\tilde{g}_\nu|_0 + |S_N \tilde{f}|_0.$$

Also,

$$-Ah + T_N \tilde{f} = -AA^* v + T_N \tilde{f} = B^* Bv - Mv + T_N \tilde{f},$$

and since $Mv = MM^{-1}(T_N \tilde{f} - [\tilde{f}]) = T_N \tilde{f} - [\tilde{f}]$, then

$$(4.6) \quad \tilde{g} = B^* Bv + [\tilde{f}] + \Psi.$$

By inequality (3.7), Lemma 3.3 and Proposition 4.1, we estimate the C^0 -norm of $B^* Bv$:

$$\begin{aligned} |B^* Bv|_0 &\leq 4|Bv|_0 \leq 4|M^{-1}BT_N \tilde{f}|_0 \\ &\leq 4C_1 |T_N B\tilde{f}|_\sigma \\ &\leq C_3 N^{\sigma+d+2} |B\tilde{f}|_0 \\ &\leq 2C_3 N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1. \end{aligned}$$

As for the estimation of $[\tilde{f}_1]$, since $|h|_1 < \frac{1}{4}$ and the rotation number is invariant under the transformation close to the identity, there exists $(\varphi_0, \theta_0) \in \mathbb{T}^{d+1}$ at which $\tilde{g}_1$ vanishes. We substitute the value (φ_0, θ_0) into the formula (4.6) for $\tilde{g}$ in the first variable and we obtain

$$0 = (B^* Bv)_1 + [\tilde{f}_1] + \tilde{f}_1 \circ H - \tilde{f}_1 + S_N \tilde{f}_1.$$

Consequently,

$$|[\tilde{f}_1]| \leq |B^* Bv|_0 + |\tilde{f} \circ H - \tilde{f}|_0 + |S_N \tilde{f}|_0 \leq 3C_3 N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1 + |S_N \tilde{f}|_0.$$

As for $[\tilde{f}_2]$, we can get the same estimation as $[\tilde{f}_1]$.

Finally,

$$|\tilde{g}|_0 \leq 6C_3 N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1 + 2|S_N \tilde{f}|_0 + \frac{1}{4} |\tilde{g}|_0,$$

which implies that

$$(4.7) \quad \begin{aligned} |\tilde{g}|_0 &\leq C_4 \{N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1 + |S_N \tilde{f}|_0\} \\ &\leq C_4 \{N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1 + |\tilde{f}|_s N^{-s+d+2}\}, \end{aligned}$$

where $s > d + 2$.

By Proposition 4.2 and the interpolation estimate

$$(4.8) \quad |\tilde{f}|_r \leq c_l |\tilde{f}|_0^{1-\frac{r}{l}} |\tilde{f}|_l^{\frac{r}{l}}$$

for $0 \leq r \leq l$, we can conclude that

$$\|G\|_l \leq c_l (1 + \|h\|_l + \|F\|_l)$$

if $|\tilde{f}|_1 < 1$ and $|h|_1 < \frac{1}{4}$. Therefore, by estimate (4.4),

$$(4.9) \quad |\tilde{g}|_r \leq c_r N^{\sigma+d+2} (1 + |\tilde{f}|_r). \quad \square$$

5. PROOF OF THEOREM 2.2

Proof. We begin with the given mappings $F_k^{(0)} = F_k$ ($k = 1, 2$), $H_0 = id$ and construct a sequence of mappings $H_n(\varphi, \theta) = (\varphi, \theta + h_n(\varphi, \theta))$, $F^{(n)} = H_{n-1}^{-1} \circ F^{(n-1)} \circ H_{n-1} = H^{-(n-1)} \circ F \circ H^{(n-1)}$, for $n = 1, 2, \dots$, inductively, where

$$H^{(n)} = H_1 \circ \dots \circ H_n.$$

Setting

$$\kappa = \frac{3}{2}, \quad \varepsilon_n = \varepsilon_{n-1}^\kappa, \quad N_n = \varepsilon_n^{-\frac{1}{2(\sigma+d+3)}}, \quad r = 8(\sigma + d + 3), \quad \sigma = 2\tau + d + 2,$$

we claim that for sufficiently small $\varepsilon_0 = \varepsilon$, the estimates

$$(5.1) \quad |\tilde{f}^{(n)}|_0 \leq \varepsilon_n, \quad |\tilde{f}^{(n)}|_r \leq \varepsilon_n^{-1}, \quad |h_n|_1 \leq \varepsilon_n^{1/2}$$

hold.

We prove (5.1) by induction. Suppose it holds for $F^{(n)}$, H_n .

In the n -th step, for simpler notation, we set $F^{(n)} = F$, $F^{(n+1)} = G$, $H_n = H$, and $N_n = N$. Then by Lemma 4.1, we have

$$(5.2) \quad |h|_r \leq C_3 N^{\sigma+d+2} |\tilde{f}|_r,$$

$$(5.3) \quad |\tilde{g}|_r \leq c_r N^{\sigma+d+2} (1 + |\tilde{f}|_r),$$

$$(5.4) \quad |\tilde{g}|_0 \leq C_4 \{N^{\sigma+d+2} |\tilde{f}|_0 |\tilde{f}|_1 + |\tilde{f}|_s N^{-s+d+2}\}.$$

By the interpolation estimate (4.8), we have

$$|\tilde{f}|_1 \leq c_r |\tilde{f}|_0^{1-\frac{1}{r}} |\tilde{f}|_r^{\frac{1}{r}},$$

which implies the first inequality of (5.1):

$$\begin{aligned} |\tilde{g}|_0 &\leq C_4 c_1 N^{\sigma+d+2} \varepsilon^{2-\frac{2}{r}} + C_4 c_r N^{-(s-d-2)} \varepsilon^{1-\frac{2s}{r}} \\ &\leq C_4 c_1 \varepsilon^{\frac{3}{2} + \frac{1}{4(\sigma+d+3)}} + C_4 c_r \varepsilon^{\frac{3}{2} + \frac{1}{4(\sigma+d+3)}} \\ &\leq \varepsilon^{\frac{3}{2}} = \varepsilon_{n+1}, \end{aligned}$$

taking $s = 2\sigma + 4d + 11$.

The third inequality of (5.1) can be easily obtained from (5.2) by taking $r = 1$:

$$|h|_1 \leq C_3 N^{\sigma+d+2} |\tilde{f}|_1 \leq C_3 c_r \varepsilon^{\frac{1}{2} + \frac{1}{4(\sigma+d+3)}} \leq \varepsilon^{\frac{1}{2}} = \varepsilon_n^{\frac{1}{2}}.$$

We can also get the second inequality easily by (5.3):

$$|\tilde{g}|_r \leq 2c_r N^{\sigma+d+2} |\tilde{f}|_r \leq 2c_r N^{\sigma+d+2} \varepsilon^{-1} \leq 2c_r \varepsilon^{-\frac{3}{2} + \frac{1}{2(\sigma+d+3)}} \leq \varepsilon^{-3/2} = \varepsilon_{n+1}^{-1}.$$

Thus for sufficiently small $|\tilde{f}|_0, |\tilde{f}|_r$ we can assure the convergence of the process and consequently the existence of a solution $h \in C^1(\mathbb{T}^{d+1}, \mathbb{R})$ with $|h|_1 < \frac{1}{4}$. Finally we show that this solution is actually in C^∞ .

For an arbitrary $s \in \mathbb{N}$, we have by (4.9),

$$|\tilde{f}^{(n+1)}|_s \leq c_s N_n^{\sigma+d+2} (1 + |\tilde{f}^{(n)}|_s) \leq \varepsilon_n^{-1/2} (1 + |\tilde{f}^{(n)}|_s),$$

which yields

$$|\tilde{f}^{(n)}|_s \leq C \prod_{m=1}^{n-1} \varepsilon_m^{-1/2} (1 + |\tilde{f}|_s) \leq \varepsilon_n^{-1} C (1 + |\tilde{f}|_s).$$

Now for $\forall k \in \mathbb{N}$, and $s = 4k$, by interpolating estimates, we have

$$|\tilde{f}^{(n)}|_k \leq c_s |\tilde{f}^{(n)}|_0^{1-\frac{1}{4}} |\tilde{f}^{(n)}|_s^{\frac{1}{4}} \leq C' \varepsilon_n^{1/2}$$

and

$$|h_n|_k \leq C_3 N_n^{\sigma+d+2} |\tilde{f}^{(n)}|_k \leq c \varepsilon_n^\delta$$

with

$$\delta = \frac{1}{2} - \frac{\sigma + d + 2}{2(\sigma + d + 3)} = \frac{1}{2(\sigma + d + 3)} > 0.$$

The estimate guarantees the convergence of $H^{(n)}$ in C^k for every $k \in \mathbb{N}^*$, and this implies that the limit $H_\infty = \lim_{n \rightarrow \infty} H^{(n)}$ belongs to C^∞ .

Thus the main theorem, Theorem 2.2, is proved. $\square$

ACKNOWLEDGEMENTS

The authors would like to thank Jiangong You and Yiqian Wang for stimulating discussions. They also would like to thank the anonymous referee for useful suggestions.

REFERENCES

1. A. Avila and S. Jitomirskaya, *The Ten Martini Problem*, Ann. of Math. (2) **170** (2009) 303-342. MR2521117 (2011a:47081)
2. A. Avila and R. Krikorian, *Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles*, Ann. of Math. (2) **164** (2006) 911-940. MR2259248 (2008h:81044)
3. D. Damjanovic, B. Fayad and R. Krikorian, *Rigidity results for commuting $SL(2, \mathbb{R})$ cocycles above circular rotations*, Preprint.
4. M. Ding, C. Grebogi, and E. Ott, *Evolution of attractors in quasiperiodically forced systems: From quasiperiodic to strange nonchaotic to chaotic*, Physical Review A **39**(5) (1989) 2593-2598.
5. B. Fayad and R. Krikorian, *Rigidity results for quasiperiodic $SL(2, \mathbb{R})$ -cocycles*, Journal of Modern Dynamics **3**(4) (2009) 497-510. MR2587083 (2011f:37046)
6. B. Fayad and K. Khanin, *Smooth linearization of commuting circle diffeomorphisms*, Ann. of Math. (2) **170** (2009) 961-980. MR2552115 (2011c:37088)
7. A. Haro and J. Puig, *Strange nonchaotic attractors in Harper maps*, Chaos **16** (2006), no. 3. MR2266838 (2007g:37025)
8. M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Etudes Sci. Publ. Math. **49** (1979) 5-233. MR538680 (81h:58039)
9. M. R. Herman, *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2*, Comment. Math. Helv. **58** (1983) 453-502. MR727713 (85g:58057)

10. L. Hernández Encinas and J. Muñoz Masqué, *A short proof of the generalized Faà di Bruno's formula*, Applied Mathematics Letters **16** (2003) 975-979. MR2005276 (2004k:26018)
11. T. Jäger, *Linearisation of conservative toral homeomorphisms*, Invent. Math. **176**(3) (2009) 601-616. MR2501297 (2011e:37095)
12. T. Jäger and J. Stark, *Towards a classification for quasiperiodically forced circle homeomorphisms*, Journal of the LMS **73**(3) (2006) 727-744. MR2241977 (2008f:37087)
13. R. Krikorian, *Réductibilité des Systèmes Produits-Croisés à Valeurs dans des Groupes Compacts*, Astérisque **259** (1999). MR1732061 (2001f:37030)
14. J. Moser, *On commuting circle mappings and simultaneous Diophantine approximations*, Math. Z. **205** (1990) 105-121. MR1069487 (92e:58120)
15. J. C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Ann. Sci. Ecole Norm. Sup. **17** (1984) 333-359. MR777374 (86j:58086)
16. J. C. Yoccoz, *Analytic linearization of circle diffeomorphisms*, in Dynamical Systems and Small Divisors (Cetraro, 1998), Lecture Notes in Math. **1784**, Springer-Verlag, New York, (2002) 125-173. MR1924912 (2004c:37073)

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