ON THE FIRST MOMENT OF THE SYMMETRIC-SQUARE L -FUNCTION

QINGFENG SUN

(Communicated by Ken Ono)

ABSTRACT. We improve the error term in the asymptotic formula for the twisted first moment of the symmetric-square L -function on the critical line.

1. INTRODUCTION

Let f be a cusp form of weight k for $SL_2(\mathbb{Z})$ with Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \quad \text{Im}(z) > 0.$$

The symmetric-square L -function associated to f is defined as

$$L(\text{sym}^2 f, s) = \zeta(2s) \sum_{n \geq 1} a_f(n^2) n^{-s}, \quad \text{Re}(s) > 1.$$

Shimura [12] proved that $L(\text{sym}^2 f, s)$ can be extended to an entire function on the complex plane and satisfies the functional equation

$$L_\infty(s) L(\text{sym}^2 f, s) = L_\infty(1-s) L(\text{sym}^2 f, 1-s),$$

where

$$(1.1) \quad L_\infty(s) = \pi^{-\frac{3}{2}s} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right).$$

Many other important properties of $L(\text{sym}^2 f, s)$ have been obtained by calculating moments of $L(\text{sym}^2 f, s)$ on the critical line (see Blomer [2], Iwaniec and Michel [6], Li [11], Kohnen and Sengupta [8] and Lau [9]). Let H_k denote the basis of weight k cusp forms for $SL_2(\mathbb{Z})$ consisting of eigenfunctions of all Hecke operators with first Fourier coefficients equal to 1. Khan [7] proved an asymptotic formula for the twisted first moment of $L(\text{sym}^2 f, s)$ over H_k on the critical line, as $k \rightarrow \infty$. As an application, he showed that given any point on the critical line and large enough k , there exists an eigenform f such that $L(\text{sym}^2 f, s)$ is non-vanishing at that point.

The aim of this paper is to improve the error term in Khan's asymptotic formula.

Received by the editors November 7, 2010.

2000 *Mathematics Subject Classification*. Primary 11F67.

Key words and phrases. Symmetric-square L -function, first moment.

The author was supported by the National Natural Science Foundation of China (Grant No. 11101239).

Theorem 1.1. *Given a real number $t \neq 0$ and an integer $m \geq 1$, we have*

$$\begin{aligned} \sum_{f \in H_k} L\left(\mathrm{sym}^2 f, \frac{1}{2}\right) \omega_f^{-1} &= \log k + C + O\left(k^{-\frac{1}{2}}\right), \\ \sum_{f \in H_k} L\left(\mathrm{sym}^2 f, \frac{1}{2} + it\right) a_f(m^2) \omega_f^{-1} &= \frac{\zeta(1+2it)}{m^{\frac{1}{2}+it}} + \frac{\zeta(1-2it)}{m^{\frac{1}{2}-it}} \frac{L_\infty\left(\frac{1}{2} - it\right)}{L_\infty\left(\frac{1}{2} + it\right)} \\ &\quad + O_{t,m}\left(k^{-\frac{1}{2}}\right), \end{aligned}$$

where $\omega_f = \frac{k-1}{2\pi^2} L(\mathrm{sym}^2 f, 1)$ and

$$C = 2\gamma - \frac{3}{2} \log \pi - \log 2 + \frac{\Gamma'\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}.$$

Here γ is Euler's constant.

The above asymptotic formulas were proved by Khan [7] with $O\left(k^{-\frac{1}{2}}\right)$ and $O_{t,m}\left(k^{-\frac{1}{2}}\right)$ replaced by $O\left(k^{-\frac{1}{20}+\epsilon}\right)$ and $O_{t,m}\left(k^{-\frac{1}{20}+\epsilon}\right)$, respectively, where $\epsilon > 0$ is arbitrary.

At the beginning of the proof of Theorem 1.1, we follow closely Khan [7]. After applying the approximate functional equation and Petersson trace formula, we obtain the so-called diagonal term and non-diagonal term. The difference comes from the treatment of the non-diagonal term. We will not appeal to the estimates for J -Bessel functions directly but use the Mellin inversion formula for J -Bessel functions. This method has already been used by Ganguly, Hoffstein and Sengupta [3] and by Lau and Tsang [10].

2. PRELIMINARIES

In this section, we list some results needed for the proof of Theorem 1.1. First, we need the approximate functional equation for $L\left(\mathrm{sym}^2 f, \frac{1}{2} + it\right)$ (see Lemma 2.3 in Khan [7]).

Lemma 2.1. *Let $G(y) = e^{y^2}$. We have*

$$(2.1) \quad L\left(\mathrm{sym}^2 f, \frac{1}{2} + it\right) = \sum_{n \geq 1} \frac{a_f(n^2)}{n^{\frac{1}{2}+it}} V_{\frac{1}{2}+it}(n) + \frac{L_\infty\left(\frac{1}{2} - it\right)}{L_\infty\left(\frac{1}{2} + it\right)} \sum_{n \geq 1} \frac{a_f(n^2)}{n^{\frac{1}{2}-it}} V_{\frac{1}{2}-it}(n),$$

where

$$(2.2) \quad V_{\frac{1}{2}+it}(n) = \frac{1}{2\pi i} \int_{(2)} \frac{L_\infty\left(\frac{1}{2} + it + y\right)}{L_\infty\left(\frac{1}{2} + it\right)} \zeta(1+2it+2y) G(y) n^{-y} \frac{dy}{y}.$$

Moreover, $V_{\frac{1}{2}+it}(n)$ satisfies

$$(2.3) \quad V_{\frac{1}{2}+it}(n) = \begin{cases} \zeta(1+2it) + i \frac{n^{it}}{2t} \frac{G(it) L_\infty\left(\frac{1}{2}\right)}{L_\infty\left(\frac{1}{2} + it\right)} + O_t\left(\frac{n}{k}\right) & \text{if } t \neq 0, \\ \frac{1}{2} \log\left(\frac{k}{n}\right) + \frac{C}{2} + O\left(\frac{n}{k}\right) & \text{if } t = 0, \end{cases}$$

where

$$(2.4) \quad C = 2\gamma - \frac{3}{2} \log \pi - \log 2 + \frac{\Gamma'(\frac{3}{4})}{2\Gamma(\frac{3}{4})}.$$

Next, we need the Petersson trace formula (see Iwaniec and Kowalski [5], Proposition 14.5).

Lemma 2.2. *We have*

$$\sum_{f \in H_k} \omega_f^{-1} a_f(m) a_f(n) = \delta_{m,n} + 2\pi i^k \sum_{c \geq 1} c^{-1} S(m, n; c) J_{k-1} \left(\frac{4\pi\sqrt{mn}}{c} \right),$$

where $\delta_{m,n} = 1$ if $m = n$ and is zero otherwise, $S(m, n; c)$ is the classical Kloosterman sum, and $J_{k-1}(x)$ is the J -Bessel function.

To deal with the non-diagonal term from the Petersson trace formula, we need the Mellin inversion formula for the J -Bessel functions (see Gradshteyn and Ryzhik [4], 6.422-6).

Lemma 2.3. *We have*

$$J_\nu(x) = \frac{1}{4\pi i} \int_{(a)} \frac{\Gamma(\frac{\nu+s}{2})}{\Gamma(1 + \frac{\nu-s}{2})} \left(\frac{x}{2}\right)^{-s} ds, \quad x > 0, \quad -\operatorname{Re}(\nu) < a < 1.$$

Finally, we need the functional equation of the Hurwitz zeta function. For $\operatorname{Re}(s) > 1$, the Hurwitz zeta function is defined as

$$(2.5) \quad \zeta(s, \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s},$$

which can be extended to a meromorphic function defined for all $s \in \mathbb{C}$ with $s \neq 1$. At $s = 1$, it has a simple pole with residue 1. $\zeta(s, \alpha)$ satisfies the following functional equation (see Apostol [1], Theorem 12.6).

Lemma 2.4. *If $0 < \alpha \leq 1$ and $\operatorname{Re}(s) > 1$, we have*

$$\zeta(1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left(e\left(-\frac{s}{4}\right) F(\alpha, s) + e\left(\frac{s}{4}\right) F(-\alpha, s) \right)$$

where $e(s) = e^{2\pi i s}$ and

$$(2.6) \quad F(\alpha, s) = \sum_{n \geq 1} e(\alpha n) n^{-s}.$$

3. PROOF OF THEOREM 1.1

By (2.1) and Lemma 2.2, we have

$$\begin{aligned}
 (3.1) \quad & \sum_{f \in H_k} L \left(\text{sym}^2 f, \frac{1}{2} + it \right) a_f(m^2) \omega_f^{-1} \\
 &= \sum_{f \in H_k} \left\{ \sum_{n \geq 1} \frac{a_f(n^2)}{n^{\frac{1}{2}+it}} V_{\frac{1}{2}+it}(n) + \frac{L_\infty \left(\frac{1}{2} - it \right)}{L_\infty \left(\frac{1}{2} + it \right)} \sum_{n \geq 1} \frac{a_f(n^2)}{n^{\frac{1}{2}-it}} V_{\frac{1}{2}-it}(n) \right\} a_f(m^2) \omega_f^{-1} \\
 &= \sum_{n \geq 1} \left\{ \frac{V_{\frac{1}{2}+it}(n)}{n^{\frac{1}{2}+it}} + \frac{L_\infty \left(\frac{1}{2} - it \right)}{L_\infty \left(\frac{1}{2} + it \right)} \frac{V_{\frac{1}{2}-it}(n)}{n^{\frac{1}{2}-it}} \right\} \sum_{f \in H_k} \omega_f^{-1} a_f(m^2) a_f(n^2) \\
 &= \sum_{n \geq 1} \left\{ \frac{V_{\frac{1}{2}+it}(n)}{n^{\frac{1}{2}+it}} + \frac{L_\infty \left(\frac{1}{2} - it \right)}{L_\infty \left(\frac{1}{2} + it \right)} \frac{V_{\frac{1}{2}-it}(n)}{n^{\frac{1}{2}-it}} \right\} \\
 &\quad \times \left\{ \delta_{m,n} + 2\pi i^k \sum_{c \geq 1} \frac{S(m^2, n^2; c)}{c} J_{k-1} \left(\frac{4\pi mn}{c} \right) \right\} \\
 &= \mathcal{D} + 2\pi i^k \left(\mathcal{R} + \frac{L_\infty \left(\frac{1}{2} - it \right)}{L_\infty \left(\frac{1}{2} + it \right)} \overline{\mathcal{R}} \right),
 \end{aligned}$$

where $\mathcal{D}$ is the diagonal term and by (2.3) and (2.4),

$$(3.2) \quad \mathcal{D} = \begin{cases} \frac{\zeta(1+2it)}{m^{\frac{1}{2}+it}} + \frac{\zeta(1-2it)}{m^{\frac{1}{2}-it}} \frac{L_\infty \left(\frac{1}{2} - it \right)}{L_\infty \left(\frac{1}{2} + it \right)} + O_{t,m}(k^{-1}) & \text{if } t \neq 0, \\ \log k + C + O(k^{-1}) & \text{if } t = 0 \text{ and } m = 1, \end{cases}$$

and $2\pi i^k \left(\mathcal{R} + \frac{L_\infty \left(\frac{1}{2} - it \right)}{L_\infty \left(\frac{1}{2} + it \right)} \overline{\mathcal{R}} \right)$ is the non-diagonal term with

$$(3.3) \quad \mathcal{R} = \sum_{n \geq 1} \sum_{c \geq 1} \frac{V_{\frac{1}{2}+it}(n)}{n^{\frac{1}{2}+it}} \frac{S(m^2, n^2; c)}{c} J_{k-1} \left(\frac{4\pi mn}{c} \right).$$

Thus to prove Theorem 1.1, we only need to estimate $\mathcal{R}$.

Taking $a = -1$ and $\nu = k - 1$ in Lemma 2.3, we have

$$(3.4) \quad J_{k-1} \left(\frac{4\pi mn}{c} \right) = \frac{1}{4\pi i} \int_{(-1)} \frac{\Gamma \left(\frac{k-1+s}{2} \right)}{\Gamma \left(\frac{k+1-s}{2} \right)} \left(\frac{2\pi mn}{c} \right)^{-s} ds.$$

By (2.2) and (3.4), $\mathcal{R}$ in (3.3) takes the form

$$\begin{aligned}
 (3.5) \quad \mathcal{R} &= \frac{1}{2(2\pi i)^2} \int_{(2)} \int_{(-1)} \frac{\zeta(1+2it+2y)}{y} \frac{L_\infty \left(\frac{1}{2} + it + y \right)}{L_\infty \left(\frac{1}{2} + it \right)} \frac{\Gamma \left(\frac{k-1+s}{2} \right)}{\Gamma \left(\frac{k+1-s}{2} \right)} \\
 &\quad \times \mathcal{L}_{t,m}(y, s) G(y) dy \frac{ds}{(2\pi m)^s},
 \end{aligned}$$

where

$$\mathcal{L}_{t,m}(y, s) = \sum_{n \geq 1} \sum_{c \geq 1} \frac{S(m^2, n^2; c)}{n^{s+y+\frac{1}{2}+it} c^{1-s}}.$$

Note that

$$\begin{aligned}\mathcal{L}_{t,m}(y,s) &= \sum_{c \geq 1} c^{s-1} \sum_{b \bmod c} S(m^2, b^2; c) \sum_{n \geq 1, n \equiv b \bmod c} n^{-s-y-\frac{1}{2}-it} \\ &= \sum_{c \geq 1} c^{-\frac{3}{2}-y-it} \sum_{b \bmod c} S(m^2, b^2; c) \sum_{\ell \geq 0} \left(\ell + \frac{b}{c} \right)^{-s-y-\frac{1}{2}-it} \\ &= \sum_{c \geq 1} c^{-\frac{3}{2}-y-it} \sum_{b \bmod c} S(m^2, b^2; c) \zeta \left(s + y + \frac{1}{2} + it, \frac{b}{c} \right),\end{aligned}$$

where $\zeta(s, \alpha)$ is the Hurwitz zeta function defined in (2.5). Moving the line of integration in s in (3.5) to $\operatorname{Re}(s) = -5$, passing a simple pole at $s = \frac{1}{2} - y - it$, we have

$$(3.6) \quad \mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2,$$

where

$$(3.7) \quad \begin{aligned}\mathcal{R}_1 &= \frac{1}{4\pi i} \int_{(2)} \frac{\zeta(1+2it+2y)}{y} \frac{L_\infty\left(\frac{1}{2}+it+y\right)}{L_\infty\left(\frac{1}{2}+it\right)} \frac{\Gamma\left(\frac{k-\frac{1}{2}-y-it}{2}\right)}{\Gamma\left(\frac{k+\frac{1}{2}+y+it}{2}\right)} \\ &\quad \times \mathcal{A}(y) (2\pi m)^{-\frac{1}{2}+y+it} G(y) dy,\end{aligned}$$

with

$$\mathcal{A}(y) = \sum_{c \geq 1} c^{-\frac{3}{2}-y-it} \sum_{b \bmod c} S(m^2, b^2; c),$$

and

$$(3.8) \quad \begin{aligned}\mathcal{R}_2 &= \frac{1}{2(2\pi i)^2} \int_{(2)} \int_{(-5)} \frac{\zeta(1+2it+2y)}{y} \frac{L_\infty\left(\frac{1}{2}+it+y\right)}{L_\infty\left(\frac{1}{2}+it\right)} \frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)} \\ &\quad \times \mathcal{L}_{t,m}(y,s) G(y) dy \frac{ds}{(2\pi m)^s}.\end{aligned}$$

First we estimate $\mathcal{R}_1$ in (3.7). Trivially, we have $|S(m^2, b^2; c)| \leq c$. Thus $\mathcal{A}(y)$ converges absolutely and

$$(3.9) \quad \mathcal{R}_1 \ll_m \int_{-\infty}^{\infty} \frac{|\zeta(5+2it+2iu)|}{|2+iu|} \left| \frac{L_\infty\left(\frac{5}{2}+it+iu\right)}{L_\infty\left(\frac{1}{2}+it\right)} \right| \left| \frac{\Gamma\left(\frac{k-\frac{5}{2}-iu-it}{2}\right)}{\Gamma\left(\frac{k+\frac{5}{2}+iu+it}{2}\right)} \right| e^{-u^2} du,$$

where by Stirling's formula,

$$(3.10) \quad \left| \frac{\Gamma\left(\frac{k-\frac{5}{2}-iu-it}{2}\right)}{\Gamma\left(\frac{k+\frac{5}{2}+iu+it}{2}\right)} \right| = \left| \frac{\Gamma\left(\frac{k-\frac{5}{2}+iu+it}{2}\right)}{\Gamma\left(\frac{k+\frac{5}{2}+iu+it}{2}\right)} \right| \ll |k+iu+it|^{-\frac{5}{2}},$$

and by (1.1),

$$(3.11) \quad \begin{aligned}\left| \frac{L_\infty\left(\frac{5}{2}+it+iu\right)}{L_\infty\left(\frac{1}{2}+it\right)} \right| &\ll \left| \frac{\Gamma\left(\frac{\frac{7}{2}+iu+it}{2}\right)}{\Gamma\left(\frac{\frac{3}{2}+it}{2}\right)} \frac{\Gamma\left(\frac{\frac{3}{2}+k+iu+it}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+k+it}{2}\right)} \frac{\Gamma\left(\frac{\frac{5}{2}+k+iu+it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+k+it}{2}\right)} \right| \\ &\ll_t |k+iu+it|^2 \left(1 + \frac{|u|}{k}\right)^{k+1} e^{\frac{\pi}{2}|u|}.\end{aligned}$$

By (3.9)-(3.11), we have

$$(3.12) \quad \mathcal{R}_1 \ll_{t,m} k^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left(1 + \frac{|u|}{k}\right)^{k+1} e^{\frac{\pi}{2}|u|-u^2} \frac{du}{|2+iu|} \ll_{t,m} k^{-\frac{1}{2}}.$$

Next we estimate $\mathcal{R}_2$ in (3.8). Denote $w = \frac{1}{2} - s - y - it$. By Lemma 2.4, we have

$$\zeta\left(s + y + \frac{1}{2} + it, \frac{b}{c}\right) = \frac{\Gamma(w)}{(2\pi)^w} \left\{ e\left(-\frac{w}{4}\right) F\left(\frac{b}{c}, w\right) + e\left(\frac{w}{4}\right) F\left(-\frac{b}{c}, w\right) \right\},$$

where $F(\alpha, s)$ is defined in (2.6). Thus

$$\begin{aligned} \mathcal{L}_{t,m}(y, s) &= \sum_{c \geq 1} c^{-\frac{3}{2}-y-it} \sum_{b \bmod c} S(m^2, b^2; c) \frac{\Gamma(w)}{(2\pi)^w} \\ &\quad \times \left\{ e\left(-\frac{w}{4}\right) F\left(\frac{b}{c}, w\right) + e\left(\frac{w}{4}\right) F\left(-\frac{b}{c}, w\right) \right\}. \end{aligned}$$

Note that $\operatorname{Re}\left(\frac{3}{2} + y + it\right) = \frac{7}{2}$ and $\operatorname{Re}(w) = \frac{7}{2}$. Thus $|\mathcal{L}_{t,m}(y, s)| \ll |\Gamma(w)| \ll 1$ and $\mathcal{R}_2$ in (3.8) is bounded by

$$\begin{aligned} (3.13) \quad \mathcal{R}_2 &\ll_m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\zeta(5 + 2it + 2iu)|}{|2 + iu|} \left| \frac{L_{\infty}\left(\frac{5}{2} + it + iu\right)}{L_{\infty}\left(\frac{1}{2} + it\right)} \right| \left| \frac{\Gamma\left(\frac{k-6+iv}{2}\right)}{\Gamma\left(\frac{k+6-iv}{2}\right)} \right| du dv \\ &\ll_{t,m} \int_{-\infty}^{\infty} |k + iu|^2 \left(1 + \frac{|u|}{k}\right)^{k+1} e^{\frac{\pi}{2}|u|-u^2} \frac{du}{|2 + iu|} \int_{-\infty}^{\infty} |k + iv|^{-6} dv \\ &\ll_{t,m} k^{-3}. \end{aligned}$$

Then Theorem 1.1 follows from (3.1), (3.2), (3.6), (3.12) and (3.13).

REFERENCES

1. T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1976. MR0434929 (55:7892)
2. V. Blomer, *On the central value of symmetric-square L -functions*, Math. Z. **260** (2008), 755–777. MR2443329 (2009j:11077)
3. S. Ganguly, J. Hoffstein and J. Sengupta, *Determining modular forms on $SL_2(\mathbb{Z})$ by central values of convolution L -functions*, Math. Ann. **345** (2009), 843–857. MR2545869 (2010i:11067)
4. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, Academic Press, New York, 2007. MR2360010 (2008g:00005)
5. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. **53**, American Mathematical Society, Providence, 2004. MR2061214 (2005h:11005)
6. H. Iwaniec and P. Michel, *The second moment of the symmetric square L -function*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 465–482. MR1833252 (2002d:11051)
7. R. Khan, *The first moment of the symmetric-square L -function*, J. Number Theory **124** (2007), 259–266. MR2321361 (2008b:11056)
8. W. Kohnen and J. Sengupta, *Nonvanishing of symmetric square L -functions of cusp forms inside the critical strip*, Proc. Amer. Math. Soc. **128** (6) (2000), 1641–1646. MR1676328 (2000j:11072)

9. Y.-K. Lau, *Non-vanishing of symmetric square L -functions*, Proc. Amer. Math. Soc. **130** (11) (2002), 3133–3139. MR1912989 (2003e:11053)
10. Y.-K. Lau and K.-M. Tsang, *A mean square formula for central values of twisted automorphic L -functions*, Acta Arith. **118** (3) (2005), 231–262. MR2168765 (2006g:11095)
11. X.-J. Li, *On the poles of Rankin-Selberg convolutions of modular forms*, Trans. Amer. Math. Soc. **348** (3) (1996), 1213–1234. MR1333393 (96h:11038)
12. G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. **31** (3) (1975), 79–98. MR0382176 (52:3064)

SCHOOL OF MATHEMATICS AND STATISTICS, SHANDONG UNIVERSITY AT WEIHAI, WEIHAI 264209,
PEOPLE'S REPUBLIC OF CHINA

E-mail address: `qfsun@mail.sdu.edu.cn`