A SHARP OPERATOR VERSION OF THE BISHOP-PHELPS THEOREM FOR OPERATORS FROM ℓ_1 TO CL-SPACES

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ABSTRACT. Acosta et al. in 2008 gave a characterization of a Banach space Y (called an approximate hyperplane series property, or AHSP for short) guaranteeing exactly that a quantitative version of the Bishop-Phelps theorem holds for bounded operators from ℓ_1 to the space Y . In this note, we give two new examples of spaces having the AHSP: the almost CL-spaces and the class of Banach spaces Y whose dual Y^* is uniformly strongly subdifferentiable on some boundary of Y . We then calculate the precise parameters associated to almost CL-spaces.

1. INTRODUCTION

The Bishop-Phelps theorem states that the set of norm-attaining functionals on a Banach space X is dense in the topological dual X^* [2]. It has been usefully extended in many directions and in the study of optimization. See Phelps' survey [13]. In 1970, B. Bollobás [3] gave a quantitative version of the Bishop-Phelps theorem (known as the Bishop-Phelps-Bollobás theorem). In 1963, J. Lindenstrauss [11] showed that for two Banach spaces X and Y , if X is reflexive, then the set $N(X, Y)$ of all norm-attaining operators is always dense in the space $B(X, Y)$ of all bounded linear operators from X to Y . He also gave a counterexample showing that it does not hold in general. J. Bourgain [4] proved that $N(X, Y)$ is dense in $B(X, Y)$ if the space X has the Radon-Nikodým property (RNP).

In this paper, the letter X will always be a real Banach space and X^* its dual. We denote by B_X (S_X , resp.) the closed unit ball (the unit sphere, resp.) of X . For a closed convex set $A \subset X$, $\text{ext } A$ stands for the set of all extreme points of A .

In 2008, Acosta, Aron, García and Maestre [1] introduced two useful notions: the Bishop-Phelps-Bollobás property (BPBP for short) and the approximate hyperplane series property (AHSP for short). Among many other things, they then showed the following insightful result: (ℓ_1, X) has the BPBP if and only if X has the AHSP. For two Banach spaces X, Y , the pair (X, Y) is said to admit the BPBP provided that for every $\varepsilon > 0$ there exist $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t \rightarrow 0} \beta(t) = 0$ such that for all $T \in S_{B(X, Y)}$, if $x_0 \in S_X$ such that $\|T(x_0)\| > 1 - \eta(\varepsilon)$, then

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there exist a point $u_0 \in S_X$ and an operator $S \in S_{B(X,Y)}$ such that

$$\|Su_0\| = 1, \|u_0 - x_0\| < \beta(\varepsilon) \text{ and } \|T - S\| < \varepsilon.$$

A Banach space X is said to have the AHSP provided that for every $\varepsilon > 0$ there exist $0 < \delta, \eta < \varepsilon$ such that for every sequence $\{x_k\} \subset S_X$ and for every sequence $\{t_k\} \subset \mathbb{R}$ of nonnegative numbers with $\sum_{k=1}^{\infty} t_k = 1$ satisfying $\|\sum_{k=1}^{\infty} t_k x_k\| > 1 - \eta$, there exist a subset $A \subset \mathbb{N}$, a subset $\{z_k : k \in A\} \subset S_X$ and a functional $x^* \in S_{X^*}$ such that

$$(1) \sum_{k \in A} t_k > 1 - \delta; (2) \|z_k - x_k\| < \varepsilon, \forall k \in A \text{ and } (3) \langle x^*, z_k \rangle = 1, \forall k \in A.$$

In this note, we show first that both real almost CL-spaces and real Banach spaces Y whose dual is uniformly strongly subdifferentiable on some boundary of Y satisfy the AHSP; hence for these spaces the pair (ℓ_1, Y) satisfies the BPBP. Indeed, if Y is an almost CL-space, $\varepsilon > 0$, and $x_0 \in S_{\ell_1}$ $T \in S_{(\ell_1, Y)}$ satisfies $\|Tx_0\| \geq 1 - \varepsilon^2$, then there exists an operator $S \in S_{(\ell_1, Y)}$ and an element $y_0 \in \ell_1$ satisfying

$$\|Sy_0\| = 1, \|S - T\| \leq \varepsilon, \text{ and } \|y_0 - x_0\| < 2\varepsilon.$$

2. THE MAIN RESULTS

The notion of a CL-space was first introduced by R. Fullerton [8] in 1960, and a generalized notion of a CL-space (i.e., an almost CL-space) was introduced by Á. Lima [10] in 1978. A Banach space is called a (an almost) CL-space provided its closed unit ball is the (closed) absolutely convex hull of each maximal convex set of the unit sphere. For example, for any nonempty set Γ the real Banach spaces $c_0(\Gamma)$, $\ell_1(\Gamma)$, $\ell_\infty(\Gamma)$, and for any measure μ the real spaces $L_1(\mu)$, $L_\infty(\mu)$, and $C(K)$ for any compact space K are CL-spaces (see, for instance, [12]). Assume that X is a Banach space and that C is a maximal convex set of the unit sphere S_X . Then there exists an extreme point x^* of the dual unit ball B_{X^*} such that $C = \{x \in B_X : \langle x^*, x \rangle = 1\}$ (see, for instance, [5]).

Theorem 2.1. *Assume that X is an almost CL-space and $0 < \varepsilon < 1$. Then for every sequence $\{x_n\} \subset B_X$ and every convex series $\sum_n t_n$ satisfying*

$$\left\| \sum_{n=1}^{\infty} t_n x_n \right\| > 1 - \varepsilon^2,$$

there exist $A \subset \mathbb{N}$, $\{z_k : k \in A\} \subset S_X$ and $x^ \in \text{ext}(B_{X^*})$ such that*

$$(1) \sum_{k \in A} t_k > 1 - \varepsilon; (2) \|z_k - x_k\| < \varepsilon, \forall k \in A \text{ and } (3) \langle x^*, z_k \rangle = 1, \forall k \in A.$$

As a consequence, X has the AHSP.

Proof. Suppose that the sequence $\{x_n\} \subset B_X$ and the convex series $\sum_n t_n$ satisfies the assumption; that is,

$$\left\| \sum_{n=1}^{\infty} t_n x_n \right\| > 1 - \varepsilon^2.$$

We can choose $0 < \eta < \varepsilon$ small enough such that

$$\left\| \sum_{n=1}^{\infty} t_n x_n \right\| > 1 - (\varepsilon - \eta)^2.$$

In view of the Hahn-Banach theorem there is $x^* \in S_{X^*}$ such that

$$\langle x^*, \sum_{n=1}^{\infty} t_n x_n \rangle = \left\| \sum_{n=1}^{\infty} t_n x_n \right\| > 1 - (\varepsilon - \eta)^2.$$

By Zorn's lemma, the (nonempty) convex subset

$$C := \{x \in B_X : \langle x^*, x \rangle = 1\}$$

is contained in a maximal convex subset of the unit sphere S_X . Hence, by changing the functional x^* by an appropriate extreme point of B_{X^*} (if needed), we can assume that C is a maximal convex subset of S_X .

We take $A := \{k \in \mathbb{N} : \langle x^*, x_k \rangle > 1 - \varepsilon + \eta\}$. It is clear that

$$\begin{aligned} 1 - (\varepsilon - \eta)^2 &\leq \langle x^*, \sum_{n=1}^{\infty} t_n x_n \rangle \\ &\leq \sum_{n=1}^{\infty} t_n + (1 - \varepsilon + \eta)(1 - \sum_{n \in A} t_n). \end{aligned}$$

Hence

$$(1) \quad 1 - \varepsilon < 1 - (\varepsilon - \eta) \leq \sum_{n \in A} t_n.$$

By using now the assumption that X is an almost CL-space, $B_X = \overline{\text{co}}(C \cup -C)$, and so there are sequences $\{z_k\} \subset C$, $\{y_k\} \subset -C$ and $\{\alpha_k\} \subset [0, 1]$ satisfying that

$$\|x_k - (\alpha_k z_k + (1 - \alpha_k) y_k)\| < \frac{\eta}{2}, \quad \forall k \in \mathbb{N}.$$

For every $k \in A$ we have

$$\begin{aligned} 1 - (\varepsilon - \eta) &\leq \langle x^*, x_k \rangle \leq \frac{\eta}{2} + \langle x^*, \alpha_k z_k + (1 - \alpha_k) y_k \rangle \\ &= \frac{\eta}{2} + 2\alpha_k - 1. \end{aligned}$$

As a consequence,

$$2(1 - \alpha_k) \leq \frac{\eta}{2} + \varepsilon - \eta, \quad \forall k \in A.$$

We deduce for every $k \in A$ that

$$\begin{aligned} \|x_k - z_k\| &\leq \|x_k - (\alpha_k z_k + (1 - \alpha_k) y_k)\| + \|(\alpha_k z_k + (1 - \alpha_k) y_k) - z_k\| \\ &< \frac{\eta}{2} + 2(1 - \alpha_k) \leq \frac{\eta}{2} + \frac{\eta}{2} + \varepsilon - \eta = \varepsilon. \end{aligned}$$

Hence the statement is proved. $\square$

In the following, we shall show that every pair (ℓ_1, Y) has the BPBP (or, equivalently, Y has the AHSP) if Y is in the class of Banach spaces whose dual norm is uniformly strongly subdifferentiable (USSD for short) on some boundary of Y . The following notions can be found in [6], [7] and [9]. A convex function f defined on a Banach space X is said to be strongly subdifferentiable (SSD) at $x \in X$ if the one-side limit $\lim_{t \rightarrow 0+} \frac{f(x+th) - f(x)}{t}$ exists uniformly for $h \in S_X$, and it is called uniformly strongly subdifferentiable (USSD) on a subset A of X if the limit above exists uniformly for $x \in A$ and $h \in S_X$. For a nonempty closed bounded convex set C in a Banach space X , the mapping $J_C : X^* \rightarrow 2^C$ is defined for $x^* \in X^*$ by $J_C(x^*) = \{x \in C : \langle x^*, x \rangle = \sup_{z \in C} \langle x^*, z \rangle\}$. If, in particular, $C = B_X$, then we simply write J_X instead of J_{B_X} . Note that in this case

$J_X(x^*) = \{x \in S_X : \langle x^*, x \rangle = \|x^*\|\}$. A subset $B \subset B_{X^*}$ is said to be a boundary (also called a James' boundary) of a Banach space X if for each $x \in X$ there is an $x^* \in B$ such that $\langle x^*, x \rangle = \|x\|$; that is, a boundary is an exact norming set of X contained in B_{X^*} . Clearly, $J_{X^*}(S_X)$ is a boundary of X .

The following proposition is a consequence of Proposition 2.2 in G. Godefroy [9] for strong subdifferentiability (SSD) of the norm. C. Franchetti and R. Payá [7] have also shown that (i) $\Leftrightarrow$ (ii).

Proposition 2.2. *Suppose that X is a Banach space. Then the following statements are equivalent:*

- (i) *the dual norm of X^* is SSD at $x^* \in X^*$;*
- (ii) *the map $J_{X^{**}}$ is norm-to-norm upper semi-continuous at X^* . That is, for every $\epsilon > 0$, there exists $\delta > 0$ such that $J_{X^{**}}(y^*) \subseteq J_{X^{**}}(x^*) + \epsilon B_{X^{**}}$ whenever $y^* \in S_{Y^*}$, satisfying $\|y^* - x^*\| < \delta$;*
- (iii) *the set $J_X(x^*)$ is nonempty, and for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$x \in B_X, \langle x^*, x \rangle > \|x^*\| - \delta \implies d(x, J_X(x^*)) < \epsilon;$$

- (iv) *for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$x^{**} \in B_{X^{**}}, \langle x^{**}, x^* \rangle > \|x^*\| - \delta \implies d(x^{**}, J_{X^{**}}(x^*)) < \epsilon.$$

Analogously, we have

Proposition 2.3. *Suppose that X is a Banach space and $B \subset S_{X^*}$ is a boundary of X . Then the following statements are equivalent:*

- (i) *the dual norm of X^* is USSD on the boundary B ;*
- (ii) *the map $J_{X^{**}}$ is norm-to-norm uniformly upper semi-continuous on B . That is, for every $\epsilon > 0$, there exists $\delta > 0$ such that $J_{X^{**}}(y^*) \subseteq J_{X^{**}}(x^*) + \epsilon B_{X^{**}}$ whenever $x^* \in B$ and $y^* \in S_{Y^*}$, satisfying $\|y^* - x^*\| < \delta$.*
- (iii) *for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, J_X(x^*)) < \epsilon$ whenever $x \in B_X$ and $x^* \in B$, satisfying $\langle x^*, x \rangle > 1 - \delta$.*

Note Proposition 2.3 (iii) entails that the Banach space X has the AHSP. As a consequence of [1], we have

Corollary 2.4. *Suppose that Y is a Banach space and that the dual norm of Y^* is USSD on a boundary B of Y . Then the pair (ℓ_1, Y) admits the BPBP.*

Acosta, Aron, García and Maestre [1] have shown that if X is uniformly convex, then X has the AHSP. Since X is uniformly convex if and only if X^* is uniformly smooth, we have

Corollary 2.5 (Acosta-Aron-García-Maestre). *Suppose that Y is a uniformly convex Banach space. Then the pair (ℓ_1, Y) has the BPBP.*

Remark 2.6. If X is an almost CL -space, then for every $\epsilon > 0$, we have $d(x, J_X(x^*)) < \epsilon$ whenever $x \in B_X$ and $x^* \in B$ satisfying $\langle x^*, x \rangle > 1 - \epsilon$, where B denotes the boundary $\text{mex}(B_{X^*}) \subset \text{ext}(B_{X^*})$ of X consisting of those extreme points x^* satisfying the fact that $C \equiv \{x \in S_X : \langle x^*, x \rangle = 1\}$ is a maximal convex set of S_X . Therefore, every almost CL -space admits the property that the dual norm is USSD on the boundary $\text{mex}(B_{X^*})$ of X .

We have proven that almost CL -spaces have the AHSP (Theorem 2.1), so in view of Theorem 4.1 of [1], for an almost CL -space Y , the pair (ℓ_1, Y) has the BPBP.

Even so, for the sake of completeness we will give a direct proof of the last fact and give functions η and β satisfying the definition of the BPBP in this case.

Theorem 2.7. *Suppose that Y is an almost CL-space, $\varepsilon > 0$, and that $x_0 \in \ell_1$ with $\|x_0\| = 1$. Let $T \in B(\ell_1, Y)$ with $\|T\| = 1$ such that $\|T(x_0)\| \geq 1 - \varepsilon^2$. Then there exist a point $y_0 \in S_{\ell_1}$ and a norm-attaining operator $S \in S_{B(\ell_1, Y)}$ such that*

$$(1) \|S(y_0)\| = 1; \quad (2) \|S - T\| \leq \varepsilon \quad \text{and} \quad (3) \|y_0 - x_0\| < 2\varepsilon.$$

Proof. Let $x_0 \in \ell_1$ with $\|x_0\| = \sum_{n=1}^{\infty} |x_0(n)| = 1$ satisfy $\|Tx_0\| > 1 - \varepsilon^2$. Let C be a maximal convex set of S_Y containing $v_0 \equiv T(x_0)/\|T(x_0)\|$. Then there exists $x^* \in \text{ext}(B_{Y^*})$ such that $C = \{u \in S_Y : \langle x^*, u \rangle = 1\}$. Therefore, $\langle x^*, Tx_0 \rangle = \|Tx_0\|$.

Let $\{e_n\}_{n=1}^{\infty}$ be the canonical basis of ℓ_1 . Applying Theorem 2.1 to the nonnegative numbers $t_n = |x_0(n)|$ and the vectors $y_n = T(\text{sgn } x_0(n)e_n)$ (for all $n \in \mathbb{N}$), there exist a set $A \subset \mathbb{N}$ and a set $\{z_n : n \in A\} \subset Y$ with $\|z_n\| = 1$ for all $n \in A$ such that

$$\sum_{n \in A} |x_0(n)| > 1 - \varepsilon \quad \text{and} \quad \|z_n - y_n\| < \varepsilon, \quad \langle x^*, z_n \rangle = 1$$

and

$$\left\| \sum_{n \in A} |x_0(n)| z_n \right\| = \sum_{n \in A} |x_0(n)|.$$

We define $S : \ell_1 \rightarrow Y$ for $x \in \ell_1$ by $S(x) = \sum_{n=1}^{\infty} x(n)u_n$, where

$$u_n = \begin{cases} \text{sgn } x_0(n)z_n, & \text{if } n \in A; \\ T(e_n), & \text{if } n \notin A. \end{cases}$$

It is clear that $\|S\| = 1$. Let $y_0 = \sum_{n \in A} x_0(n)e_n / \sum_{n \in A} |x_0(n)|$. Then

$$\|y_0\| = 1, \quad \|S(y_0)\| = \left\| \sum_{n \in A} x_0(n)u_n \right\| / \sum_{n \in A} |x_0(n)| = \left\| \sum_{n \in A} |x_0(n)| z_n \right\| / \sum_{n \in A} |x_0(n)| = 1,$$

$$\|y_0 - x_0\| \leq \|y_0 - \sum_{n \in A} x_0(n)e_n\| + \left\| \sum_{n \in A} x_0(n)e_n - x_0 \right\|$$

$$= (1 - \left\| \sum_{n \in A} x_0(n)e_n \right\|) + \left\| \sum_{n \in A} x_0(n)e_n - x_0 \right\|$$

$$= \sum_{n \notin A} |x_0(n)| + \left\| \sum_{x \notin A} x_0(n)e_n \right\| < \varepsilon + \varepsilon = 2\varepsilon,$$

and

$$\|S - T\| = \sup_n \|(S - T)(e_n)\| = \sup_{n \in A} \|z_n - y_n\| \leq \varepsilon.$$

□

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