REFINEMENTS OF LOWER BOUNDS FOR POLYGAMMA FUNCTIONS

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ABSTRACT. In the paper, some lower bounds for polygamma functions are refined. Moreover, several open problems are posed.

1. INTRODUCTION AND MAIN RESULTS

It is well-known that the classical Euler gamma function

$$(1) \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for $x > 0$, the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and the polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are a series of important special functions and have many extensive applications in many branches such as statistics, probability, number theory, theory of 0-1 matrices, graph theory, combinatorics, physics, engineering, and other mathematical sciences.

We begin by summarizing several results which motivated this paper.

In [6, Corollary 2], the inequality

$$(2) \quad \psi'(x)e^{\psi(x)} < 1$$

for $x > 0$ was deduced. We observe that the inequality (2) can be rearranged as

$$(3) \quad \psi'(x) < e^{-\psi(x)}.$$

In [4, Lemma 1.2], the inequality (2) was recovered.

In [1, Theorem 4.8], by the aid of the inequality

$$(4) \quad [\psi'(x)]^2 + \psi''(x) > 0$$

for $x > 0$ or, say,

$$(5) \quad \sqrt{\frac{|\psi''(x)|}{(2-1)!}} < \psi'(x),$$

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the inequality (2) was generalized to

$$(6) \quad (n-1)! \exp(-n\psi(x+1)) < |\psi^{(n)}(x)| < (n-1)! \exp(-n\psi(x))$$

for $x > 0$ and $n \in \mathbb{N}$, which can be restated as

$$(7) \quad e^{-\psi(x+1)} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < e^{-\psi(x)}$$

for $x > 0$ and $n \in \mathbb{N}$.

In [3, Theorem 2.1], the left-hand-side inequality in (6) was refined to

$$(8) \quad (n-1)! \exp\left(-n\psi\left(x + \frac{1}{2}\right)\right) < |\psi^{(n)}(x)| < (n-1)! \exp(-n\psi(x)),$$

which can be reformulated as

$$(9) \quad e^{-\psi(x+1/2)} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < e^{-\psi(x)}$$

for $x > 0$ and $n \in \mathbb{N}$.

Furthermore, the function $\psi^{(n)}(x)$ was alternatively bounded in [3, Theorem 2.2] by

$$(10) \quad (n-1)! \left[\frac{\psi^{(k)}(x+1/2)}{(-1)^{k-1}(k-1)!} \right]^{n/k} < |\psi^{(n)}(x)| < (n-1)! \left[\frac{\psi^{(k)}(x)}{(-1)^{k-1}(k-1)!} \right]^{n/k},$$

which can be rewritten as

$$(11) \quad \sqrt[k]{\frac{|\psi^{(k)}(x+1/2)|}{(k-1)!}} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < \sqrt[k]{\frac{|\psi^{(k)}(x)|}{(k-1)!}}$$

for $x > 0$ and $1 \leq k \leq n-1$.

For more information on this topic, please refer to the expository and survey article [15] and plenty of closely related references therein.

The main aim of this paper is to further refine the left-hand-side inequalities in (8) and (10) or their variants (9) and (11).

Our main results are stated in the following theorem.

Theorem 1. *For $n = 1, 2$, the inequality*

$$(12) \quad \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} > e^{-\psi(1/\ln(1+1/x))}$$

holds on $(0, \infty)$. For $n \in \mathbb{N}$ and $1 \leq k \leq n-1$, the inequality

$$(13) \quad \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} > \sqrt[k]{\frac{|\psi^{(k)}(1/\ln(1+1/x))|}{(k-1)!}}$$

is valid on $(0, \infty)$.

2. LEMMAS

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 ([7, 10, 12, 21]). *For $k \in \mathbb{N}$, the inequalities*

$$(14) \quad \ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}$$

and

$$(15) \quad \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}$$

are valid on $(0, \infty)$.

Lemma 2. *The inequality*

$$(16) \quad \psi'(t) < e^{1/t} - 1$$

holds on $(0, \infty)$.

Proof. For the sake of convenience, denote $e^{1/t} - \psi'(t)$ by $h(x)$. It is clear that

$$(17) \quad \lim_{t \rightarrow \infty} h(t) = 1.$$

A direct calculation reveals that

$$\begin{aligned} h(t+1) - h(t) &= e^{1/(t+1)} - e^{1/t} + \psi'(t) - \psi'(t+1) \\ &= e^{1/(t+1)} - e^{1/t} + \frac{1}{t^2} \end{aligned}$$

and

$$\begin{aligned} e^{1/t} - e^{1/(t+1)} &= \int_0^1 \frac{1}{(t+u)^2} e^{1/(t+u)} \, du \\ &> \int_0^1 \frac{1}{(t+u)^2} \left[1 + \frac{1}{t+u} + \frac{1}{2(t+u)^2} \right] \, du \\ &> \frac{6t(t+1)^3 + 1}{6t^3(t+1)^3} \\ &> \frac{1}{t^2} \end{aligned}$$

for $t \in (0, \infty)$. Hence, by the limit (17) and mathematical induction, we have

$$(18) \quad h(t) > h(t+1) > h(t+2) > \cdots > h(t+k) > \lim_{k \rightarrow \infty} h(t+k) = 1,$$

which is equivalent to the inequality (16). $\square$

3. PROOF OF THEOREM 1

Now we turn our attention to proving Theorem 1.

Letting $\frac{1}{\ln(1+1/x)} = t$ in (12) and rearranging yield

$$(19) \quad e^{n\psi(t)} \left| \psi^{(n)} \left(\frac{1}{e^{1/t} - 1} \right) \right| > (n-1)!$$

for $t \in (0, \infty)$.

Utilizing the left-hand-side inequalities in (14) and (15) gives

$$\begin{aligned} e^{n\psi(t)} \left| \psi^{(n)} \left(\frac{1}{e^{1/t} - 1} \right) \right| &> e^{n(\ln t - 1/t)} \left[(n-1)! (e^{1/t} - 1)^n + \frac{n!}{2} (e^{1/t} - 1)^{n+1} \right] \\ &= (n-1)! \frac{(e^{1/t} - 1)^n}{e^{n[\ln(1/t) + 1/t]}} \left[\frac{n}{2} (e^{1/t} - 1) + 1 \right] \\ &= (n-1)! \frac{(e^u - 1)^n}{u^n e^{nu}} \left[\frac{n}{2} (e^u - 1) + 1 \right], \end{aligned}$$

where $u = \frac{1}{t} > 0$. So, in order to prove (19), it is sufficient to show

$$(20) \quad \frac{(e^u - 1)^n}{u^n e^{nu}} \left[\frac{n}{2} (e^u - 1) + 1 \right] \geq 1, \quad u > 0,$$

that is,

$$(21) \quad (e^u - 1)^n [n(e^u - 1) + 2] \geq 2u^n e^{nu}, \quad u > 0.$$

Let

$$f_n(u) = (e^u - 1)^n [n(e^u - 1) + 2] - 2u^n e^{nu}$$

on $(0, \infty)$. A straightforward differentiation gives

$$\begin{aligned} f_1'(u) &= 2e^u(e^u - 1 - u) > 0, \\ f_2'(u) &= 2e^u[3e^{2u} - 2e^u(u^2 + u + 2) + 1], \\ \left[\frac{f_2'(u)}{2e^u} \right]' &= 2e^u(3e^u - u^2 - 3u - 3) > 0. \end{aligned}$$

Hence, the derivative $f_2'(u)$ is also positive on $(0, \infty)$. Since $f_n(0) = 0$ and the functions $f_1(u)$ and $f_2(u)$ are strictly increasing on $(0, \infty)$, it is readily obtained that the functions $f_1(u)$ and $f_2(u)$ are strictly positive on $(0, \infty)$. This shows that inequalities (20) and (21) are valid on $(0, \infty)$ for $n = 1, 2$. As a result, the inequality (12) is valid on $(0, \infty)$ for $n = 1, 2$.

Letting $\frac{1}{\ln(1+1/x)} = t$ in (13) leads to

$$(22) \quad \sqrt[n]{\frac{|\psi^{(n)}(1/(e^{1/t} - 1))|}{(n-1)!}} > \sqrt[k]{\frac{|\psi^{(k)}(t)|}{(k-1)!}}$$

for $t > 0$, where $n \in \mathbb{N}$ and $1 \leq k \leq n-1$. In [3, Lemma 1.2], the inequality

$$(23) \quad (-1)^n \psi^{(n+1)}(x) < \frac{n}{\sqrt[n]{(n-1)!}} [(-1)^{n-1} \psi^{(n)}(x)]^{1+1/n}$$

for $x > 0$ and $n \in \mathbb{N}$ was turned out, which can be restated more significantly as

$$(24) \quad \sqrt[n+1]{\frac{|\psi^{(n+1)}(x)|}{n!}} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}},$$

an equivalence of the right-hand-side inequalities in (10) and (11). Therefore, it is sufficient to show

$$(25) \quad \lim_{n \rightarrow \infty} \sqrt[n]{\frac{|\psi^{(n)}(1/(e^{1/t} - 1))|}{(n-1)!}} \geq \psi'(t)$$

for $t > 0$.

Making use of the double inequality (15), it is easy to acquire that

$$(26) \quad (e^{1/t} - 1) \sqrt[k]{\frac{k}{2}(e^{1/t} - 1) + 1} < \sqrt[k]{\frac{|\psi^{(k)}(1/(e^{1/t} - 1))|}{(k-1)!}} < (e^{1/t} - 1) \sqrt[k]{k(e^{1/t} - 1) + 1}$$

for $t \in (0, \infty)$ and $k \in \mathbb{N}$. Hence,

$$(27) \quad \lim_{k \rightarrow \infty} \sqrt[k]{\frac{|\psi^{(k)}(1/(e^{1/t} - 1))|}{(k-1)!}} = e^{1/t} - 1.$$

By virtue of inequality (16), inequality (25) follows, so inequality (13) is proved.

4. REMARKS

In this section, we would like to supply several remarks on Theorem 1.

Remark 1. Since

$$x < \frac{1}{\ln(1+1/x)} < x + \frac{1}{2},$$

the psi function $\psi(x)$ is strictly increasing, and $|\psi^{(n)}(x)|$ for $n \in \mathbb{N}$ are strictly decreasing on $(0, \infty)$. Then the left-hand-side inequalities in (8) and (9) for $n = 1, 2$ and the left-hand-side inequalities in (10) and (11) are respectively refined, say nothing of the left-hand-side inequality in (6) for $n = 1, 2$.

Remark 2. The inequality (12) would be invalid if n were big enough. In other words, the inequality (12) is not valid for all $n \in \mathbb{N}$. Otherwise, the inequality

$$(28) \quad \lim_{n \rightarrow \infty} \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} = \frac{1}{x} \geq e^{-\psi(1/\ln(1+1/x))}$$

would be valid on $(0, \infty)$. However, the reversed inequality of (28) holds on $(0, \infty)$. (Why? See Remark 4 below.) This situation motivates us to pose an open problem: What is the largest positive integer n such that inequality (12) holds on $(0, \infty)$?

Remark 3. Rewriting (2) and (12) for $n = 1$ leads to

$$(29) \quad e^{-\psi(L(x, x+1))} < \psi'(x) < e^{-\psi(L(x, x))}$$

for $x > 0$, where

$$(30) \quad L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b \end{cases}$$

stands for the logarithmic mean for positive numbers a and b . Since the logarithmic mean $L(a, b)$ is strictly increasing with respect to both $a > 0$ and $b > 0$ and the psi function $\psi(x)$ is also strictly increasing on $(0, \infty)$, inequalities (6), (8), (9), (12), and (29) stimulate us to naturally ask the following question: What are the best scalars $p(n) \geq 0$ and $q(n) > 0$ such that the inequality

$$(31) \quad e^{-\psi(L(x, x+q(n)))} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < e^{-\psi(L(x, x+p(n)))}$$

is valid on $(0, \infty)$?

Similarly, inequalities (10), (11), and (13) motivate us to pose the following open problem: What are the best constants $p(n, k) \geq 0$ and $0 < q(n, k) \leq 1$ such that the inequality

$$(32) \quad \sqrt[k]{\frac{|\psi^{(k)}(L(x, x + q(n, k)))|}{(k-1)!}} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < \sqrt[k]{\frac{|\psi^{(k)}(L(x, x + p(n, k)))|}{(k-1)!}}$$

holds on $(0, \infty)$ for $1 \leq k \leq n-1$.

Remark 4. Letting $\frac{1}{\ln(1+1/x)} = t$ in the reversed version of the inequality (28) and taking the logarithm yield

$$(33) \quad \psi(t) + \ln(e^{1/t} - 1) < 0$$

on $(0, \infty)$, an inequality established in [3, Theorem 2.8] and [4, Theorem 2]. The increasing monotonicity of the function in the left-hand-side of the inequality (33) was presented in [2, 11, 17] respectively. The strict concavity and some other generalizations of the function in the inequality (33) was discussed in [11] recently.

Remark 5. The case $n = 2$ and $k = 1$ in (13) is

$$(34) \quad \psi''(x) + [\psi'(1/\ln(1+1/x))]^2 < 0$$

on $(0, \infty)$. This refines the inequality

$$(35) \quad \psi''(x) + \left[\psi'\left(x + \frac{1}{2}\right)\right]^2 < 0$$

on $(0, \infty)$, the special case $n = 2$ and $k = 1$ of the inequality (10). The inequality (35) was also refined and generalized in [19] in another direction.

The inequality (4), a special case with $n = 1$ of the inequality (23), has been generalized to the complete monotonicity and many other cases. For more information, please refer to [8, 13, 18, 19] and closely related references therein.

Remark 6. The generalized logarithmic mean $L(p; a, b)$ of order $p \in \mathbb{R}$ for positive numbers a and b with $a \neq b$ is defined in [5, p. 385] by

$$(36) \quad L(p; a, b) = \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p}, & p \neq -1, 0; \\ \frac{b-a}{\ln b - \ln a}, & p = -1; \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, & p = 0. \end{cases}$$

It is known from [22, 23] that $L(p; a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$. See also [9, 16] and closely related references therein. Furthermore, we can pose the following more general open problem: What are the best scalars $\lambda(n)$, $\mu(n)$, $p(n)$ and $q(n)$ such that the inequality

$$(37) \quad e^{-\psi(L(\lambda(n); x, x+q(n)))} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} < e^{-\psi(L(\mu(n); x, x+p(n)))}$$

is valid on $(0, \infty)$? What are the best constants $\lambda(n, k)$, $\mu(n, k)$, $p(n, k)$ and $q(n, k)$ such that the inequality

$$(38) \quad \sqrt[k]{\frac{|\psi^{(k)}(L(\lambda(n, k); x, x + q(n, k)))|}{(k-1)!}} < \sqrt[n]{\frac{|\psi^{(n)}(x)|}{(n-1)!}} \\ < \sqrt[k]{\frac{|\psi^{(k)}(L(\mu(n, k); x, x + p(n, k)))|}{(k-1)!}}$$

holds on $(0, \infty)$ for $1 \leq k \leq n-1$?

Remark 7. Finally, an alternative proof of the inequality (12) for $n = 1$ is provided as follows. Letting $\frac{1}{\ln(1+1/x)} = t$ in (12) results in

$$(39) \quad \sqrt[n]{\frac{|\psi^{(n)}(1/(e^{1/t} - 1))|}{(n-1)!}} > e^{-\psi(t)}$$

for $t > 0$ and $n \in \mathbb{N}$. By the inequality

$$(40) \quad 1 + \frac{\alpha x}{1 + (1 - \alpha)x} \leq (1 + x)^\alpha \leq 1 + \alpha x$$

for $x > -1$ and $0 \leq \alpha \leq 1$ (see [14, p. 128] and [24, p. 533]) we have

$$(41) \quad \sqrt[k]{\frac{k}{2}(e^{1/t} - 1) + 1} \geq 1 + \frac{e^{1/t} - 1}{2 + (k-1)(e^{1/t} - 1)}, \quad t > 0.$$

Combining this with the left-hand-side inequality in (26) reveals that it suffices to show

$$(42) \quad 1 + \frac{e^{1/t} - 1}{2 + (k-1)(e^{1/t} - 1)} > \frac{e^{-\psi(t)}}{e^{1/t} - 1}, \quad t > 0,$$

that is,

$$(43) \quad k < \frac{1}{e^{-\psi(t)}/(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} + 1, \quad t > 0.$$

By the left-hand-side inequality in (14), it follows that

$$\begin{aligned} \frac{1}{e^{-\psi(t)}/(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} &> \frac{1}{e^{-(\ln t - 1/t)}/(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} \\ &= \frac{1}{e^{1/t}/t(e^{1/t} - 1) - 1} - \frac{2}{e^{1/t} - 1} \\ &= \frac{e^{2u} - 2e^u u - 1}{(e^u - 1)(ue^u - e^u + 1)} \\ &> 0 \end{aligned}$$

and

$$\lim_{u \rightarrow \infty} \frac{e^{2u} - 2e^u u - 1}{(e^u - 1)(ue^u - e^u + 1)} = 0,$$

where $u = \frac{1}{t}$. Hence, we obtain that $k \leq 1$. The inequality (12) for $n = 1$ is proved.

Remark 8. This is a revised version of the preprint [20].

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