ON COMMUTING MATRICES AND EXPONENTIALS

CLÉMENT DE SEGUINS PAZZIS

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ABSTRACT. Let A and B be matrices of $M_n(\mathbb{C})$. We show that if $\exp(A)^k \exp(B)^l = \exp(kA + lB)$ for all integers k and l , then $AB = BA$. We also show that if $\exp(A)^k \exp(B) = \exp(B) \exp(A)^k = \exp(kA + B)$ for every positive integer k , then the pair (A, B) has property L of Motzkin and Taussky.

As a consequence, if G is a subgroup of $(M_n(\mathbb{C}), +)$ and $M \mapsto \exp(M)$ is a homomorphism from G to $(GL_n(\mathbb{C}), \times)$, then G consists of commuting matrices. If S is a subsemigroup of $(M_n(\mathbb{C}), +)$ and $M \mapsto \exp(M)$ is a homomorphism from S to $(GL_n(\mathbb{C}), \times)$, then the linear subspace $\text{Span}(S)$ of $M_n(\mathbb{C})$ has property L of Motzkin and Taussky.

1. INTRODUCTION

1.1. Notation and definition.

- i) We denote by $\mathbb{N}$ the set of non-negative integers.
- ii) If $M \in M_n(\mathbb{C})$, we denote by e^M or $\exp(M)$ its exponential, by $\text{Sp}(M)$ its set of eigenvalues.
- iii) The $n \times n$ complex matrices A, B are said to be simultaneously triangularizable if there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular.
- iv) A pair (A, B) of complex $n \times n$ matrices is said to have property L if for a special ordering $(\lambda_i)_{1 \leq i \leq n}, (\mu_i)_{1 \leq i \leq n}$ of the eigenvalues of A, B , the eigenvalues of $xA + yB$ are $(x\lambda_i + y\mu_i)_{1 \leq i \leq n}$ for all values of the complex numbers x, y .

1.2. The problem. It is well-known that the exponential is not a group homomorphism from $(M_n(\mathbb{C}), +)$ to $(GL_n(\mathbb{C}), \times)$ if $n \geq 2$. Nevertheless, when A and B are commuting matrices of $M_n(\mathbb{C})$, one has

$$(1) \quad e^{A+B} = e^A e^B = e^B e^A.$$

However (1) is not a sufficient condition for the commutativity of A with B , nor even for A and B to be simultaneously triangularizable. Still, if

$$(2) \quad \forall t \in \mathbb{R}, e^{tA} e^{tB} = e^{tB} e^{tA},$$

or

$$(3) \quad \forall t \in \mathbb{R}, e^{t(A+B)} = e^{tA} e^{tB},$$

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then a power series expansion at $t = 0$ shows that $AB = BA$. In the 1950's, pairs of matrices (A, B) of small size such that $e^{A+B} = e^A e^B$ have been under extensive scrutiny [3, 4, 6, 7, 9, 10]. More recently, Wermuth [16, 17] and Schmoeger [14, 15] studied the problem of adding extra conditions on the matrices A and B for the commutativity of e^A with e^B to imply the commutativity of A with B . A few years ago, Bourgeois (see [1]) investigated, for small n , the pairs $(A, B) \in M_n(\mathbb{C})^2$ that satisfy

$$(4) \quad \forall k \in \mathbb{N}, \quad e^{kA+B} = e^{kA} e^B = e^B e^{kA}.$$

The main interest in this condition lies in the fact that, contrary to Conditions (2) and (3), it is not possible to use it to obtain information on A and B based only on the local behavior of the exponential around 0. Bourgeois showed that Condition (4) implies that A and B are simultaneously triangularizable if $n = 2$, and produced a proof that this also holds when $n = 3$. This last result is however false, as the following counterexample, communicated to us by Jean-Louis Tu, shows: consider the matrices

$$A_1 := 2i\pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1 := 2i\pi \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Notice that A_1 and B_1 are not simultaneously triangularizable since they share no eigenvector (indeed, the eigenspaces of A_1 are the lines spanned by the three vectors of the canonical basis, and none of them is stabilized by B_1). However, for every $t \in \mathbb{C}$, a straightforward computation shows that the characteristic polynomial of $tA_1 + B_1$ is

$$X(X - 2i\pi(t + 2))(X - 2i\pi(2t + 3)).$$

Then for every $t \in \mathbb{N}$, the matrix $tA_1 + B_1$ has three distinct eigenvalues in $2i\pi\mathbb{Z}$, hence is diagonalizable with $e^{tA_1+B_1} = I_3$. In particular $e^{B_1} = I_3$, and on the other hand, $e^{A_1} = I_3$. This shows that Condition (4) holds.

It then appears that one should strengthen Bourgeois' condition as follows in order to obtain at least the simultaneous triangularizability of A and B :

$$(5) \quad \forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kA} e^{lB}.$$

Notice immediately that this condition implies that e^A and e^B commute. Indeed, if Condition (5) holds, then

$$e^B e^A = (e^{-A} e^{-B})^{-1} = (e^{-A-B})^{-1} = e^{A+B} = e^A e^B.$$

Therefore Condition (5) is equivalent to

$$(6) \quad \forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kA} e^{lB} = e^{lB} e^{kA}.$$

Here is our main result.

Theorem 1. *Let $(A, B) \in M_n(\mathbb{C})^2$ be such that for all $(k, l) \in \mathbb{Z}^2$, $e^{kA+lB} = e^{kA} e^{lB}$. Then $AB = BA$.*

The following corollary is straightforward.

Theorem 2. *Let G be a subgroup of $(M_n(\mathbb{C}), +)$ and assume that $M \mapsto \exp(M)$ is a homomorphism from $(G, +)$ to $(GL_n(\mathbb{C}), \times)$. Then, for all $(A, B) \in G^2$, $AB = BA$.*

The key to the proof of Theorem 1 is

Proposition 3. *Let $(A, B) \in M_n(\mathbb{C})^2$. Assume that, for every $(k, l) \in \mathbb{Z}^2$, the matrix $kA + lB$ is diagonalizable and $\text{Sp}(kA + lB) \subset \mathbb{Z}$. Then $AB = BA$.*

For subsemigroups of $(M_n(\mathbb{C}), +)$, Theorem 2 surely fails. A very simple counterexample is indeed given by the semigroup generated by

$$A := \begin{bmatrix} 0 & 0 \\ 0 & 2i\pi \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 1 \\ 0 & 2i\pi \end{bmatrix}.$$

One may however wonder whether a subsemigroup S on which the exponential is a homomorphism must be simultaneously triangularizable. Obviously the additive semigroup generated by the matrices A_1 and B_1 above is a counterexample. Nevertheless, we will prove a weaker result, which rectifies and generalizes Bourgeois' results [1].

Proposition 4. *Let $(A, B) \in M_n(\mathbb{C})^2$ be such that $\forall k \in \mathbb{N}$, $e^{kA+B} = e^{kA}e^B = e^Be^{kA}$. Then (A, B) has property L.*

Note that the converse is obviously false.

The proofs of Theorem 1 and of Proposition 4 have largely similar parts, so they will be tackled simultaneously. There are three main steps.

- We will prove Proposition 4 in the special case where $\text{Sp}(A) \subset 2i\pi\mathbb{Z}$ and $\text{Sp}(B) \subset 2i\pi\mathbb{Z}$. This will involve a study of the matrix pencil $z \mapsto A + zB$. We will then easily derive Proposition 3 using a refinement of the Motzkin-Taussky theorem.
- We will handle the more general case $\text{Sp}(A) \subset 2i\pi\mathbb{Z}$ and $\text{Sp}(B) \subset 2i\pi\mathbb{Z}$ in Theorem 1 by using the Jordan-Chevalley decompositions of A and B together with Proposition 3.
- In the general case, we will use an induction to reduce the situation to the previous one, both for Theorem 1 and Proposition 4.

In the last section, we will prove a sort a generalized version of Proposition 4 for additive semigroups of matrices (see Theorem 11).

2. ADDITIVE GROUPS AND SEMIGROUPS OF MATRICES WITH AN INTEGRAL SPECTRUM

2.1. Notation.

- i) We denote by Σ_n the group of permutations of $\{1, \dots, n\}$, make it act on $\mathbb{C}^n$ by $\sigma.(z_1, \dots, z_n) := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$, and consider the quotient set $\mathbb{C}^n/\Sigma_n$. The class of a list $(z_1, \dots, z_n) \in \mathbb{C}^n$ in $\mathbb{C}^n/\Sigma_n$ will be denoted by $[z_1, \dots, z_n]$.
- ii) For $M \in M_n(\mathbb{C})$, we denote by $\chi_M(X) \in \mathbb{C}[X]$ its characteristic polynomial, and we set

$$\text{OSp}(M) := [z_1, \dots, z_n], \quad \text{where } \chi_M(X) = \prod_{k=1}^n (X - z_k).$$

- iii) Given an integer $N \geq 1$, we set $\mathbb{U}_N(z) := \{\zeta \in \mathbb{C} : \zeta^N = z\}$.

2.2. Definition.

Definition 1 (A reformulation of Motzkin-Taussky's Property L [11]).

A pair $(A, B) \in M_n(\mathbb{C})^2$ has property L when there are n linear forms $f_1, \dots, f_n$ on $\mathbb{C}^2$ such that

$$\forall (x, y) \in \mathbb{C}^2, \operatorname{OSp}(xA + yB) = [f_k(x, y)]_{1 \leq k \leq n}.$$

Using the fact that the eigenvalues are continuous functions of the coefficients, it is obvious that a pair $(A, B) \in M_n(\mathbb{C})^2$ has property L if and only if there are affine maps $f_1, \dots, f_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that

$$\forall z \in \mathbb{C}, \operatorname{OSp}(A + zB) = [f_k(z)]_{1 \leq k \leq n}.$$

2.3. Property L for pairs of matrices with an integral spectrum. We denote by $\mathcal{K}(\mathbb{C})$ the quotient field of the integral domain $H(\mathbb{C})$ of entire functions (i.e. analytic functions from $\mathbb{C}$ to $\mathbb{C}$). Considering $\operatorname{id}_{\mathbb{C}}$ as an element of $\mathcal{K}(\mathbb{C})$, we may view $A + \operatorname{id}_{\mathbb{C}} B$ as a matrix of $M_n(\mathcal{K}(\mathbb{C}))$. We define the **generic number** p of eigenvalues of the pencil $z \mapsto A + zB$ as the number of the distinct eigenvalues of $A + \operatorname{id}_{\mathbb{C}} B$ in an algebraic closure of $\mathcal{K}(\mathbb{C})$. A complex number z is called **regular** when $A + zB$ has exactly p distinct eigenvalues, and **exceptional** otherwise. In a neighborhood of 0, the spectrum of $A + zB$ may be classically described with Puiseux series as follows (see [2, chapter 7]): there exists a radius $r > 0$, an integer $q \in \{1, \dots, n\}$, positive integers $d_1, \dots, d_q$ such that $n = d_1 + \dots + d_q$, and analytic functions $f_1, \dots, f_q$ defined on a neighborhood of 0 such that

$$\forall z \in \mathbb{C} \setminus \{0\}, \quad |z| < r \Rightarrow \chi_{A+zB}(X) = \prod_{k=1}^q \prod_{\zeta \in \mathbb{U}_{d_k}(z)} (X - f_k(\zeta)).$$

We may now prove the following result.

Proposition 5. *Let $(A, B) \in M_n(\mathbb{C})^2$. Assume that $\operatorname{Sp}(kA + B) \subset \mathbb{Z}$ for every $k \in \mathbb{N}$. Then (A, B) has property L.*

Proof. With the above notation, we prove that $f_1, \dots, f_q$ are polynomial functions. For instance, consider f_1 and its power series expansion

$$f_1(z) = \sum_{j=0}^{+\infty} a_j z^j.$$

Set $N := d_1$ for convenience. Let k_0 be a positive integer such that $\frac{1}{k_0} < r$. For every integer $k \geq k_0$, $k f_1(k^{-\frac{1}{N}})$ is an eigenvalue of $kA + B$; hence it is an integer. Therefore one has: for every integer $k \geq k_0$,

$$(k+1)f_1\left((k+1)^{-\frac{1}{N}}\right) - kf_1\left(k^{-\frac{1}{N}}\right) \in \mathbb{Z}.$$

For every integer $k \geq k_0$, the following equality holds:

$$(k+1)f_1\left((k+1)^{-\frac{1}{N}}\right) - kf_1\left(k^{-\frac{1}{N}}\right) = a_0 + \sum_{j \in \mathbb{N} \setminus \{0, N\}} a_j \left((k+1)^{1-\frac{j}{N}} - k^{1-\frac{j}{N}}\right).$$

Assume that $a_j \neq 0$ for some $j \geq 1$ with $j \neq N$, and define s as the smallest such j . On the one hand, one has for every integer $j \in \mathbb{N}$,

$$(k+1)^{1-\frac{j}{N}} - k^{1-\frac{j}{N}} = k^{1-\frac{j}{N}} \left(\left(1 + \frac{1}{k}\right)^{1-\frac{j}{N}} - 1 \right) \underset{k \rightarrow +\infty}{\sim} k^{1-\frac{j}{N}} \frac{1-\frac{j}{N}}{k} = \left(1 - \frac{j}{N}\right) k^{-\frac{j}{N}}.$$

On the other hand, when $k \rightarrow +\infty$, one has

$$\begin{aligned} \sum_{j=s+N+1}^{+\infty} a_j k^{1-\frac{j}{N}} &= o\left(k^{-\frac{s}{N}}\right) \quad \text{and} \\ \sum_{j=s+N+1}^{+\infty} a_j (k+1)^{1-\frac{j}{N}} &= o\left(k^{-\frac{s}{N}}\right). \end{aligned}$$

It follows that

$$\sum_{j \in \mathbb{N} \setminus \{0, N\}} a_j \left((k+1)^{1-\frac{j}{N}} - k^{1-\frac{j}{N}} \right) \underset{k \rightarrow +\infty}{\sim} a_s \left(1 - \frac{s}{N} \right) k^{-\frac{s}{N}}.$$

The sequence $\left((k+1)f_1((k+1)^{-\frac{1}{N}}) - kf_1(k^{-\frac{1}{N}}) - a_0 \right)_{k \geq k_0}$ is discrete, converges to 0 and is not ultimately zero. This is a contradiction. Therefore $\forall j \in \mathbb{N} \setminus \{0, N\}$, $a_j = 0$. In the same way, one shows that, for every $k \in \{1, \dots, q\}$, there exists a $b_k \in \mathbb{C}$ such that $f_k(z) = f_k(0) + b_k z^{d_k}$ in a neighborhood of 0. It follows that, in a neighborhood of 0,

$$\chi_{A+zB}(X) = \prod_{k=1}^q (X - f_k(0) - b_k z)^{d_k}.$$

Therefore we found affine maps $g_1, \dots, g_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that, in a neighborhood of 0,

$$\chi_{A+zB}(X) = \prod_{k=1}^n (X - g_k(z)).$$

The coefficients of these polynomials are polynomial functions of z that coincide on a neighborhood of 0; therefore

$$\forall z \in \mathbb{C}, \chi_{A+zB}(X) = \prod_{k=1}^n (X - g_k(z)).$$

The pair (A, B) has property L, and Proposition 5 is proven. $\square$

2.4. Commutativity for subgroups of diagonalizable matrices with an integral spectrum. Given a matrix $M \in M_n(\mathbb{C})$ and an eigenvalue λ of it, recall that the **eigenprojection** of M associated to λ is the projection onto $\text{Ker}(M - \lambda I_n)^n$ alongside $\text{im}(M - \lambda I_n)^n = \sum_{\mu \in \text{Sp}(M), \mu \neq \lambda} \text{Ker}(M - \mu I_n)^n$.

Here, we derive Proposition 3 from Proposition 5. We start by explaining how Kato's proof [8, p. 85, Theorem 2.6] of the Motzkin-Taussky theorem [12] leads to the following refinement.

Theorem 6 (Refined Motzkin-Taussky theorem). *Let $(A, B) \in M_n(\mathbb{C})^2$ be a pair of matrices which satisfies property L. Assume that B is diagonalizable and that $A + z_0 B$ is diagonalizable for every exceptional point z_0 of the matrix pencil $z \mapsto A + zB$. Then $AB = BA$.*

Proof. We refer to the line of reasoning of [8, p. 85, Theorem 2.6] and explain how it may be adapted to prove Theorem 6. Denote by p the generic number of eigenvalues of $z \mapsto A + zB$, and by $f_1, \dots, f_p$ the p distinct affine maps such that

$\forall z \in \mathbb{C}$, $\mathrm{Sp}(A + zB) = \{f_1(z), \dots, f_p(z)\}$. Denote by Ω the (open) set of regular points of $z \mapsto A + zB$, i.e.

$$\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} : \exists (i, j) \in \{1, \dots, p\}^2 : i \neq j \text{ and } f_i(z) = f_j(z)\}.$$

For $z \in \Omega$ and $i \in \{1, \dots, p\}$, denote by $\Pi_i(z)$ the eigenprojection of $A + zB$ associated to the eigenvalue $f_i(z)$. Then $z \mapsto \Pi_i(z)$ is holomorphic on Ω for any $i \in \{1, \dots, p\}$ (see [8, II.1.4]). Let $z_0 \in \mathbb{C} \setminus \Omega$. Then $A + z_0B$ is diagonalizable and hence [8, p. 82, Theorem 2.3] shows that z_0 is a regular point for each map $z \mapsto \Pi_i(z)$. We deduce that the functions $(\Pi_i)_{i \leq p}$ are restrictions of entire functions. Since B is diagonalizable, these functions are bounded at infinity (see the last paragraph of [8, p. 85]) and Liouville's theorem yields that they are constant. By a classical continuity argument (see [8, II.1.4, formula (1.16)]), we deduce that each eigenprojection of B is a sum of some projections chosen among the $(\Pi_i(0))_{i \leq p}$. As B is diagonalizable, it is a linear combination of the $(\Pi_i(0))_{i \leq p}$, which all commute with $A + zB$ for any regular z . Therefore $AB = BA$. $\square$

We now turn to the proof of Proposition 3.

Proof of Proposition 3. Let $(A, B) \in M_n(\mathbb{C})^2$. Assume that, for every $(k, l) \in \mathbb{Z}^2$, the matrix $kA + lB$ is diagonalizable and $\mathrm{Sp}(kA + lB) \subset \mathbb{Z}$. Proposition 5 then shows that (A, B) has property L. For $k \in \{1, \dots, n\}$, choose $f_k : (y, z) \mapsto \alpha_k y + \beta_k z$ such that

$$\forall (y, z) \in \mathbb{C}^2, \mathrm{OSp}(yA + zB) = [f_k(y, z)]_{1 \leq k \leq n}.$$

Since $\mathrm{Sp}(A) = \{\alpha_1, \dots, \alpha_n\}$ and $\mathrm{Sp}(B) = \{\beta_1, \dots, \beta_n\}$, the families $(\alpha_k)_{k \leq n}$ and $(\beta_k)_{k \leq n}$ are made of integers. It follows that the exceptional points of the matrix pencil $z \mapsto A + zB$ are rational numbers. As the matrix $A + \frac{l}{k}B = \frac{1}{k}(kA + lB)$ is diagonalizable for every $(k, l) \in (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z}$, the refined Motzkin-Taussky theorem implies that $AB = BA$. $\square$

We now deduce the following special case of Theorem 1.

Lemma 7. *Let $(A, B) \in M_n(\mathbb{C})^2$ be such that for all $(k, l) \in \mathbb{Z}^2$, $e^{kA+lB} = I_n$. Then $AB = BA$.*

Proof. Recall that the solutions of the equation $e^M = I_n$ are the diagonalizable matrices M such that $\mathrm{Sp}(M) \subset 2i\pi\mathbb{Z}$ (see [5, Theorem 1.27]). In particular, for every $(k, l) \in \mathbb{Z}^2$, the matrix $kA + lB$ is diagonalizable and $\mathrm{Sp}(kA + lB) \subset 2i\pi\mathbb{Z}$. Setting $A' := \frac{1}{2i\pi}A$ and $B' := \frac{1}{2i\pi}B$, we deduce that (A', B') satisfies the assumptions of Proposition 3. It follows that $A'B' = B'A'$, and hence $AB = BA$. $\square$

3. THE CASE $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ AND $\mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$ IN THEOREM 1

Proposition 8. *Let $(A, B) \in M_n(\mathbb{C})^2$ be such that $\forall (k, l) \in \mathbb{Z}^2$, $e^{kA+lB} = e^{kA}e^{lB}$, $\mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ and $\mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$. Then $AB = BA$.*

Proof. We consider the Jordan-Chevalley decompositions $A = D + N$ and $B = D' + N'$, where D and D' are diagonalizable, N and N' are nilpotent and $DN = ND$ and $D'N' = N'D'$. Clearly, for every integer k , $kA = kD + kN$ (resp. $kB = kD' + kN'$) is the Jordan-Chevalley decomposition of kA (resp. of kB), and $\mathrm{Sp}(kD) = \mathrm{Sp}(kA) = k \mathrm{Sp}(A) \subset 2i\pi\mathbb{Z}$ (resp. $\mathrm{Sp}(kD') = \mathrm{Sp}(kB) = k \mathrm{Sp}(B) \subset 2i\pi\mathbb{Z}$). This shows that

$$e^{kA} = e^{kN} \quad \text{and} \quad e^{kB} = e^{kN'}.$$

Condition (6) may be written as

$$\forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kN} e^{lN'} = e^{lN'} e^{kN}.$$

Note in particular that e^N and $e^{N'}$ commute. Since N is nilpotent, we have

$$N = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} (e^N - I_n)^k.$$

That shows that N is a polynomial in e^N . Similarly N' is a polynomial in $e^{N'}$. Therefore,

$$NN' = N'N.$$

The above condition yields

$$\forall (k, l) \in \mathbb{Z}^2, \quad e^{kA+lB} = e^{kN+lN'}.$$

For any $(k, l) \in \mathbb{Z}^2$, $kN + lN'$ is nilpotent since N and N' are commuting nilpotent matrices. Hence $kN + lN'$ is a polynomial in $e^{kN+lN'}$. Since $kA + lB$ commutes with e^{kA+lB} , it commutes with $kN + lN'$. Therefore

$$e^{kD+lD'} = e^{kA+lB} e^{-kN-lN'} = I_n.$$

In particular, this yields that $kD + lD'$ is diagonalizable with $\text{Sp}(kD + lD') \subset 2i\pi\mathbb{Z}$, and the Jordan-Chevalley decomposition of $kA + lB$ is $kA + lB = (kD + lD') + (kN + lN')$ as $kN + lN'$ commutes with $kA + lB$.

By Lemma 7, the matrices D and D' commute. In particular (D, D') has property L, which yields affine maps $f_1, \dots, f_n$ from $\mathbb{C}$ to $\mathbb{C}$ such that

$$\forall z \in \mathbb{C}, \quad \text{OSp}(D + zD') = [f_k(z)]_{1 \leq k \leq n}.$$

The set

$$E := \{k \in \mathbb{Z} : \exists (i, j) \in \{1, \dots, n\}^2 : f_i \neq f_j \text{ and } f_i(k) = f_j(k)\}$$

is clearly finite. We may choose two distinct elements a and b in $\mathbb{Z} \setminus E$. The following equivalence holds:

$$\forall (i, j) \in \{1, \dots, n\}^2, \quad f_i(a) = f_j(a) \Leftrightarrow f_i = f_j \Leftrightarrow f_i(b) = f_j(b).$$

Since D and D' are simultaneously diagonalizable, it easily follows that $D + aD'$ is a polynomial in $D + bD'$ and conversely $D + bD'$ is a polynomial in $D + aD'$. Hence $N + aN'$ and $N + bN'$ both commute with $D + aD'$ and $D + bD'$. Since $N + aN'$ and $N + bN'$ commute with one another, we deduce that $A + aB = (D + aD') + (N + aN')$ commutes with $A + bB = (D + bD') + (N + bN')$. Since $a \neq b$, we conclude that $AB = BA$. $\square$

4. PROOFS OF THEOREM 1 AND PROPOSITION 4

Definition 2. Let $(A, B) \in M_n(\mathbb{C})^2$.

- i) (A, B) is said to be decomposable if there exists a non-trivial decomposition $\mathbb{C}^n = F \oplus G$ in which F and G are invariant linear subspaces for both A and B .
- ii) In the sequel, we consider, for $k \in \mathbb{N} \setminus \{0\}$, the function

$$\gamma_k : (\lambda, \mu) \in \text{Sp}(e^A) \times \text{Sp}(e^B) \mapsto \lambda^k \mu \in \mathbb{C}.$$

- iii) For $\lambda \in \mathbb{C}$, we denote by $C_\lambda(M)$ the characteristic subspace of M with respect to λ , i.e. $C_\lambda(M) = \text{Ker}(M - \lambda I_n)^n$.

Lemma 9. *Assume that A satisfies Condition*

$$(7) \quad \forall (\lambda, \mu) \in \mathrm{Sp}(A)^2, \lambda - \mu \in 2i\pi\mathbb{Q} \Rightarrow \lambda - \mu \in 2i\pi\mathbb{Z}.$$

Then there exists $k \in \mathbb{N} \setminus \{0\}$ such that γ_k is one-to-one.

Proof. Assume that for every $k \in \mathbb{N} \setminus \{0\}$, there are distinct pairs (λ, μ) and (λ', μ') in $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ such that $\lambda^k \mu = (\lambda')^k \mu'$. Since $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ is finite and $\mathbb{N} \setminus \{0\}$ is infinite, we may then find distinct pairs (λ, μ) and (λ', μ') in $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$ and distinct non-zero integers a and b such that

$$\lambda^a \mu = (\lambda')^a \mu' \quad \text{and} \quad \lambda^b \mu = (\lambda')^b \mu'.$$

All those eigenvalues are non-zero and $(\frac{\lambda}{\lambda'})^{a-b} = 1$ with $a \neq b$. It follows that $\frac{\lambda}{\lambda'}$ is a root of unity. However $\lambda = e^\alpha$ and $\lambda' = e^\beta$ for some $(\alpha, \beta) \in \mathrm{Sp}(A)^2$, which shows that $(a-b)(\alpha-\beta) \in 2i\pi\mathbb{Z}$. Condition (7) yields $\alpha - \beta \in 2i\pi\mathbb{Z}$; hence $\lambda = \lambda'$. It follows that $\mu = \mu'$, in contradiction with $(\lambda, \mu) \neq (\lambda', \mu')$. $\square$

Lemma 10. *Assume that γ_1 is one-to-one and that (A, B) satisfies Equality (5) (resp. Equality (4)). Then the characteristic subspaces of e^A and e^B are stabilized by A and B .*

Proof. Notice that $A + B$ commutes with e^{A+B} , hence commutes with $e^A e^B$. It thus stabilizes the characteristic subspaces of $e^A e^B$. Let us show that

$$(8) \quad \forall \mu \in \mathrm{Sp}(e^B), C_\mu(e^B) = \bigoplus_{\lambda \in \mathrm{Sp}(e^A)} C_{\lambda\mu}(e^A e^B).$$

• Since e^B and e^A commute, e^A stabilizes the characteristic subspaces of e^B . Considering the characteristic subspaces of the endomorphism of $C_\mu(e^B)$ induced by e^A , we find

$$\forall \mu \in \mathrm{Sp}(e^B), C_\mu(e^B) = \bigoplus_{\lambda \in \mathrm{Sp}(e^A)} [C_\lambda(e^A) \cap C_\mu(e^B)].$$

• Let $(\lambda, \mu) \in \mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$. Since e^A and e^B commute, they both stabilize $C_\lambda(e^A) \cap C_\mu(e^B)$ and induce simultaneously triangularizable endomorphisms of $C_\lambda(e^A) \cap C_\mu(e^B)$ each with a sole eigenvalue, respectively λ and μ : it follows that

$$C_\lambda(e^A) \cap C_\mu(e^B) \subset C_{\lambda\mu}(e^A e^B).$$

• Finally, the application $(\lambda, \mu) \mapsto \lambda\mu$ is one-to-one on $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$. Therefore

$$C_{\lambda\mu}(e^A e^B) \cap C_{\lambda'\mu'}(e^A e^B) = \{0\}$$

for all distinct pairs (λ, μ) and (λ', μ') in $\mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B)$.

One has

$$\mathbb{C}^n = \bigoplus_{\mu \in \mathrm{Sp}(e^B)} C_\mu(e^B) = \bigoplus_{\mu \in \mathrm{Sp}(e^B)} \bigoplus_{\lambda \in \mathrm{Sp}(e^A)} [C_\lambda(e^A) \cap C_\mu(e^B)]$$

and $\mathbb{C}^n$ is the sum of all the characteristic subspaces of $e^A e^B$. We deduce that

$$\forall (\lambda, \mu) \in \mathrm{Sp}(e^A) \times \mathrm{Sp}(e^B), C_{\lambda\mu}(e^A e^B) = C_\lambda(e^A) \cap C_\mu(e^B).$$

This gives Equality (8).

We deduce that $A + B$ stabilizes every characteristic subspace of e^B . However this is also true of B since it commutes with e^B . Hence both A and B stabilize the characteristic subspaces of e^B . Symmetrically, every characteristic subspace of e^A is stabilized by both A and B . $\square$

Proof of Theorem 1 and Proposition 4. We use an induction on n . Both Theorem 1 and Proposition 4 obviously hold for $n = 1$, so we fix $n \geq 2$ and assume that they hold for any pair $(A, B) \in M_k(\mathbb{C})^2$ with $k \in \{1, \dots, n-1\}$. Let $(A, B) \in M_n(\mathbb{C})^2$ satisfy Equality (5) (resp. Equality (4)). Assume first that (A, B) is decomposable. Then there exists $p \in \{1, \dots, n-1\}$, a non-singular matrix $P \in GL_n(\mathbb{C})$ and square matrices A_1, B_1, A_2, B_2 respectively in $M_p(\mathbb{C})$, in $M_p(\mathbb{C})$, in $M_{n-p}(\mathbb{C})$ and in $M_{n-p}(\mathbb{C})$ such that

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P^{-1} \quad \text{and} \quad B = P \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} P^{-1}.$$

Since the pair (A, B) satisfies Equality (5) (resp. Equality (4)), it easily follows that this is also the case of (A_1, B_1) and (A_2, B_2) ; hence the induction hypothesis yields that (A_1, B_1) and (A_2, B_2) are commuting pairs (resp. have property L). Therefore (A, B) is also a commuting pair (resp. has property L).

From that point on, we assume that (A, B) is indecomposable. We may also assume that A satisfies Condition (7). Indeed, consider in general the finite set

$$\mathcal{E} := \mathbb{Q} \cap \frac{1}{2i\pi} \{ \lambda - \mu \mid (\lambda, \mu) \in \text{Sp}(A)^2 \}.$$

Since its elements are rational numbers, we may find some integer $p > 0$ such that $p\mathcal{E} \subset \mathbb{Z}$. Replacing A with pA , we notice that (pA, B) still satisfies Equality (5) (resp. Equality (4)) and that it is a commuting pair (resp. satisfies property L) if and only if (A, B) is a commuting pair (resp. satisfies property L).

Assume now that A satisfies Condition (7) as well as all the previous assumptions; i.e. (A, B) is indecomposable and satisfies Equality (5) (resp. Equality (4)). By Lemma 9, we may choose $k \in \mathbb{N} \setminus \{0\}$ such that γ_k is one-to-one. Replacing A with kA , we lose no generality in assuming that γ_1 is one-to-one.

We can conclude: if e^B has several eigenvalues, Lemma 10 contradicts the assumption that (A, B) is indecomposable. It follows that e^B has a sole eigenvalue, and for the same reason this is also true of e^A . Choosing $(\alpha, \beta) \in \mathbb{C}^2$ such that $\text{Sp}(e^A) = \{e^\alpha\}$ and $\text{Sp}(e^B) = \{e^\beta\}$, we find that $\exp(A - \alpha I_n)$ and $\exp(B - \beta I_n)$ both have 1 as their sole eigenvalue. We deduce that $\text{Sp}(A - \alpha I_n) \subset 2i\pi\mathbb{Z}$ and $\text{Sp}(B - \beta I_n) \subset 2i\pi\mathbb{Z}$. Set $A' := A - \alpha I_n$ and $B' := B - \beta I_n$. We now conclude the proofs of Theorem 1 and Proposition 4 by considering the two cases separately.

• **Case 1.** (A, B) satisfies Equality (5). The pair (A', B') clearly satisfies Equality (5). Proposition 8 yields that A' commutes with B' ; hence $AB = BA$.

• **Case 2.** (A, B) satisfies Equality (4). The pair (A', B') obviously satisfies Equality (4). The matrices $e^{A'}$ and $e^{B'}$ commute and are therefore simultaneously triangularizable (see [13, Theorem 1.1.5]). Moreover, they have 1 as their sole eigenvalue. Therefore $e^{kA' + B'} = (e^{A'})^k e^{B'}$ has 1 as sole eigenvalue for every $k \in \mathbb{N}$. Proposition 5 shows that $(\frac{1}{2i\pi}A', \frac{1}{2i\pi}B')$ has property L, which clearly entails that (A, B) has property L.

Thus Theorem 1 and Proposition 4 are proven. $\square$

5. ADDITIVE SEMIGROUPS ON WHICH THE EXPONENTIAL IS A HOMOMORPHISM

Notation 3. We denote by $\mathbb{Q}_+$ the set of non-negative rational numbers.

Definition 4. A linear subspace V of $M_n(\mathbb{C})$ has property L when there are n linear forms $f_1, \dots, f_n$ on V such that

$$\forall M \in V, \text{OSp}(M) = [f_k(M)]_{1 \leq k \leq n}.$$

In this short section, we prove the following result.

Theorem 11. *Let S be a subsemigroup of $(M_n(\mathbb{C}), +)$ and assume that $M \mapsto \exp(M)$ is a homomorphism from $(S, +)$ to $(\text{GL}_n(\mathbb{C}), \times)$. Then $\text{span}(S)$ has property L.*

By Proposition 4, it suffices to establish the following lemma.

Lemma 12. *Let S be a subsemigroup of $(M_n(\mathbb{C}), +)$. Assume that every pair $(A, B) \in S^2$ has property L. Then the linear subspace $\text{span}(S)$ has property L.*

Proof. Let $(A_1, \dots, A_r)$ be a basis of $\text{span}(S)$ formed of elements of S . For every $j \in \{1, \dots, r\}$, we choose a list $(a_1^{(j)}, \dots, a_n^{(j)}) \in \mathbb{C}^n$ such that

$$\text{OSp}(A_j) = [a_k^{(j)}]_{1 \leq k \leq n}.$$

Since, for every $(p_1, \dots, p_r) \in \mathbb{N}^r$, the pair $\left(\sum_{k=1}^{j-1} p_k A_k, A_j\right)$ has property L for every $j \in \{2, \dots, r\}$, by induction we obtain a list $(\sigma_1, \dots, \sigma_r) \in (\Sigma_n)^r$ such that

$$\text{OSp}\left(\sum_{j=1}^r p_j A_j\right) = \left[\sum_{j=1}^r p_j a_{\sigma_j(k)}^{(j)}\right]_{1 \leq k \leq n}.$$

Multiplying by inverses of positive integers, we readily generalize this as follows: for every $(z_1, \dots, z_r) \in (\mathbb{Q}_+)^r$, there exists a list $(\sigma_1, \dots, \sigma_r) \in (\Sigma_n)^r$ such that

$$\text{OSp}\left(\sum_{j=1}^r z_j A_j\right) = \left[\sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)}\right]_{1 \leq k \leq n}.$$

Now, we prove the following property, depending on $l \in \{0, \dots, r\}$, by downward induction:

$\mathcal{P}(l)$: For every $(z_1, \dots, z_l) \in (\mathbb{Q}_+)^l$, there exists a list $(\sigma_1, \dots, \sigma_r) \in (\Sigma_n)^r$ satisfying

$$\forall (z_{l+1}, \dots, z_r) \in \mathbb{C}^{r-l}, \text{OSp}\left(\sum_{j=1}^r z_j A_j\right) = \left[\sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)}\right]_{1 \leq k \leq n}.$$

In particular, $\mathcal{P}(r)$ is precisely what we have just proven, whilst $\mathcal{P}(0)$ means that there exists a list $(\sigma_1, \dots, \sigma_r) \in (\Sigma_n)^r$ such that, for every $(z_1, \dots, z_r) \in \mathbb{C}^r$,

$$\text{OSp}\left(\sum_{j=1}^r z_j A_j\right) = \left[\sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)}\right]_{1 \leq k \leq n}.$$

Therefore $\mathcal{P}(0)$ implies that $\text{span}(S)$ has property L.

Let $l \in \{1, \dots, r\}$ be such that $\mathcal{P}(l)$ holds, and fix $(z_1, \dots, z_{l-1}) \in (\mathbb{Q}_+)^{l-1}$. By $\mathcal{P}(l)$, for every $z_l \in \mathbb{Q}_+$, we may choose a list $(\sigma_1^{z_l}, \dots, \sigma_r^{z_l}) \in (\Sigma_n)^r$ such that

$$\forall (z_{l+1}, \dots, z_r) \in \mathbb{C}^{r-l}, \text{OSp}\left(\sum_{j=1}^r z_j A_j\right) = \left[\sum_{j=1}^r z_j a_{\sigma_j^{z_l}(k)}^{(j)}\right]_{1 \leq k \leq n}.$$

Since $(\Sigma_n)^r$ is finite and $\mathbb{Q}_+ \cap (0, 1)$ is infinite, some list $(\sigma_1, \dots, \sigma_r) \in (\Sigma_n)^r$ equals $(\sigma_1^{z_l}, \dots, \sigma_r^{z_l})$ for infinitely many values of z_l in $\mathbb{Q}_+ \cap (0, 1)$. Fixing $(z_{l+1}, \dots, z_r) \in \mathbb{C}^{r-l}$, we deduce the identity

$$\forall z_l \in \mathbb{C}, \quad \chi_{\sum_{j=1}^r z_j A_j}(X) = \prod_{k=1}^n \left(X - \sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)}\right)$$

by remarking that, on both sides, the coefficients of the polynomials are polynomials in z_l . Hence

$$\forall (z_l, \dots, z_r) \in \mathbb{C}^{r-l+1}, \text{OSp}\left(\sum_{j=1}^r z_j A_j\right) = \left[\sum_{j=1}^r z_j a_{\sigma_j(k)}^{(j)}\right]_{1 \leq k \leq n}.$$

This proves that $\mathcal{P}(l-1)$ holds. $\square$

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