DIFFERENTIABILITY, POROSITY AND DOUBLING IN METRIC MEASURE SPACES

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ABSTRACT. We show that if a metric measure space admits a differentiable structure, then porous sets have measure zero and hence the measure is pointwise doubling. We then give a construction to show that if we require only an approximate differentiable structure, the measure need no longer be pointwise doubling.

1. BASIC DEFINITIONS

Let (X, d) be a complete separable metric space equipped with a locally finite Borel measure μ . We begin by recalling the notion of a differentiable structure on (X, d, μ) introduced by Cheeger [1]. We call a metric measure space which admits such a structure a differentiability space and define a weaker notion of an approximate differentiability space.

Definition 1.1. If $U \subset X$ is Borel measurable and $\varphi: X \rightarrow \mathbb{R}^n$ is Lipschitz, we say that (U, φ) is a chart on (X, d, μ) of dimension $n \in \mathbb{N}$. A map $f: X \rightarrow \mathbb{R}$ is differentiable with respect to a chart (U, φ) of dimension n at $x_0 \in U$ if there exists a unique derivative $df(x_0) \in \mathbb{R}^n$ (depending on the chart) such that

$$\limsup_{X \ni x \rightarrow x_0} \frac{|f(x) - f(x_0) - df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

We say that (X, d, μ) is a differentiability space if there exists a countable decomposition of X into charts so that any Lipschitz $f: X \rightarrow \mathbb{R}$ is differentiable at almost every point of every chart.

We say that (X, d, μ) is an approximate differentiability space if the above holds but with the limit replaced by an approximate limit. We say that a function $g: X \rightarrow \mathbb{R}$ has an approximate limit $l \in \mathbb{R}$ at $x_0 \in X$ if for every $\varepsilon > 0$,

$$\lim_{r \downarrow 0} \frac{\mu\{x \in B(x_0, r) : |g(x) - l| > \varepsilon\}}{\mu(B(x_0, r))} = 0.$$

We observe that since (X, d) is separable for almost every x_0 , we have $\mu(B(x_0, r)) > 0$ for all $r > 0$.

Remark 1.2. Let B be a closed ball in $\mathbb{R}^n$. Observe that a function f has a derivative in B at x_0 if and only if for every $\epsilon \in \mathbb{Q}_+$ there exists a $\delta \in \mathbb{Q}_+$ and $q \in B$ with

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rational coordinates so that

$$\sup_{d(x, x_0) < \delta} \frac{|f(x) - f(x_0) - q \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} < \epsilon.$$

However, the set of x_0 that satisfy this for a particular q, δ and ϵ is an open subset of X . Thus, by taking countable unions and intersections over δ, ϵ and $q \in B$, we see that the set of points where the derivative of f exists and lies in B is a Borel set. Thus df is Borel measurable and the set of points of differentiability is also Borel.

When we are interested in approximate differentiability a similar decomposition yields the set of $x_0 \in X$ so that

$$\mu\{x \in B(x_0, \delta) : \frac{|f(x) - f(x_0) - q \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} < \epsilon\} > \eta \mu(B(x_0, \delta))$$

for some $\delta, \epsilon, \eta \in \mathbb{Q}_+$ and rational q . This is also an open set, and so countable intersections and unions over δ, ϵ, η and $q \in B$ also show that the approximate derivative exists on a Borel set and is a Borel function.

Examples of differentiability spaces include Euclidean spaces, the Heisenberg group [1] and Laakso spaces [2]. As Keith remarks, rectifiable sets in metric spaces are approximate differentiability spaces with Hausdorff measure [3]. Keith has shown that if the measure μ on X is globally doubling, namely balls have finite positive measure and there exists $C \geq 1$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X$ and $r > 0$, then the notions of differentiability space and approximate differentiability space coincide [4].

Definition 1.3. For $\eta > 0$ we say that a set $S \subset X$ is η -porous at $x_0 \in S$ if there exist $x_n \rightarrow x_0$ with

$$d(x_n, S) > \eta d(x_n, x_0)$$

and that such an x_n is a witness of porosity for x_0 . Further we call S porous if it is porous at each of its points and define for each $x_0 \in S$,

$$\eta(x_0) = \sup\{\eta : S \text{ is } 2\eta \text{ porous at } x_0\}$$

so that for every $x_0 \in S$, S is $\eta(x_0)$ -porous at x_0 . Finally, for $r > 0$, we define the function $\sigma_r : S \rightarrow \mathbb{R}$ by

$$\sigma_r(x_0) = \sup\{d(x, S) < r : x \text{ is a } \eta(x_0)\text{-witness of porosity for } x_0\}.$$

Remark 1.4. Observe that for a porous set S and $C > 0$, $\eta(x_0) > C$ if and only if there exists an $m \in \mathbb{N}$ and for every $n \in \mathbb{N}$ an $x \in X$ with $d(x, x_0) < 1/n$ and

$$d(x, S) > (2C + 1/m)d(x, x_0).$$

This final condition is open in x_0 , and so an arbitrary union over $x \in X$ followed by a countable intersection over n and union over m shows that $x_0 \mapsto \eta(x_0)$ is a Borel function. For any $r > 0$ a similar decomposition shows that σ_r is also Borel.

Definition 1.5. We say that μ is pointwise doubling at $x \in X$ if

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty.$$

Note that an equivalent condition is obtained if 2 is replaced by any other enlargement factor greater than one.

We will see that porous sets in differentiability spaces necessarily have measure zero (Theorem 2.4) and thus the underlying measure is pointwise doubling almost everywhere (Corollary 2.6). We then show that any approximate differentiability space which gives measure zero to porous sets is necessarily a differentiability space (Proposition 2.8). Finally, we show, by construction of an example, that the measure underlying an approximate differentiability space need not be pointwise doubling in a set of positive measure (Section 3).

2. POROUS SETS IN DIFFERENTIABILITY SPACES

We begin with a characterisation of the uniqueness of derivatives with respect to a chart.

Lemma 2.1. *Let (U, φ) be an n -dimensional chart in a metric measure space (X, d, μ) and $x_0 \in U$. Given $f: X \rightarrow \mathbb{R}$, a derivative of f with respect to (U, φ) at x_0 (if one exists) is unique if and only if there exists a $\lambda > 0$ and $X \ni x_m \rightarrow x_0$ so that for any $v \in \mathbb{R}^n$,*

$$(2.1) \quad \lim_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \geq \lambda \|v\|.$$

In particular, the uniqueness of derivatives at x_0 depends only on the chart and is independent of the function we differentiate.

Remark 2.2. Notice that if the derivatives at x_0 are unique, then for any $f: X \rightarrow \mathbb{R}$ with derivative $df(x_0)$ at x_0 we have

$$\liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i}, x_0)} \geq \lambda \|df(x_0)\|.$$

Remark 2.3. Note that any chart (U, φ) with respect to which Lipschitz functions have a nonunique derivative may be decomposed into a finite union of new charts (U_i, φ_i) with respect to which derivatives exist and are unique.

To see this, observe that if derivatives at $x_0 \in U$ are not unique, then there exists $D \neq 0$ such that

$$\limsup_{x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot D|}{d(x, x_0)} = 0.$$

Thus we can write, up to first order at x_0 , one of the coordinates of $(\varphi(x) - \varphi(x_0))$ as a linear combination of the others. This implies that we can omit this coordinate from φ and Lipschitz maps will still have derivatives at x_0 .

Repeatedly decompose the points of U where nonuniqueness occurs into subsets corresponding to which coordinates of φ should be omitted. If at some stage no more coordinates can be omitted at a point x_0 , then the derivatives there are unique. This process terminates as if all coordinates of φ are omitted (so φ is identically zero) and derivatives are automatically unique.

We thus obtain a finite collection of new charts (U_i, φ_i) with respect to which derivatives exist and are unique. The chart map φ_i in each of these new charts is a subcollection (possibly empty) of the coordinates of the original map φ .

Proof. First suppose that the conclusion holds and that a function $f: X \rightarrow \mathbb{R}$ has two derivatives D and $D' \in \mathbb{R}^n$. Then by the definition of the derivative and the

triangle inequality we see that

$$\limsup_{M \ni x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot (D - D')|}{d(x, x_0)} = 0$$

and so $\|D - D'\| = 0$.

To show the converse observe that, since φ is Lipschitz, the left-hand side of (2.1) is homogeneous and continuous in v , regardless of our choice of x_m . Therefore it suffices to show the existence of a sequence so that (2.1) gives positive values to nonzero v . For this we find a sequence x_m so that for every $0 \leq i < n$,

$$\lim_{m \rightarrow \infty} \frac{\varphi(x_{mn+i}) - \varphi(x_0)}{d(x_{mn+i}, x_0)}$$

exists and forms a basis of $\mathbb{R}^n$.

Suppose no such sequence exists. Then for every sequence so that the above limits exist the limit vectors must belong to some $(n-1)$ -dimensional subspace V of $\mathbb{R}^n$. Then for $v^\perp \neq 0$ orthogonal to V we have

$$\limsup_{X \ni x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v^\perp|}{d(x, x_0)} = 0,$$

and so for any $f: X \rightarrow \mathbb{R}$ with derivative $df(x_0)$ at x_0 , $df(x_0) + v^\perp$ is another derivative, contradicting our hypothesis. $\square$

We may now prove our first result.

Theorem 2.4. *Any porous set S in a Lipschitz differentiability space is null.*

Proof. It suffices to show that for an n -dimensional chart (U, φ) any porous $S \subset U$ is null. Given such an S , σ_r is a Borel function and so there exists an $S' \subset S$ with $\mu(S') \geq \mu(S)/2$ and a sequence $r_m \rightarrow 0$ monotonically so that for every $x_0 \in S'$ there exist $\eta(x_0)$ -witnesses y_m with

$$(2.2) \quad r_{m+1} < d(y_m, S)/2 < 3d(y_m, S)/2 < r_m.$$

In particular for any $y \in B(y_m, d(y_m, S)/2)$ we have

$$r_{m+1} < d(y, S) < r_m.$$

We first partition these regions of $X \setminus S$ into $n+1$ cases that will allow us to define nondifferentiable Lipschitz functions. For $0 \leq i < n+1$ let R_i be the set of $x \in X$ with

$$d(x, S) \in \bigcup_{m \in \mathbb{N}} (r_{m(n+1)+i+1}, r_{m(n+1)+i}).$$

Then, for $y \notin S$, define the Lipschitz map $g_y: X \rightarrow \mathbb{R}$ by

$$g_y(x) = \max(d(y, S)/2 - d(y, x), 0),$$

a “cone” centred at y of radius $d(y, S)/2$. Finally for $0 \leq i < n+1$ define f_i to be the 1-Lipschitz map

$$f_i = \sup\{g_y: y \text{ an } \eta(x_0)\text{-witness of some } x_0 \in S', y \in R_i \text{ satisfies (2.2) for some } m\}.$$

Note that by the above observation this satisfies $f_i(x) = 0$ for any $x \notin R_i$. We may now partition S' into sets of nondifferentiability for some f_i .

For almost every $x_0 \in S'$ let $x_m \rightarrow x_0$ be as in the conclusion of Lemma 2.1 for x_0 . Then by the pigeon-hole principle, there exists an i and $m_k \rightarrow \infty$ so that

for every $k \in \mathbb{N}$ and $0 \leq j < n$ either $d(x_{m_k n+j}, S) = 0$ or $x_{m_k n+j} \notin R_i$. Thus $f(x_{m_k n+j}) = f(x_0) = 0$ and so by Lemma 2.1, $df_i(x_0) = 0$ if it exists.

In addition, since $x_0 \in S'$, we may find for any m an $\eta(x_0)$ -witness x of x_0 satisfying (2.2) and so $\limsup |f(x) - f(x_0)|/d(x, x_0) \geq \eta(x_0)/2$. This would require $df(x_0) \neq 0$ and so f_i cannot be differentiable at x_0 .

We have therefore decomposed S' into a finite union of sets of nondifferentiability and a set of nonuniqueness. Therefore S' and hence S are null. $\square$

Remark 2.5. If $S \subset \mathbb{R}^n$ is porous, then the distance function $d(x, S)$ provides a very easy example of a Lipschitz map differentiable nowhere on S . However in the more general setting such a function may be a coordinate chart and so the more involved construction given above is required. To see this take $S \subset \mathbb{R}$ to be a nonempty, closed null set. We show that $\mathbb{R}$ with coordinate chart $d(x, S)$ is a Lipschitz differentiability space.

Let (a, b) be a connected component of $\mathbb{R} \setminus S$, so that at most one of a or b is infinite. Then every $x_0 \in ((a+b)/2, b)$ has a neighbourhood where $d(x, S) - d(x_0, S) = x - x_0$, and so if a Lipschitz function f has Euclidean derivative df at x_0 , f also has derivative df with respect to d and vice versa. Similarly every $x_0 \in (a, (a+b)/2)$ has a neighbourhood where $d(x, S) - d(x_0, S) = -(x - x_0)$, and so such an f has derivative $-df$ with respect to d . Thus any Lipschitz function has a unique derivative almost everywhere with respect to $d(x, S)$.

Corollary 2.6. *Theorem 3.6(iv) of [5] gives a decomposition $X = P \cup S \cup N$, where S is σ -porous, N is null and μ is pointwise doubling at any point of P . Thus a Lipschitz differentiability space is pointwise doubling almost everywhere. Thus, by the standard proof, (X, d, μ) is a Vitali space.*

Corollary 2.7. *Any subset S of a Lipschitz differentiability space X is also a Lipschitz differentiability space with respect to the charts obtained by restricting charts for X to S . Moreover, the derivative of any Lipschitz $f: S \rightarrow \mathbb{R}$ agrees with the derivative of any extension $f: X \rightarrow \mathbb{R}$ almost everywhere in S .*

Proof. Let (U, φ) be a chart in X and suppose that $S \subset U$. Any Lipschitz map $f: S \rightarrow \mathbb{R}$ extends to a Lipschitz map on X and so we have a candidate for the derivative of f at almost every point of S ; we must just check for uniqueness. As seen above, this is a property of S and nonuniqueness occurs at any x_0 such that

$$\limsup_{S \ni x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot D|}{d(x, x_0)} = 0$$

for some $D \in \mathbb{R}^n$ with $D \neq 0$. However for almost every such x_0 the uniqueness of derivatives in X provides a sequence $x_m \rightarrow x_0$ in X and an $\eta > 0$ so that

$$\frac{|(\varphi(x_m) - \varphi(x_0)) \cdot D|}{d(x_m, x_0)} > \eta.$$

Thus, for sufficiently large m , the ball of radius $\eta d(x_m, x_0)/(2|D| \text{Lip } \varphi)$ around x_m does not belong to S so that the set of such x_0 is porous and so null. Thus S is a Lipschitz differentiability space. $\square$

Since any differentiability space is also an approximate differentiability space we see that admitting an approximate differentiable structure and porous sets being null is a necessary condition for a space to admit a differentiable structure. The following proposition shows this condition is also sufficient.

Proposition 2.8. *Suppose (X, d, μ) is a metric measure space in which porous sets have measure zero. Then for almost every $x_0 \in X$ we have that whenever $g: X \rightarrow \mathbb{R}$ is a Lipschitz function such that*

$$\operatorname{aplim}_{x \rightarrow x_0} g(x)/d(x, x_0) = 0,$$

then

$$\lim_{x \rightarrow x_0} g(x)/d(x, x_0) = 0.$$

Proof. In Proposition 3.5 of [4], Keith proves a variant of this result under the different hypothesis that μ is globally doubling. An analysis of his proof shows the weaker condition that for almost every x and all $\varepsilon > 0$,

$$\liminf_{y \rightarrow x} \frac{\mu(B(y, \varepsilon d(y, x)))}{\mu(B(x, 2d(y, x)))} > 0$$

suffices. As in [5] let $\gamma(\mu, x, r, \delta)$ be the supremum of the set

$$\{s > 0 : \exists z \text{ such that } d(x, z) + s \leq r \text{ and } \mu(B(z, s)) \leq \delta \mu(B(x, r))\}$$

and define the upper porosity of μ at x by

$$\overline{\operatorname{por}}(\mu, x) = \lim_{\delta \downarrow 0} \limsup_{r \downarrow 0} \frac{\gamma(\mu, x, r, \delta)}{r}.$$

By Theorem 3.6 of [5], if porous sets have measure zero, then $\overline{\operatorname{por}}(\mu, x) = 0$ for almost every x . This immediately implies the condition which suffices for Keith's proof. $\square$

3. A NONDOUBLING APPROXIMATE DIFFERENTIABILITY SPACE

It is easy to see that the measure in approximate differentiability spaces need not give measure zero to porous sets. Indeed, $\mathbb{R}^2$ with one-dimensional Hausdorff measure restricted to a line is an example of an approximate differentiability space which gives full measure to a porous set. We show, by construction of a counterexample, that the measure underlying an approximate differentiability space need not even be pointwise doubling.

The space we construct is based on Laakso's spaces [2]. We start with a simple base space, in which distinct points are not connected by (continuous) paths, equipped with a nondoubling measure. We then take the product of this space with an interval and Lebesgue measure. We identify different points in the base space at different heights on the interval so as to make the quotient space path connected, and we equip it with the path metric. Since we cannot move horizontally inside the base space and have to travel vertically to move between points, the quotient acquires a differentiable structure somewhat like Euclidean space. Due to the choice of identifications the product measure is still nondoubling at almost every point.

For clarity we will work with a pseudometric, which satisfies the requirements of a metric except distances between distinct points may be 0, rather than a metric on the quotient space. The following remark explains how these are equivalent.

Remark 3.1. Suppose a set M is equipped with a pseudometric d' . Then we can define balls, Borel sets and so on in (M, d') as in any metric space. Suppose (M, d') is complete, separable and equipped with a locally finite Borel measure μ . Then the notions of differentiability space and approximate differentiability space also make sense for (M, d') .

Define an equivalence relation on (M, d') by $x \sim y$ if $d'(x, y) = 0$. The set of equivalence classes $\widetilde{M} := \{[x] : x \in M\}$ is a metric space with well-defined distance $d([x], [y]) = d'(x, y)$. Let $i: M \rightarrow \widetilde{M}$ denote the projection $x \mapsto [x]$. Notice that $i(B_{d'}(x, r)) = B_d([x], r)$ and $i^{-1}(B_d([x], r)) = B_{d'}(x, r)$ for $x \in X$ and $r > 0$. It is then easy to see that $(\widetilde{M}, d)$ is complete and separable and $i_*(\mu)$ is a locally finite Borel measure on $(\widetilde{M}, d)$. If μ on (M, d') is nonpointwise doubling almost everywhere, then $i_*(\mu)$ on $(\widetilde{M}, d)$ is also nonpointwise doubling almost everywhere. Further, if (M, d', μ) is a (approximate) differentiability space, then $(\widetilde{M}, d, i_*(\mu))$ is also a (approximate) differentiability space. This follows because Lipschitz functions on (M, d') are constant on equivalence classes, which implies, using uniqueness of derivatives in (M, d', μ) , that the derivative of f is constant on equivalence classes, hence gives rise to a function defined almost everywhere on $\widetilde{M}$.

Definition 3.2. We define the set

$$M := \{(a_1, a_2, a_3, \dots) : a_i \in \{1, \dots, i\}\}$$

and equip M with the metric $d_M(a, b) = 1/2^k$ where $k = \min\{i : a_i \neq b_i\}$.

If $k \geq 1$ we call a set of the form

$$\{(c_1, c_2, c_3, \dots, c_k, a_{k+1}, a_{k+2}, \dots) : a_i \in \{1, \dots, i\} \forall i \geq k+1\}$$

for $c_1, c_2, \dots, c_k$ fixed an island of level k . Note that there are $k!$ islands of level k .

Notice that M is an ultrametric space. That is, the distance d_M satisfies the strong triangle inequality $d_M(x, y) \leq \max(d_M(x, z), d_M(z, y))$ for all $x, y, z \in M$. This implies that any two balls in M are either disjoint or one is contained inside the other. Balls in M are islands and the Borel sigma algebra on M consists of countable unions of disjoint islands and points. Hence we can define a Borel probability measure ν on M which gives the same measure $\nu_k = 1/k!$ to any island of level k .

Notice that if $a \in M$, then $B_M(a, 1/2^k)$ is the island of level k containing a . Hence, for any $a \in M$,

$$\nu(B_M(a, 1/2^{k+1}))/\nu(B_M(a, 1/2^k)) = \nu_{k+1}/\nu_k \rightarrow 0$$

as $k \rightarrow \infty$, so the measure ν is not pointwise doubling.

However, since any two balls are either disjoint or one is contained inside the other, (M, d_M, ν) is a Vitali space and consequently the density theorem holds in (M, d_M, ν) .

The space M is compact. To see this note that given a sequence of elements of M we can find a subsequence for which each coordinate is eventually constant. This immediately implies that the subsequence converges.

Let $I = [0, 1]$. We now introduce a pseudometric on $M \times I$ which gives rise to the quotient space described.

Definition 3.3. For an integer $k \geq 0$ define $I_k := \{n/2^k : n \in \mathbb{N}, 0 \leq n \leq 2^k\} \subset I$. Define a pseudometric d_p on $M \times I$ by

$$d_p((x, t), (y, s)) := \inf\{|t - u| + |u - s| : u \in I_k, d_M(x, y) < 1/2^k\}.$$

We call subsets of the form $M \times \{n/2^k\}$, for some $n \in \mathbb{N}$, jump levels of size k and denote the open ball with centre (x, t) and radius $r > 0$ in $(M \times I, d_p)$ by $B_p((x, t), r)$. The height of a point (x, t) is its coordinate $t \in I$.

Notice that geometrically the pseudometric d_p corresponds to the shortest path joining points and moving only vertically, except at jump levels of size k , where we can jump horizontally inside islands of level k .

Balls $B_p((x, t), r)$, $(x, t) \in M \times I$ and $r > 0$, are countable unions of overlapping rectangles. The rectangles in this union are centred on jump levels and are of the form

$$B_M(x, 1/2^k) \times (u - r + |t - u|, u + r - |t - u|),$$

where $u \in I_k$ and $|t - u| < r$. Notice that this implies that $\nu \times \mathcal{L}^1$ is a Borel measure on $(M \times I, d_p)$.

We will also use the maximum metric $d_\infty((x, t), (y, s)) = \max(d_M(x, y), |t - s|)$ on $M \times I$. We show that $d_p((x, t), (y, s)) \leq 3d_\infty((x, t), (y, s))$ for all $(x, t), (y, s) \in M \times I$. Fix (x, t) and (y, s) in $M \times I$. If $x = y$, then the inequality is obvious as both distances are equal to $|t - s|$. Suppose not and suppose $d_M(x, y) = 1/2^{k+1}$ for some fixed k . Then x and y belong to the same island of level k but distinct islands of level $k + 1$. We can find $u \in I_k$ with $|t - u| \leq 1/2^{k+1}$. Hence,

$$d_p((x, t), (y, s)) \leq 1/2^{k+1} + 1/2^{k+1} + |s - t| \leq 3d_\infty(x, y).$$

Notice that this implies that $i: (M \times I, d_\infty) \rightarrow (M \times I, d_p)$ is Lipschitz. Since M and I are separable, the collection of d_∞ Borel sets is the smallest sigma algebra containing products of Borel sets in M and I .

Since (M, d_M) and I are compact it follows that the product space $(M \times I, d_\infty)$ is also compact. Since $i: (M \times I, d_\infty) \rightarrow (M \times I, d_p)$ is continuous and surjective we see that $(M \times I, d_p)$ is compact, hence complete and separable.

4. NONPOINTWISE DOUBLING

Theorem 4.1. *The measure $\nu \times \mathcal{L}^1$ on $(M \times I, d_p)$ is nonpointwise doubling almost everywhere. Indeed, for almost every $(x, t) \in M \times I$,*

$$\limsup_{r \downarrow 0} \frac{\nu \times \mathcal{L}^1(B_p((x, t), 4r))}{\nu \times \mathcal{L}^1(B_p((x, t), r))} = \infty.$$

Proof. The main idea is that, on small scales, rectangles centred on larger jump levels have much greater measure than those with similar height and centred on smaller jump levels. Let

$$E_k = \{t \in I : |t - u| \geq 1/2^{k+2} \forall u \in I_k\}.$$

We claim that $\nu \times \mathcal{L}^1(M \times \limsup E_k) = 1$, where $\limsup E_k = \bigcap_{k \geq 1} \bigcup_{l \geq k} E_l$ denotes those points in infinitely many of the sets E_k .

To see this it suffices to show that for any fixed k the set $\bigcap_{l \geq k} (E_l)^c$ has measure 0. We notice that if $t \in I$, $|t - n/2^l| < 1/2^{l+2}$ for a fixed $n \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ such that $|t - m/2^{l+1}| < 1/2^{l+3}$, then $|t - n/2^l| < 1/2^{l+3}$. Hence, repeatedly applying this observation, we see that $\bigcap_{l \geq k} (E_l)^c = I_k$, so has measure 0.

Fix $(x, t) \in M \times \limsup E_k$ and an increasing sequence $n_k \rightarrow \infty$ such that $t \in E_{n_k}$ for every k . We cannot reach a jump level of size n_k within vertical distance strictly less than $1/2^{n_k+2}$ of (x, t) . Hence

$$B_p((x, t), 1/2^{n_k+2}) \subset B_M(x, 1/2^{n_k+1}) \times (t - 1/2^{n_k+2}, t + 1/2^{n_k+2}),$$

which implies that

$$\nu \times \mathcal{L}^1(B_p((x, t), 1/2^{n_k+2})) \leq 1/2^{n_k+1} \nu_{n_k+1}.$$

We note that for each fixed k there exists $u \in I_{n_k}$ with $|t - u| \leq 1/2^{n_k+1}$. Hence

$$B_M(x, 1/2^{n_k}) \times (u - 1/2^{n_k+1}, u + 1/2^{n_k+1}) \subset B_p((x, t), 1/2^{n_k}),$$

which implies that

$$\nu \times \mathcal{L}^1(B_p((x, t), 1/2^{n_k})) \geq 1/2^{n_k} \nu_{n_k}.$$

Putting these inequalities together we observe that

$$\frac{\nu \times \mathcal{L}^1(B_p((x, t), 1/2^{n_k}))}{\nu \times \mathcal{L}^1(B_p((x, t), 1/2^{n_k+2}))} \geq 2 \frac{\nu_{n_k}}{\nu_{n_k+1}} \rightarrow \infty$$

as $k \rightarrow \infty$. Hence $\nu \times \mathcal{L}^1$ is not pointwise doubling at (x, t) . $\square$

5. APPROXIMATE DIFFERENTIABILITY

We now show that $(M \times I, d_p, \nu \times \mathcal{L}^1)$ is an approximate differentiability space. Denote by $h: (M \times I, d_p) \rightarrow \mathbb{R}$ the height map defined by $h(x, t) = t$. It is easy to see from the definition of d_p that h is Lipschitz.

Theorem 5.1. *The space $(M \times I, d_p, \nu \times \mathcal{L}^1)$ is an approximate differentiability space. The structure consists of the single chart $(M \times I, h)$ and for each Lipschitz map and almost every $(x_0, t_0) \in M \times I$ the derivative of f at (x_0, t_0) is given by*

$$df(x_0, t_0) := \lim_{u \rightarrow 0} \frac{f(x_0, t_0 + u) - f(x_0, t_0)}{u}.$$

Remark 5.2. While it will take an effort to prove existence of derivatives with respect to h it is not hard to see uniqueness. If $a_1, a_2 \in \mathbb{R}$ are possible candidates for derivatives of a given function at $(x_0, t_0) \in M \times I$, then by the triangle inequality,

$$\operatorname{aplim}_{(x,t) \rightarrow (x_0, t_0)} \frac{|(a_1 - a_2)(t - t_0)|}{d_p((x, t), (x_0, t_0))} = 0.$$

To see that this implies that $a_1 = a_2$, it suffices to observe that for each $r > 0$ the points $(x, t) \in B_p((x_0, t_0), r)$ satisfying $|t - t_0| > r/2$ comprise at least half the measure of $B_p((x_0, t_0), r)$ and satisfy $|t - t_0| > r/2 > d_p((x, t), (x_0, t_0))/2$.

To prove Theorem 5.1 we show that balls in $(M \times I, d_p)$ are on small scales well approximated by at most three, not necessarily disjoint, rectangles.

Lemma 5.3. *Fix $\varepsilon > 0$, $x \in M$ and $t \in I \setminus \bigcup_{k=1}^{\infty} I_k$. Choose $r > 0$ sufficiently small so that $(t - r, t + r) \cap I_k \neq \emptyset$ implies $k > 0$ and $\nu_{k+1}/\nu_k < \varepsilon/2$. Then there exist at most three rectangles $R_i \subset B_p((x, t), r)$ of the form*

$$R_i = B_M(x, 1/2^{k_i}) \times (t_i - r_i, t_i + r_i),$$

where $t_i \in I_{k_i}$, $r_i = r - |t - t_i| > 0$, $k_1 = \min\{k \in \mathbb{N} : (t - r/2, t + r/2) \cap I_k \neq \emptyset\}$ and $k_2, k_3 < k_1$, if defined, such that

$$\frac{\nu \times \mathcal{L}^1(B_p((x, t), r) \setminus \bigcup_i R_i)}{\nu \times \mathcal{L}^1(B_p((x, t), r))} < \varepsilon.$$

Proof. The main idea is that rectangles centred on larger jump levels and relatively close to the centre of a ball contain most of the measure. Fix $(x, t) \in M \times I$, $\varepsilon > 0$ and $r > 0$ as in the statement of the lemma. Define k_1 corresponding to the largest jump level we can reach within height $r/2$ above or below (x, t) . Since $t \notin \bigcup_{k=1}^{\infty} I_k$, the intersection $(t - r/2, t + r/2) \cap I_{k_1}$ consists of a single point, t_1 say.

Since $(t-r/2, t+r/2) \cap I_{k_1-1} = \emptyset$ and elements of I_{k_1-1} are spaced only a distance $1/2^{k_1-1}$ apart we deduce that $r < 1/2^{k_1-1}$. It may happen that $[t+r/2, t+r) \cap I_{k_1-1}$ is empty, but as before this intersection is at most a single point. If so, write $[t+r/2, t+r) \cap I_{k_1-1} = \{t_2\}$ and let

$$k_2 = \min\{k \in \mathbb{N} : [t+r/2, t+r) \cap I_k \neq \emptyset\}.$$

Similarly, if the intersection is nonempty, let $(t-r, t-r/2] \cap I_{k_1-1} = \{t_3\}$ and

$$k_3 = \min\{k \in \mathbb{N} : (t-r, t-r/2] \cap I_k \neq \emptyset\}.$$

For those $1 \leq i \leq 3$ for which we have defined t_i let $r_i = r - |t - t_i| > 0$ and define $R_i \subset B_p((x, t), r)$ to be the rectangle

$$R_i = B_M(x, 1/2^{k_i}) \times (t_i - r_i, t_i + r_i).$$

We estimate the measure of the region in $B_p((x, t), r)$ not covered by the defined rectangles. Notice that

$$B_p((x, t), r) \setminus \bigcup_i R_i \subset B_M(x, 1/2^{k_1+1}) \times (t-r, t+r).$$

Using that $R_1 \subset B_p((x, t), r)$ and $r_1 > r/2$ this implies that

$$\frac{\nu \times \mathcal{L}^1(B_p((x, t), r) \setminus \bigcup_i R_i)}{\nu \times \mathcal{L}^1(B_p((x, t), r))} \leq \frac{2r\nu_{k_1+1}}{r\nu_{k_1}} < \varepsilon. \quad \square$$

Our proof of Theorem 5.1 will use the density theorem for vertical lines and rectangles to construct paths along which the derivative of f is almost constant. We then show that for most points, changes in f are well approximated by the product of the derivative and the change in height.

Lemma 5.4. *The density theorem with respect to rectangles holds for d_∞ Borel measurable sets in $M \times I$. That is, if $A \subset M \times I$ is d_∞ Borel measurable, then for almost every $(x, t) \in A$ for all $\varepsilon > 0$ there exists $R > 0$ such that $0 < u, v < R$ implies*

$$\frac{\nu \times \mathcal{L}^1(B_M(x, u) \times (t-v, t+v) \setminus A)}{2\nu(B_M(x, u))} < \varepsilon.$$

Each d_∞ Borel measurable function $g: M \times I \rightarrow \mathbb{R}$ is approximately continuous almost everywhere. That is, for almost every $(x, t) \in M \times I$ for all $\varepsilon > 0$ there exists $R > 0$ such that $0 < u, v < R$ implies

$$\frac{\nu \times \mathcal{L}^1\{(y, s) \in B_M(x, u) \times (t-v, t+v) : |g(y, s) - g(x, t)| > \varepsilon\}}{2\nu(B_M(x, u))} < \varepsilon.$$

The first part of the lemma follows from Theorem 4 of [6] by Bruckner and Weiss or can be proven using the density theorem in M and I . The second part then follows by applying the density theorem similarly to 2.9.13 of [7].

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Let $f: (M \times I, d_p) \rightarrow \mathbb{R}$ be Lipschitz. Since $i: (M \times I, d_\infty) \rightarrow (M \times I, d_p)$ is Lipschitz this implies that f is Lipschitz as a function on $(M \times I, d_\infty)$. It follows easily from Fubini's theorem that df , as defined in the statement of the theorem, exists almost everywhere in $M \times I$ and can be extended to a d_∞ Borel measurable function on $M \times I$. Thus we can use approximate continuity of df along vertical lines and for rectangles. Fix $\varepsilon > 0$. We begin by using the density theorem to construct paths along which df does not vary too much.

Claim 1. Define, for each $(x, t) \in M \times I$,

$$D(x, t, \varepsilon) := \{(y, s) \in M \times I : |df(y, s) - df(x, t)| \leq \varepsilon\}.$$

For each k the following are d_∞ Borel subsets of $M \times I$:

$$\begin{aligned} E_k^1 &:= \{(x, t) : \mathcal{L}^1\{s \in (t-r, t+r) : (x, s) \in D(x, t, \varepsilon)\} \\ &\geq (2-\varepsilon)r \ \forall \ 0 < r < 1/k\}, \end{aligned}$$

$$\begin{aligned} E_k^2 &:= \{(x, t) : \nu \times \mathcal{L}^1(B_M(x, r) \times (t-s, t+s) \cap E_k^1 \cap D(x, t, \varepsilon)) \\ &\geq (2-\varepsilon)s\nu(B_M(x, r)) \ \forall \ 0 < r, s < 1/k\}, \end{aligned}$$

$$\begin{aligned} E_k^3 &:= \{(x, t) : \mathcal{L}^1\{s \in (t-r, t+r) : (x, s) \in E_k^2 \cap D(x, t, \varepsilon)\} \\ &\geq (2-\varepsilon^2)r \ \forall \ 0 < r < 1/k\}. \end{aligned}$$

Furthermore $\bigcup_{k=1}^\infty E_k^3$ is a set of full measure in $M \times I$.

Proof. For a fixed q let

$$D_q = \{(y, s) : |df(y, s) - q| < \varepsilon\}.$$

Then for any Borel $A \subset I$ and $r, s, q \in \mathbb{R}$ the functions

$$(x, t) \mapsto \mathcal{L}^1\{s \in (t-r, t+r) : (x, s) \in D_q \cap A\}$$

and

$$(x, t) \mapsto \nu \times \mathcal{L}^1(B_M(x, r) \times (t-s, t+s) \cap D_q \cap A)$$

are defined on a product of separable metric spaces, Borel measurable in the first variable by Fubini's theorem and continuous in the second and so are Borel measurable as a function of two variables. By taking countable intersections over appropriate $q, r, s \in \mathbb{Q}$ with $A = M \times I$, E_k^1 and E_k^2 show the measurability of E_k^1 , E_k^2 and E_k^3 respectively. We illustrate this for the sets E_k^1 . Notice that

$$E_k^1 = \bigcap_{0 < r < 1/k, r \in \mathbb{Q}} \{(x, t) : \mathcal{L}^1\{s \in (t-r, t+r) : |df(x, s) - df(x, t)| > \varepsilon\} < \varepsilon r\}.$$

The sets inside this countable intersection are Borel measurable. Indeed, for fixed $\alpha > 0$,

$$\begin{aligned} &\{(x, t) : \mathcal{L}^1\{s \in (t-r, t+r) : |df(x, s) - df(x, t)| > \varepsilon\} > \alpha\} \\ &= \bigcup_{\eta > \varepsilon, \eta \in \mathbb{Q}} \bigcap_{n=1}^\infty \bigcup_{q \in \mathbb{Q}} \{(x, t) : |df(x, t) - q| < 1/n\} \\ &\quad \cap \{(x, t) : \mathcal{L}^1\{s \in (t-r, t+r) : |df(x, s) - q| > \eta\} > \alpha\} \end{aligned}$$

is a Borel measurable set.

By various applications of Fubini's theorem and approximate continuity for vertical lines and rectangles we see that $\bigcup_{k=1}^\infty E_k^1, \bigcup_{k=1}^\infty E_k^2$ and hence $\bigcup_{k=1}^\infty E_k^3$ are of full measure. $\square$

Now fix $(x_0, t_0) \in E_k^3$ for some k with $t_0 \notin \bigcup_{l=1}^\infty I_l$. Choose $R < 1/k$ small enough that $(t_0 - R, t_0 + R) \cap I_s \neq \emptyset$ implies $1/2^s < 1/k$ and $\nu_{s+1}/\nu_s < \varepsilon/2$. Fix $0 < r < R$.

As in Lemma 5.3 we can find at most three, not necessarily disjoint, rectangles $R_i \subset B_p((x_0, t_0), r)$ of the form

$$R_i = B_M(x_0, 1/2^{k_i}) \times (t_i - r_i, t_i + r_i),$$

where $t_i \in I_{k_i}$, $r_i = r - |t_0 - t_i| > 0$, $k_1 = \min\{k \in \mathbb{N} : (t_0 - r/2, t_0 + r/2) \cap I_k \neq \emptyset\}$ and $k_2, k_3 < k_1$, if defined, such that

$$\frac{\nu \times \mathcal{L}^1(B_p((x_0, t_0), r) \setminus \bigcup_i R_i)}{\nu \times \mathcal{L}^1(B_p((x_0, t_0), r))} < \varepsilon.$$

The vertical paths we use will, for each i , join (x_0, t_0) to (x_0, t_i) and, for most points $(x, t) \in R_i$, (x, t) to (x, t_i) . Thus we need to consider points $(x, t) \in R_i$ whose distance to (x_0, t_0) is given by the length of these paths.

Let

$$G_i = \{(x, t) \in R_i : d_p((x, t), (x_0, t_0)) = |t - t_i| + |t_i - t_0|\}$$

and define

$$W_i = (B_M(x_0, 1/2^{k_i}) \setminus B_M(x_0, 1/2^{k_i+1})) \times (t_i - r_i, t_i + r_i) \subset R_i.$$

Notice that

$$\nu \times \mathcal{L}^1(R_i \setminus W_i) = 2r_i \nu_{k_i+1} \leq \varepsilon r_i \nu_{k_i} \leq \varepsilon \nu \times \mathcal{L}^1(R_i)$$

and $\bigcup_i W_i \subset \bigcup_i G_i$. Thus

$$\nu \times \mathcal{L}^1\left(\bigcup_i R_i \setminus \bigcup_i G_i\right) \leq \nu \times \mathcal{L}^1\left(\bigcup_i R_i \setminus \bigcup_i W_i\right) \leq 3\varepsilon \nu \times \mathcal{L}^1(B_p((x_0, t_0), r)).$$

Now fix i . We show that for most points in G_i the change in f is well approximated by the product of the derivative and the change in height.

Claim 2. Let

$$L(x_0, t_0, \varepsilon) := \{(x, t) : \frac{|f(x, t) - f(x_0, t_0) - df(x_0, t_0)(t - t_0)|}{d_p((x_0, t_0), (x, t))} > (4\text{Lip} f + 4)\varepsilon\}.$$

Then $\nu \times \mathcal{L}^1(G_i \cap L(x_0, t_0, \varepsilon)) \leq \varepsilon \nu \times \mathcal{L}^1(R_i)$.

Proof. There are two cases depending on the shape and position of R_i in relation to (x_0, t_0) .

Suppose $r_i \leq \varepsilon|t_i - t_0|$ and let $(x, t) \in G_i$ so that $|t_i - t_0| \leq d_p((x_0, t_0), (x, t))$. We calculate, since $d_p((x_0, t_i), (x, t_i)) = 0$ implies that $f(x_0, t_i) = f(x, t_i)$,

$$\begin{aligned} |f(x, t) - f(x_0, t_0) - df(x_0, t_0)(t - t_0)| & \\ & \leq |f(x, t) - f(x, t_i)| + |df(x_0, t_0)(t - t_i)| \\ & \quad + |f(x_0, t_i) - f(x_0, t_0) - df(x_0, t_0)(t_i - t_0)| \\ & \leq 2\text{Lip} f|t - t_i| + \left| \int_{t_0}^{t_i} (df(x_0, s) - df(x_0, t_0))ds \right| \\ & \leq 2\text{Lip} f r_i + \varepsilon|t_i - t_0| + 2\text{Lip} f \varepsilon|t_i - t_0| \\ & \leq (4\text{Lip} f + 1)\varepsilon d_p((x_0, t_0), (x, t)) \end{aligned}$$

using the fundamental theorem of calculus along vertical lines and the fact that $(x_0, t_0) \in E_k^3 \subset E_k^1$ implies that

$$\mathcal{L}^1\{s \in (t_0 - |t_0 - t_i|, t_0 + |t_0 - t_i|) : (x_0, s) \notin D(x_0, t_0, \varepsilon)\} \leq \varepsilon|t_i - t_0|.$$

Suppose $r_i > \varepsilon|t_i - t_0|$. Then the rectangle R_i is large enough to be well approximated by the rectangles from our use of the density theorem.

Since $(x_0, t_0) \in E_k^3$ we can find t'_i with $|t'_i - t_i| < \varepsilon^2|t_i - t_0| < \varepsilon r_i$, $|df(x_0, t'_i) - df(x_0, t_0)| < \varepsilon$ and $(x_0, t'_i) \in E_k^2$. Thus if we define

$$A_i = B_M(x_0, 1/2^{k_i}) \times (t'_i - r_i, t'_i + r_i),$$

then

$$\nu \times \mathcal{L}^1(A_i \cap E_k^1 \cap D(x_0, t'_i, \varepsilon)) \geq (2 - \varepsilon)r_i\nu_{k_i}.$$

Notice that $D(x_0, t'_i, \varepsilon) \subset D(x_0, t_0, 2\varepsilon)$. Since $G_i \subset R_i$ and R_i is well approximated by A_i ,

$$\begin{aligned} & \nu \times \mathcal{L}^1(G_i \setminus (E_k^1 \cap D(x_0, t_0, 2\varepsilon))) \\ & \leq \nu \times \mathcal{L}^1(A_i \setminus (E_k^1 \cap D(x_0, t'_i, \varepsilon))) + \nu \times \mathcal{L}^1(R_i \setminus A_i) \\ & \leq \varepsilon r_i \nu_{k_i} + \nu_{k_i} |t'_i - t_i| \\ & \leq \varepsilon r_i \nu_{k_i} + \varepsilon r_i \nu_{k_i} \\ & = \varepsilon \nu \times \mathcal{L}^1(R_i). \end{aligned}$$

Now fix $(x, t) \in G_i \cap E_k^1 \cap D(x_0, t_0, 2\varepsilon)$ and notice that $|df(y, s) - df(x_0, t_0)| > 3\varepsilon$ implies $(y, s) \notin D(x, t, \varepsilon)$. We calculate

$$\begin{aligned} & |f(x, t) - f(x_0, t_0) - df(x_0, t_0)(t - t_0)| \\ & \leq \left| \int_{t_0}^{t_i} (df(x_0, s) - df(x_0, t_0)) ds \right| + \left| \int_{t_i}^t (df(x, s) - df(x_0, t_0)) ds \right| \\ & \leq (2\text{Lip}f + 1)\varepsilon d_p((x_0, t_0), (x, t)) + 3\varepsilon|t - t_i| + 2\varepsilon\text{Lip}f|t - t_i| \\ & \leq (4\text{Lip}f + 4)\varepsilon d_p((x_0, t_0), (x, t)). \end{aligned}$$

The estimation of the first integral is the same as earlier. Estimation of the second uses the fact that $(x, t) \in E_k^1$ as for (x_0, t_0) before. $\square$

To conclude we estimate

$$\begin{aligned} & \nu \times \mathcal{L}^1(B_p((x_0, t_0), r) \cap L(x_0, t_0, \varepsilon)) \\ & \leq \nu \times \mathcal{L}^1(B_p((x_0, t_0), r) \setminus \bigcup_i G_i) + \sum_i \nu \times \mathcal{L}^1(G_i \cap L(x_0, t_0, \varepsilon)) \\ & \leq \nu \times \mathcal{L}^1(B_p((x_0, t_0), r) \setminus \bigcup_i R_i) + \nu \times \mathcal{L}^1(\bigcup_i R_i \setminus \bigcup_i G_i) + \sum_i \varepsilon \nu \times \mathcal{L}^1(R_i) \\ & \leq 7\varepsilon \nu \times \mathcal{L}^1(B_p((x_0, t_0), r)). \end{aligned}$$

We have shown that given $\varepsilon > 0$, for almost every $(x_0, t_0) \in M \times I$ there exists $R > 0$ such that $0 < r < R$ implies that

$$\nu \times \mathcal{L}^1(B_p((x_0, t_0), r) \cap L(x_0, t_0, \varepsilon)) \leq 7\varepsilon \nu \times \mathcal{L}^1(B_p((x_0, t_0), r)).$$

By taking a countable intersection of such points for $\varepsilon_n = 1/n$ we see that f is approximately differentiable almost everywhere. $\square$

6. FURTHER DISCUSSION

There are some related questions we can ask about approximate differentiability spaces and nondoubling measures. In our example $\nu \times \mathcal{L}^1$ was still, at almost every point, doubling on arbitrarily small scales. That is, for almost every $(x_0, t_0) \in M \times I$,

$$\liminf_{r \downarrow 0} \frac{\nu \times \mathcal{L}^1(B_p((x_0, t_0), 4r))}{\nu \times \mathcal{L}^1(B_p((x_0, t_0), r))} < \infty.$$

It is not clear whether there is an approximate differentiability space X with measure μ that is, at almost every point, nondoubling on all small scales. That is, for some enlargement factor C ,

$$\lim_{r \downarrow 0} \frac{\mu(B(x_0, Cr))}{\mu(B(x_0, r))} = \infty$$

for almost every $x \in X$.

If we adjust the definition of M by splitting each island into a very large number of subislands and define a different equivalence relation on $M \times I$ so that at a jump level of order k we identify corresponding points in distinct islands of level $k+1$ that are inside the same island of level k , then the measure $\nu \times \mathcal{L}^1$ is nondoubling on all small scales. However, as we identify fewer points (only finitely many at each level) there seems no obvious way to use the density theorem in horizontal directions to construct paths joining points along which the derivative is approximately constant. Thus, it is not clear whether this alternate construction also gives an approximate differentiability space.

We can also consider an alternative definition of approximate differentiability space in which the approximate limit defining the derivative is defined in a Lipschitz-invariant way. That is, in a complete separable metric space X with locally finite measure μ we say that $g: X \rightarrow \mathbb{R}$ has invariant approximate limit $l \in \mathbb{R}$ at $x_0 \in X$ if for every $C, \varepsilon > 0$,

$$\lim_{r \downarrow 0} \frac{\mu\{x \in B(x_0, Cr) : |g(x) - l| > \varepsilon\}}{\mu(B(x_0, r))} = 0.$$

It is not clear whether there is any invariant approximate differentiability space in which the measure is nondoubling. There seems no obvious way to construct such a space using density theorems and approximate continuity like before. One can however show that the measure cannot be nondoubling on all small scales. Precisely, for all $C > 0$ and almost every $x_0 \in X$,

$$\liminf_{r \downarrow 0} \frac{\mu(B(x_0, Cr))}{\mu(B(x_0, r))} < \infty.$$

The proof of this involves using a contradiction argument to construct a Lipschitz function not invariantly approximately differentiable on a set of positive measure.

The idea is to find a fixed radius $r > 0$ such that multiplying r by a fixed factor $\eta > 0$ greatly reduces the measure of the corresponding ball centred on most points. We then use a covering theorem ([7], 2.8.4) to find a disjoint subcollection of the balls with radius r so that each ball in the original collection meets a ball in the subcollection in such a way that the corresponding reduced balls have comparable measure. We may build cones on the reduced balls in the disjoint subcollection and later repeat the process at smaller scales, as collectively they have small measure. Now around most centres c we can find a ball $B(c, \eta r)$ for which the expanded

ball $B(c, r)$ contains a ball $B(c', \eta r)$ on which a cone is defined and $B(c, \eta r)$ and $B(c', \eta r)$ have comparable measure. By either adding cones or not at smaller and smaller scales, similar to the proof of Theorem 2.4, we construct a Lipschitz function whose gradient is large for enough measure on some scales and small for enough measure on other scales. Done carefully this leads to a Lipschitz function which is not invariantly approximately differentiable on a set of positive measure.

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