GRADIENT ESTIMATE ON CONVEX DOMAINS AND APPLICATIONS

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ABSTRACT. By solving the Skorokhod equation for reflecting diffusion processes on a convex domain, gradient estimates for the associated Neumann semigroup are derived. As applications, functional/Harnack inequalities are established for the Neumann semigroup. When the domain is bounded, the gradient estimates are applied to the study of Riesz transforms and regularity of the inhomogeneous Neumann problems on convex domains.

1. INTRODUCTION

The main purpose of this paper is to study gradient estimates and applications of the Neumann semigroup on a domain Ω in $\mathbb{R}^d$. When the domain is C^2 , the topic has been well studied by using both stochastic analysis (cf. [18, 27, 28, 29]) and an analytic approach (cf. [19, 31, 30]). Both arguments need conditions on the curvature and the second fundamental form of the boundary, since when the boundary is merely Lipschitzian, the second fundamental form is no longer available; thus known results in the above-mentioned references do not apply.

In this paper we adopt a probabilistic approach by solving the Skorokhod equation for the reflecting diffusion process on convex domains. By using the Itô formula for the constructed reflecting diffusion process and a weak dissipative condition, we establish the L^1 and L^2 gradient estimates of the corresponding Neumann semigroup.

Let $\Omega \subset \mathbb{R}^d$ be a closed domain and let

$$L = \frac{1}{2}\phi^2\Delta + Z \cdot \nabla,$$

where $\phi \in C^1(\Omega)$ is bounded with $\inf_{\Omega} \phi^2 > 0$, and $Z \in C^1(\Omega; \mathbb{R}^d)$ with $|Z(x)| \leq c(1 + |x|)$ for some constant $c > 0$. We assume that

$$(1.1) \quad d|\phi(x) - \phi(y)|^2 + 2\langle Z(x) - Z(y), x - y \rangle \leq K|x - y|^2, \quad x, y \in \Omega$$

holds for some constant $K \in \mathbb{R}$.

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Let N be the inward pointing unit normal vector field of $\partial\Omega$, the boundary of Ω . For any $f \in \mathcal{B}_b(\Omega)$, the set of all bounded measurable functions on Ω , consider the Neumann problem on Ω :

$$(1.2) \quad \begin{cases} \partial_t u(t, \cdot) = Lu(t, \cdot), & Nu(t, \cdot)|_{\partial\Omega} = 0, \quad t > 0; \\ u(0, \cdot) = f. \end{cases}$$

To characterize the solution of (1.2), we first investigate the reflecting diffusion process generated by L on Ω . Let B_t be the d -dimensional Brownian motion on a complete filtered probability space with probability measure $\mathbb{P}$.

Theorem 1.1. *Let Ω be convex. For any $x \in \Omega$, there exists a unique adapted continuous process $(X_t, l_t)_{t \geq 0}$ on $\Omega \times [0, \infty)$ such that l_t is an increasing process with $l_0 = 0$ which increases only when $X_t \in \partial\Omega$, and*

$$(1.3) \quad X_t = x + \int_0^t \phi(X_s)^2 dB_s + \int_0^t Z(X_s) ds + \int_0^t N(X_s) dl_s, \quad t \geq 0$$

holds a.s.

The solution to (1.3) is known as the reflecting diffusion process generated by L on Ω . To indicate the dependence on the initial data x , we shall also denote the solution to (1.3) by (X_t^x, l_t^x) . By the uniqueness condition we conclude that X_t^x is a strong Markov process on Ω with transition semigroup P_t given by

$$P_t f(x) := \mathbb{E} f(X_t^x), \quad t \geq 0, x \in \Omega, f \in \mathcal{B}_b(\Omega).$$

It is easy to see by the Itô formula that a smooth solution to (1.2) has to satisfy

$$u(t, x) = P_t f(x), \quad t \geq 0, x \in \Omega.$$

Thus, P_t is nothing but the Neumann semigroup generated by L on Ω .

Due to Theorem 1.1 and a coupling argument, the weak dissipative condition (1.1) implies the following gradient estimates on P_t .

Theorem 1.2. *Let Ω be convex. For any $f \in C_b^1(\Omega)$,*

$$(1.4) \quad |\nabla P_t f|^2 \leq e^{Kt} P_t |\nabla f|^2, \quad t \geq 0.$$

If moreover ϕ is constant, then

$$(1.5) \quad |\nabla P_t f| \leq e^{Kt/2} P_t |\nabla f|, \quad t \geq 0.$$

To derive functional inequalities for P_t using the argument of Bakry and Emery [4], we need the following Kolmogorov equation:

$$(1.6) \quad \frac{d}{dt} P_t f = P_s L P_{t-s} f, \quad s \in [0, t], f \in C_0^2(\Omega), Nf|_{\partial\Omega} = 0.$$

When $s = t$ the equation follows from the Itô formula directly. In general, if $P_{t-s} f$ is (C^2) smooth, then the Itô formula for the process $\{P_{t-s} f(X_r)\}_{r \geq 0}$ implies the desired equation for this s . If in particular

$$(1.7) \quad Z = \frac{1}{2}(\nabla \varphi^2 + \varphi^2 \nabla V)$$

holds for some bounded $V \in C^2(\Omega)$, then P_t is symmetric in $L^2(\Omega; \mu)$ for $\mu(dx) := e^{V(x)} dx$. In this case the generator $(L, \mathcal{D}(L))$ is a negatively definite self-adjoint operator, and for any $t > 0$ and $f \in C_0(\Omega)$ one has $P_t f \in \bigcap_{n \geq 1} \mathcal{D}(L^n)$; that is, $P_t f$ is C^∞ -smooth.

As consequences of the above gradient estimates, we have the following Poincaré/log-Sobolev type inequalities and a sharp uniform gradient estimate of P_t .

Corollary 1.3. *Let Ω be convex and (1.6) hold. Then:*

- (1) $P_t f^2 - (P_t f)^2 \leq \frac{\|\phi\|_\infty^2 (e^{Kt} - 1)}{K} P_t |\nabla f|^2, \quad f \in C_b^1(\Omega), t \geq 0.$
- (2) $|\nabla P_t f|^2 \leq \frac{K}{(\inf \phi^2)(1 - e^{-Kt})} \{P_t f^2 - (P_t f)^2\}, \quad f \in \mathcal{B}_b(\Omega), t > 0.$
- (3) *If ϕ is constant, then*

- $$P_t f^2 \log f^2 - (P_t f^2) \log P_t f^2 \leq \frac{2\phi^2 (e^{Kt} - 1)}{K} P_t |\nabla f|^2, \quad f \in C_b^1(\Omega), t \geq 0,$$
- $$|\nabla P_t f^2|^2 \leq \frac{2K}{(1 - e^{-Kt})\phi^2} \{P_t f^2 \log f^2 - (P_t f^2) \log P_t f^2\} P_t f^2, \quad f \in \mathcal{B}_b(\Omega), t > 0.$$
- (4) *If (1.7) holds for some $V \in C_b^2(\Omega)$, then there exists a constant $C > 0$ independent of t such that*

$$\|\nabla P_t\|_{1 \rightarrow \infty} \leq \frac{C}{(1 \wedge t)^{(d+1)/2}}$$

holds, where $\|\cdot\|_{1 \rightarrow \infty}$ is the operator norm from $L^1(\Omega; \mu)$ to $L^\infty(\Omega; \mu)$.

Finally, according to [24, Proof of Theorem 2.1] and [26, Proof of Lemma 2.1], the gradient estimates (1.4) and (1.5) imply the following Harnack type inequalities (1.8) and (1.9) respectively. These Harnack inequalities have a number of applications to heat kernel estimates and entropy-cost inequalities, for which we refer to e.g. [14, 24, 10].

Corollary 1.4. *In the situation of Corollary 1.3, let $t > 0$ be fixed. Then for any $f \in \mathcal{B}_b(\Omega)$ with $f \geq 1$,*

$$(1.8) \quad P_t \log f(x) \leq \log P_t f(y) + \frac{K|x-y|^2}{4(\inf \phi^2)(1 - e^{-Kt})}, \quad x, y \in \Omega$$

holds. If ϕ is constant, then for any $p > 1$ and positive $f \in \mathcal{B}_b(\Omega)$,

$$(1.9) \quad (P_t f)^p(x) \leq (P_t f^p(y)) \exp \left[\frac{Kp|x-y|^2}{4(p-1)\phi^2(1 - e^{-Kt})} \right], \quad x, y \in \Omega$$

holds.

The proofs of Theorems 1.1, 1.2 and Corollary 1.3 will be given in Section 2. In Section 3, we introduce an application of our results to the study of Riesz transforms and regularity of the inhomogeneous Neumann problems on convex domains following the line of work in [12].

2. PROOFS

2.1. Proof of Theorem 1.1. By the convexity of Ω and the weak dissipative condition (1.1), it is easy to see that for every $x \in \Omega$, the equation (1.3) has at most one solution. Indeed, let (X_t, l_t) and $(\tilde{X}_t, \tilde{l}_t)$ be solutions of (1.3). Since l_t and $\tilde{l}_t$ are local times of X_t and $\tilde{X}_t$ on $\partial\Omega$ respectively, it suffices to show that $X_t = \tilde{X}_t$. Since Ω is convex,

$$(2.1) \quad \langle y - z, N(y) \rangle \leq 0, \quad y \in \partial\Omega, z \in \Omega.$$

Noting that l_t (resp. $\tilde{l}_t$) increases only when $X_t \in \partial\Omega$ (resp. $\tilde{X}_t \in \partial\Omega$), this implies that

$$\langle X_t - \tilde{X}_t, N(X_t)dl_t - N(\tilde{X}_t)d\tilde{l}_t \rangle \leq 0.$$

Combining this with the Itô formula and (1.1), we obtain

$$\begin{aligned} d|X_t - \tilde{X}_t|^2 &= \{2\langle X_t - \tilde{X}_t, Z(X_t) - Z(\tilde{X}_t) \rangle + d|\phi(X_t) - \phi(\tilde{X}_t)|^2\}dt \\ &\quad + 2\langle (\phi(X_t) - \phi(\tilde{X}_t))dB_t, X_t - \tilde{X}_t \rangle \\ &\leq K|X_t - \tilde{X}_t|^2dt + 2\langle (\phi(X_t) - \phi(\tilde{X}_t))dB_t, X_t - \tilde{X}_t \rangle. \end{aligned}$$

This implies that

$$\mathbb{E}|X_t - \tilde{X}_t|^2 \leq |X_0 - \tilde{X}_0|^2 e^{Kt} = 0, \quad t \geq 0.$$

As soon as the uniqueness is confirmed, by the Yamada-Watanabe principle [32], it remains to prove the existence of a weak solution. We shall complete the proof by using known results for the reflecting Brownian motion on compact convex domains, an approximation argument, the Girsanov theorem and a time-change argument. Although the idea of the proof is rather fundamental in stochastic analysis, we did not find any reference which explicitly includes the following details for SDEs with reflection on a non-smooth domain.

(a) Let Ω be compact and $L = \frac{1}{2}\Delta$. By using the regularity of the associated Dirichlet form, [6, Theorem 4.4] ensures the existence of a reflecting Brownian motion. According to [5, Theorem 1], there exists a d -dimensional Brownian motion $\tilde{W}_t$ and a continuous adapted process $(\tilde{X}_t, \tilde{l}_t)$ on $\Omega \times [0, \infty)$ such that $\tilde{l}_t$ is an increasing process with $\tilde{l}_0 = 0$ which increases only when $\tilde{X}_t \in \partial\Omega$, and

$$(2.2) \quad \tilde{X}_t = x + \tilde{W}_t + \int_0^t N(\tilde{X}_s)d\tilde{l}_s, \quad t \geq 0$$

holds.

(b) Let Ω be non-compact and $L = \frac{1}{2}\Delta$. In this case we may construct the solution by a simple approximation argument. Let

$$\Omega_n = \Omega \cap \{z : |z - x| \leq n\}, \quad n \geq 1.$$

Then for each n , Ω_n is a compact convex domain. By (a) and the Yamada-Watanabe principle, there exists a unique solution to the equation

$$(2.3) \quad X_t^{(n)} = x + B_t + \int_0^t N^{(n)}(X_s^{(n)})dl_s^{(n)}, \quad t \geq 0,$$

where $N^{(n)}$ is the inward pointing unit vector field of $\partial\Omega_n$. Let

$$\tau_n = \inf\{t \geq 0 : |X_t^{(n)} - x| \geq n\}.$$

By the uniqueness of a strong solution to (2.3), we have $(X_t^{(n)}, l_t^{(n)}) = (X_t^{(m)}, l_t^{(m)})$ for $t \leq \tau_n \wedge \tau_m$. In particular, τ_n is increasing in n . By the convexity of the boundary and the Itô formula, we obtain from (2.3) that

$$|X_t^{(n)} - x|^2 \leq td + 2 \int_0^t \langle X_s^{(n)}, dB_s \rangle, \quad t \geq 0.$$

This implies that

$$n\mathbb{P}(\tau_n \leq t) \leq \mathbb{E}|X_{t \wedge \tau_n} - x|^2 \leq td.$$

Therefore, $\tau_n \uparrow \infty$ as $n \uparrow \infty$. This enables us to observe that

$$(X_t, l_t) := \sum_{n=1}^{\infty} (X_t^{(n)}, l_t^{(n)}) 1_{[\tau_{n-1}, \tau_n)}, \quad \tau_0 := 0$$

solves the equation

$$(2.4) \quad X_t = x + B_t + \int_0^t N(X_s) dl_s, \quad t \geq 0.$$

(c) Let (X_t, l_t) solve (2.4). Then

$$X_t = x + \tilde{B}_t + \int_0^t Z(X_s) ds + \int_0^t N(X_s) dB_s, \quad t \geq 0$$

holds for

$$\tilde{B}_t := B_t - \int_0^t Z(X_s) ds, \quad t \geq 0.$$

Since Z has linear growth, it is easy to see that

$$t \mapsto R_t := \exp \left[\int_0^t \langle Z(X_s), dB_s \rangle - \frac{1}{2} \int_0^t |Z(X_s)|^2 ds \right]$$

is a martingale. By the Girsanov theorem, for any $T > 0$ the process $\{\tilde{B}_t\}_{t \leq T}$ is a d -dimensional Brownian motion under the probability $R_T \mathbb{P}$. Therefore, under the probability measure $R_T \mathbb{P}$, $(X_t, l_t)_{t \leq T}$ is a weak solution of (1.3) for $\phi \equiv 1$. By the Yamada-Watanabe principle, this equation has a unique strong solution up to time T . This proves the desired assertion for $\phi \equiv 1$ since $T > 0$ is arbitrary.

(d) Since $\inf \phi^2 > 0$ and Z has linear growth, $\frac{Z}{\phi^2}$ has linear growth as well. By (c), let (X_t, l_t) solve the equation

$$(2.5) \quad X_t = x + B_t + \int_0^t \frac{Z(X_s)}{\phi(X_s)^2} ds + \int_0^t N(X_s) dl_s, \quad t \geq 0.$$

Let

$$\tau(t) = \int_0^t \frac{ds}{\phi(X_s)^2}, \quad \sigma(t) = \tau^{-1}(t), \quad t \geq 0.$$

Since $\inf \phi^2 > 0$ and ϕ is bounded, there exists a constant $C > 1$ such that $\frac{t}{C} \leq \sigma(t) \leq Ct$, $t \geq 0$. Obviously,

$$\sigma'(t) = \phi(X_{\sigma(t)})^2, \quad t \geq 0.$$

Thus,

$$(2.6) \quad \int_0^{\sigma(t)} \frac{Z(X_s)}{\phi(X_s)^2} ds = \int_0^t \frac{Z(X_{\sigma(s)})}{\phi(X_{\sigma(s)})^2} d\sigma(s) = \int_0^t Z(X_{\sigma(s)}) ds.$$

Moreover, let $\tilde{l}_s = l_{\sigma(s)}$, which is a continuous increasing process with support included in $\{s \geq 0 : X_{\sigma(s)} \in \partial\Omega\}$. We have

$$(2.7) \quad \int_0^{\sigma(t)} N(X_s) dl_s = \int_0^t N(X_{\sigma(s)}) d\tilde{l}_s, \quad t \geq 0.$$

Finally, let

$$\tilde{B}_t = \int_0^t \frac{1}{\phi(X_{\sigma(s)})} dB_{\sigma(s)}, \quad t \geq 0.$$

We have

$$(2.8) \quad B_{\sigma(t)} = \int_0^t \phi(X_{\sigma(s)}) d\tilde{B}_s, \quad t \geq 0.$$

Since $\tilde{B}_t$ is a continuous martingale with quadratic process

$$\langle \tilde{B} \rangle_t = \int_0^t \frac{1}{\phi(X_{\sigma(s)})^2} d\langle B \rangle_{\sigma(s)} = \int_0^t \frac{d\sigma(s)}{\phi(X_{\sigma(s)})^2} = t,$$

it is indeed a Brownian motion. Therefore, combining (2.5), (2.6), (2.7) and (2.8), we conclude that $(\tilde{X}_t, \tilde{l}_t) := (X_{\sigma(t)}, l_{\sigma(t)})$ is a weak solution of (1.3).

2.2. Proofs of Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. For any $x \in \Omega$, let (X_t^x, l_t^x) be the unique solution to (1.3) ensured by Theorem 1.1. As explained in the beginning of subsection 2.1, by (1.1), (2.1) and using the Itô formula, we obtain

$$(2.9) \quad d|X_t^x - X_t^y|^2 \leq K|X_t^x - X_t^y|^2 dt + 2\langle X_t^x - X_t^y, (\phi(X_t^x) - \phi(X_t^y)) dB_t \rangle.$$

This implies that

$$\mathbb{E}|X_t^x - X_t^y|^2 \leq |x - y|^2 e^{Kt}, \quad t \geq 0, x, y \in \Omega.$$

Therefore,

$$\begin{aligned} |\nabla P_t f(x)| &:= \limsup_{y \rightarrow x} \frac{|P_t f(y) - P_t f(x)|}{|x - y|} \leq \limsup_{y \rightarrow x} \frac{\mathbb{E}|f(X_t^x) - f(X_t^y)|}{|x - y|} \\ &\leq \limsup_{y \rightarrow x} \left(\mathbb{E} \frac{|f(X_t^x) - f(X_t^y)|^2}{|X_t^x - X_t^y|^2} \right)^{1/2} \left(\frac{\mathbb{E}|X_t^x - X_t^y|^2}{|x - y|^2} \right)^{1/2} = (P_t |\nabla f|^2)^{1/2} e^{Kt/2}. \end{aligned}$$

This proves (1.4).

If ϕ is constant, then the martingale term in (2.9) disappears, so that

$$|X_t^x - X_t^y| \leq |x - y| e^{Kt/2}, \quad t \geq 0$$

holds. Therefore,

$$\begin{aligned} |\nabla P_t f(x)| &:= \limsup_{y \rightarrow x} \frac{|P_t f(y) - P_t f(x)|}{|x - y|} \\ &\leq e^{Kt/2} \limsup_{y \rightarrow x} \frac{\mathbb{E}|f(X_t^x) - f(X_t^y)|}{|X_t^x - X_t^y|} = e^{Kt/2} P_t |\nabla f|. \end{aligned} \quad \square$$

Proof of Corollary 1.3. Let $t > 0$ and $f \in C_0^\infty(\Omega)$. By (1.4) and (1.6),

$$(2.10) \quad \frac{d}{ds} P_s (P_{t-s} f)^2 = P_s \{ \phi^2 |\nabla P_{t-s} f|^2 \} \geq (\inf \phi^2) e^{Ks} |\nabla P_t f|^2$$

holds for $s \in [0, t]$. Integrating both sides over $[0, t]$, we prove (2). Moreover, (1.4) also implies that

$$P_s \{ \phi^2 |\nabla P_{t-s} f|^2 \} \leq e^{(t-s)K} \|\phi\|_\infty^2 P_t |\nabla f|^2.$$

Combining this with the first equality in (2.10) we prove (1).

When ϕ is constant, (3) can be proved in the same way by using (1.5) in place of (1.4) and by considering

$$\frac{d}{ds} P_s \{ (P_{t-s} f^2) \log P_{t-s} f^2 \}.$$

Finally, if Z_i is given in (4), then P_t is symmetric in $L^2(\Omega; \mu)$. Since Ω is convex, the classical Nash inequality is implied by [11, Theorem 3.2.9]; i.e., there exists a constant $C > 0$ such that

$$\int_{\Omega} f(x)^2 dx \leq C + C \left(\int_{\Omega} |\nabla f|^2(x) dx \right)^{d/(d+2)}, \quad f \in W^{2,1}(\Omega; dx), \quad \int_{\Omega} |f|(x) dx = 1.$$

Since V is bounded, the same inequality holds for $\mu(dx)$ in place of dx for some different constant $C > 0$. Therefore,

$$(2.11) \quad \|P_t\|_{1 \rightarrow \infty} \leq \frac{c}{(1 \wedge t)^{d/2}}, \quad t > 0$$

holds for some constant $c > 0$. Combining this with (2) for $P_t f$ in place of f , we prove (4) immediately. $\square$

3. APPLICATIONS: RIESZ TRANSFORMS AND REGULARITY OF THE INHOMOGENEOUS NEUMANN PROBLEMS ON CONVEX DOMAINS

Throughout this section, we let Ω be a bounded convex domain in $\mathbb{R}^d$, and let

$$(3.1) \quad L = \frac{1}{2} \varphi^2 \Delta + Z \cdot \nabla \quad \text{such that } Z \text{ is given in (1.7).}$$

Thus, the generator $(L, \mathcal{D}(L))$ of P_t in $L^2(\Omega; \mu)$ is a negatively definite self-adjoint operator with discrete spectrum. Let $\mathcal{L}_N = -L$, which is thus a positive definite self-adjoint operator such that $P_t = e^{-\mathcal{L}_N t}$, $t \geq 0$.

It is well-known that (2.11) self-improves into the off-diagonal upper estimates; i.e., there exists a constant $c_1 > 0$ such that the kernel $p_t(x, y)$ of P_t satisfies

$$(3.2) \quad |p_t(x, y)| \leq \frac{C}{|B^{\Omega}(x, \sqrt{t})|} \exp\left(-\frac{|x-y|^2}{c_1 t}\right), \quad t > 0, x, y \in \Omega.$$

See, for example, Theorem 1.1 of [15]. Here, $|E|$ denotes the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^d$, and for $x \in \mathbb{R}^d$ and $r > 0$ we have set $B^{\Omega}(x, r) := \{y \in \Omega : |x - y| < r\}$.

Lemma 3.1. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Then there exist constants $C, c_2, c_3 > 0$ such that the kernel $p_t(x, y)$ of P_t satisfies*

$$(3.3) \quad |\nabla p_t(x, y)| \leq \begin{cases} C t^{-(d+1)/2} \exp\left(-\frac{|x-y|^2}{c_2 t}\right), & 0 < t \leq 1, \\ C e^{-c_3 t} \exp\left(-\frac{|x-y|^2}{c_2 t}\right), & t > 1 \end{cases}$$

for all $x, y \in \Omega$.

Proof. Using $\frac{\mu}{\mu(\Omega)}$ to replace μ , we may assume that μ is a probability measure. From the self-improvement property (see, for example, Theorem 4.9 of [9]), it suffices to show that there exist constants $C, c_4 > 0$ such that

$$(3.4) \quad |\nabla p_t(x, y)| \leq \begin{cases} C t^{-(d+1)/2}, & 0 < t \leq 2, \\ C e^{-c_4 t}, & t > 2 \end{cases}$$

for all $x, y \in \Omega$.

Indeed, we observe that from Corollary 1.3(4), there exists a constant $C_1 > 0$ such that

$$\|\nabla P_t\|_{1 \rightarrow \infty} \leq \frac{C_1}{(1 \wedge t)^{(d+1)/2}}, \quad t > 0.$$

On the other hand, since Ω is compact, $\mathcal{L}_N$ has a spectral gap $\lambda_1(> 0)$, which is known as the first Neumann eigenvalue of $\mathcal{L}_N$ on Ω . We have

$$\mu(|P_t \hat{f}|^2) \leq e^{-\lambda_1 t} \mu(|\hat{f}|^2), \quad t \geq 0, f \in L^2(\Omega; \mu),$$

where $\hat{f} := f - \mu(f)$. Therefore, for any $t > 2$ and $f \in L^1(\Omega; \mu)$ with $\|f\|_1 \leq 1$,

$$\begin{aligned} \|\nabla P_t f\|_{1 \rightarrow \infty} &= \|\nabla P_1(P_{t-1} \hat{f})\|_{\infty} \leq \|P_1\|_{1 \rightarrow \infty} \|P_{t-1} \hat{f}\|_1 \\ &\leq C_1 \|P_{t-2}(P_1 \hat{f})\|_2 \leq C_1 e^{-\lambda_1 t} \|P_1 \hat{f}\|_2 \leq 2C_1 \|P_1\|_{1 \rightarrow 2} e^{-\lambda_1 t}. \end{aligned}$$

Since due to (2.11) one has $\|P_1\|_{1 \rightarrow 2} < \infty$, estimate (3.4) follows readily, and the proof of Lemma 3.1 is finished. $\square$

Before continuing we need to clarify some notation. Recall that $\mu dx = e^{V(x)} dx$. Since $V \in C^2(\Omega)$ is bounded, we have that $L^p(\Omega; \mu) = L^p(\Omega; dx)$. In the next two sections, we do not have to distinguish integrals w.r.t. μdx and dx , except if the self-adjointness and integration by parts formula are concerned.

3.1. Riesz transforms on bounded convex domains. Since $\phi \in C^1(\Omega)$ is bounded with $\inf_{\Omega} \phi^2 > 0$, and $V \in C^2(\Omega)$ is bounded, we have that for every $f \in L^2(\Omega)$,

$$\begin{aligned} \int_{\Omega} |\nabla \mathcal{L}_N^{-1/2} f(x)|^2 dx &\leq C \left(\inf_{\Omega} \phi^2 \right)^{-1} \int_{\Omega} \phi^2 \nabla \mathcal{L}_N^{-1/2} f(x) \overline{\nabla \mathcal{L}_N^{-1/2} f(x)} \mu dx \\ &\leq C Q(\mathcal{L}_N^{-1/2} f, \mathcal{L}_N^{-1/2} f) \\ &= C \int_{\Omega} |f(x)|^2 \mu dx \\ &\leq C \|f\|_{L^2(\Omega)}^2, \end{aligned}$$

where

$$Q(u, v) := \frac{1}{2} \int_{\Omega} \phi^2 \nabla u \overline{\nabla v} \mu dx, \quad u, v \in W^{1,2}(\Omega),$$

and the latter equality follows from the fact that Q is symmetric. Hence $\mathcal{D}(Q) = \mathcal{D}(\mathcal{L}_N^{1/2})$ and $Q(u, v) = (\mathcal{L}_N^{1/2} u, \mathcal{L}_N^{1/2} v)$ (see, for example, Theorem 8.1 of [23]). Therefore, the Riesz transform $T = \nabla \mathcal{L}_N^{-1/2}$ is bounded on $L^2(\Omega)$.

In terms of the gradient estimate (3.3) of the kernel of the semigroup $P_t, t > 0$, we have the following result. For its proof, it is almost identical to that of Theorem 1.4 of [3] and is omitted here.

Theorem 3.2. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Then the Riesz transform $T = \nabla \mathcal{L}_N^{-1/2}$ is of weak type $(1, 1)$ and bounded on $L^p(\Omega)$ for all $1 < p < \infty$. That is, there exists $C_p, 1 \leq p < \infty$ such that*

$$\|\nabla \mathcal{L}_N^{-1/2}\|_{L^p(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}$$

and

$$\|\nabla \mathcal{L}_N^{-1/2}\|_{L^{1,\infty}(\Omega)} \leq C_1 \|f\|_{L^1(\Omega)}.$$

3.2. Regularity of the inhomogeneous Neumann problems on convex domains. Given an open, bounded subset Ω of $\mathbb{R}^d$ and $f \in C^\infty(\overline{\Omega})$, we define the Neumann Green operator $\mathbb{G}_N$ as the solution operator $C^\infty(\overline{\Omega}) \ni f \mapsto u = \mathbb{G}_N(f) \in W^{1,2}(\Omega)$ for the Neumann problem

$$(3.5) \quad \begin{cases} -Lu &= f & \text{in } \Omega, \\ \partial_\nu u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where it is also assumed that $\int_\Omega f \mu dx = 0$. Above, $\nu(x)$ denotes the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$, and $\partial_\nu = \nu \cdot \nabla$ stands for the normal derivative. The main result of this section is:

Theorem 3.3. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Let $\mathbb{G}_N$ be the Neumann Green operator for the problem (3.5). Then the operators*

$$(3.6) \quad \frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n,$$

originally defined on $\{f \in C^\infty(\overline{\Omega}) : \int_\Omega f \mu dx = 0\}$, can be extended to bounded operators on $L^p(\Omega)$ for $1 < p \leq 2$.

If $p = 1$, then the operators $\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}$, $i, j = 1, \dots, n$, are also of weak type $(1, 1)$.

The strategy employed in the proof of Theorem 3.3 is to continue the line of study in [12] where X.T. Duong et al. establish regularity results for the Green operator, regarded as the inverse of the Neumann Laplacian on bounded convex domains. Indeed, the kernel of the operator $\nabla^2 L^{-1}$ is formally given by

$$(3.7) \quad \int_0^\infty \nabla_x^2 p_t(x, y) dt,$$

and the second-order derivative $\nabla_x^2 p_t(x, y)$ of the kernel $p_t(x, y)$, associated with the heat semigroup e^{-tL} , does not satisfy a Hölder condition in a spatial variable. The loss of Hölder continuity is compensated by a more subtle built-in regularity property inherited from the semigroup e^{-tL} , and by the availability of certain regularity estimates of the solution to the Neumann problem which is specific to convex domains. For more information about the background of this topic, we refer the reader to [1, 2, 12, 17, 16, 20, 22, 21] and the references therein.

3.2.1. L^2 -estimate. The following coercive estimates in convex domains are going to play a basic role in subsequent developments, whose proof follows the lines of Theorem 3.1.3.3 in [16] and Theorem 2.1 in [1].

Lemma 3.4. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Let $f \in L^2(\Omega)$ have bounded support, and let u be the unique function in $W^{1,2}(\Omega)$ satisfying (3.5). Then $u \in W^{2,2}(\Omega)$ and for any $\psi \in C^\infty(\mathbb{R}^d)$ it follows that*

$$(3.8) \quad \int_\Omega \psi^2 |\nabla^2 u|^2 dx \leq C \int_\Omega (|\nabla \psi|^2 + \psi^2) |\nabla u|^2 dx + C \int_\Omega \psi^2 f^2 dx$$

for some constant $C > 0$, depending on φ and V of L , not depending on ψ . Moreover, one also has

$$(3.9) \quad \int_\Omega |\nabla^2 u|^2 dx \leq C \int_\Omega |\nabla u|^2 dx + C \int_\Omega f^2 dx.$$

Proof. Begin by assuming that $\partial\Omega$ and f are C^∞ . Kadlec's formula on p. 134 of [16] implies that

$$\int_{\Omega} \left(\operatorname{div}(\psi \varphi^2 e^V \nabla u) \right)^2 dx = b + \int_{\Omega} \sum_{i,j} (\psi \varphi^2 e^V \nabla u_i)_j (\psi \varphi^2 e^V \nabla u_j)_i dx,$$

where b is a boundary integral, and the subscripts i and j denote derivatives with respect to the coordinates x_i and x_j . Because $\nu \cdot (\psi \varphi^2 e^V \nabla u) = \psi \varphi^2 e^V \partial_\nu u = 0$ on $\partial\Omega$, b is the integral over the boundary of the second fundamental form evaluated at the tangential component of $\psi \varphi^2 e^V \nabla u$. The only thing we need to know about b is that it is non-negative. Hence,

$$\begin{aligned} \int_{\Omega} \sum_{i,j} \left(\psi^2 \varphi^4 e^{2V} u_{ij}^2 + 2(\psi \varphi^2 e^V)_j u_i (\psi \varphi^2 e^V)_i u_{ij} + (\psi \varphi^2 e^V)_i (\psi \varphi^2 e^V)_j u_i u_j \right) dx \\ \leq \int_{\Omega} \left(\psi \operatorname{div}(\varphi^2 e^V \nabla u) + \nabla \psi \cdot \varphi^2 e^V \nabla u \right)^2 dx. \end{aligned}$$

Observe that $\operatorname{div}(\varphi^2 e^V \nabla u) = 2e^V Lu$ and $(\psi \varphi^2 e^V)_i = (\psi_i \varphi^2 + 2\psi Z_i) e^V$. Since $\phi \in C^1(\Omega)$ is bounded with $\inf_{\Omega} \phi^2 > 0$, and $Z \in C^1(\Omega; \mathbb{R}^d)$ with $|Z(x)| \leq c(1+|x|)$ for some constant $c > 0$, and $V \in C^2(\Omega)$ is bounded, we use the inequality $2xy \leq (1/C)x^2 + Cy^2$ to obtain

$$(3.10) \quad \int_{\Omega} \psi^2 |\nabla^2 u|^2 dx \leq C \int_{\Omega} (|\nabla \psi|^2 + |\psi|^2) |\nabla u|^2 dx + C \int_{\Omega} \psi^2 f^2 dx$$

for some constant C , depending on φ and V . Finally, (3.9) follows from (3.8) by taking $\psi \equiv 1$.

A routine limiting argument yields the result for all convex bounded domains Ω . See, for example, p. 149 of [16]. $\square$

Proposition 3.5. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Let $\mathbb{G}_N$ be the Neumann Green operator for the problem (3.5). Then $\nabla^2 \mathbb{G}_N f \in L^2(\Omega)$ and*

$$(3.11) \quad \int_{\Omega} |\nabla^2 \mathbb{G}_N f|^2 dx \leq \int_{\Omega} |f|^2 dx$$

for some constant $C > 0$ independent of f .

Proof. To estimate (3.11), from (3.9) it reduces to showing that there exists a constant $C > 0$ independent of f such that

$$(3.12) \quad \int_{\Omega} |\nabla \mathbb{G}_N(f)|^2 dx \leq C \int_{\Omega} |f|^2 dx.$$

Indeed, based on the gradient estimate (3.4) of the heat kernel $p_t(x, y)$ in Lemma 3.1, we have

$$|\nabla \mathbb{G}_N(x, y)| \leq \frac{C}{|x - y|^{d-1}}, \quad \forall x, y \in \Omega.$$

Estimate (3.12) follows readily. This proves Proposition 3.5. $\square$

3.2.2. Proof of Theorem 3.3.

3.2.2.1. *Singular integrals.* The following theorem is Theorem 1, [13].

Theorem 3.6. *Let T be bounded linear operator from $L^2(\Omega)$ into $L^2(\Omega)$ with Ω . Assume that there exists a class of operators A_t , $t > 0$, defined on $L^2(\Omega)$, which is represented by kernels $a_t(x, y)$ in the sense that*

(a) $A_t u(x) = \int_{\Omega} a_t(x, y) u(y) dy$ for any function $u \in L^2(\Omega) \cap L^1(\Omega)$, and the kernels $a_t(x, y)$ satisfy the following condition:

$$|a_t(x, y)| \leq \frac{1}{|B^{\Omega}(y, \sqrt{t})| \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{n+\beta}}, \quad \forall x, y \in \Omega, \quad \forall t > 0, \quad \beta > 0.$$

(b) *The composite operator $T(I - A_t)$ has a measurable kernel $\tilde{K}_t(x, y)$, and there exist constants C and $c > 0$ such that*

$$(3.13) \quad \int_{x \in \Omega: |x-y| \geq c\sqrt{t}} |\tilde{K}_t(x, y)| dx \leq C, \quad y \in \Omega.$$

Moreover, the operator T is of weak type $(1, 1)$. Hence T can be extended from $L^2(\Omega) \cap L^p(\Omega)$ to a bounded operator on $L^p(\Omega)$ for all $1 < p \leq 2$.

3.2.2.2. *Estimates for the space derivatives of the heat kernel.* Recall that $p_t(x, y)$ is the kernel of $e^{-t\mathcal{L}_N}$, which satisfies estimate (3.2). Since Gaussian upper bounds carry over from heat kernels to their time derivative of its kernels (see [11] and [23]), we have

$$(3.14) \quad \left| \frac{d}{dt} p_t(x, y) \right| \leq \frac{C}{t^{1/2} |B^{\Omega}(x, \sqrt{t})|} \exp\left(-\frac{|x-y|^2}{c_5 t}\right), \quad t > 0, x, y \in \Omega$$

for some constant $c_5 > 0$. Note that $p_t(\cdot, y)$ and their time derivatives $\frac{d}{dt} p_t(\cdot, y)$ belong to the domain of $\mathcal{D}(\mathcal{L}_N)$. In particular, $p_t(\cdot, y) \in W^{1,2}(\Omega)$. When Ω is a bounded convex domain, it follows from Proposition 3.5 that for each fixed $y \in \Omega$, the function $u := p_t(\cdot, y) \in W^{2,2}(\Omega)$ is the unique solution of the Neumann problem (3.5) with $f := -\frac{d}{dt} p_t(\cdot, y)$. In particular, $\nabla_x^2 p_t(x, y) \in L^2(\Omega)$.

Lemma 3.7. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. Then there exists $\gamma > 0$ and a positive constant $C = C(\gamma)$ such that for each $y \in \Omega$,*

$$(3.15) \quad \int_{\Omega} |\nabla_x^2 p_t(x, y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx \leq C t^{-2} |B^{\Omega}(y, \sqrt{t})|^{-1}, \quad t > 0.$$

Proof. Recall that c_1, c_2 and c_5 are three constants appearing in (3.2), (3.3) and (3.14), respectively. Let $c_0 = \max\{c_1, c_2, c_5\}$, and fix a $\gamma \in (0, \frac{1}{8c_0})$ and $y \in \Omega$. We apply Lemma 3.4 with

$$u := p_t(\cdot, y), \quad f := -\frac{d}{dt} p_t(\cdot, y) \quad \text{and} \quad \psi := e^{\gamma \frac{|\cdot-y|^2}{2t}}$$

to obtain (for some fixed $\gamma' \in (0, \gamma)$)

$$\begin{aligned} \int_{\Omega} |\nabla_x^2 p_t(x, y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx &\leq \frac{C}{t} \int_{\Omega} |\nabla_x p_t(x, y)|^2 e^{\gamma' \frac{|x-y|^2}{t}} dx \\ &\quad + C \int_{\Omega} |\nabla_x p_t(x, y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx \\ &\quad + C \int_{\Omega} \left| \frac{d}{dt} p_t(x, y) \right|^2 e^{\gamma \frac{|x-y|^2}{t}} dx \\ &=: J_t^{(1)}(y) + J_t^{(2)}(y) + J_t^{(3)}(y). \end{aligned}$$

It follows from the gradient estimate (3.3) of $p_t(x, y)$ that

$$J_t^{(1)}(y) \leq Ct^{-2} |B^{\Omega}(y; \sqrt{t})|^{-1}.$$

We then use the estimate (3.14) to obtain

$$J_t^{(3)}(y) \leq Ct^{-2} |B^{\Omega}(y; \sqrt{t})|^{-1}.$$

For the term $J_t^{(2)}(y)$, we estimate it by considering two cases:

Case 1. When $0 < t \leq 1$, we use Lemma 3.1 to obtain

$$\begin{aligned} J_t^{(2)}(y) &\leq C \int_{\Omega} t^{-(d+1)} e^{-2 \frac{|x-y|^2}{c_2 t}} e^{\gamma \frac{|x-y|^2}{t}} dx \\ &\leq Ct^{-1} |B^{\Omega}(y; \sqrt{t})|^{-1} \\ &\leq Ct^{-2} |B^{\Omega}(y; \sqrt{t})|^{-1}. \end{aligned}$$

Case 2. When $t > 1$, using Lemma 3.1 again, we have

$$J_t^{(2)}(y) \leq C \int_{\Omega} e^{-2c_3 t} e^{-2 \frac{|x-y|^2}{c_2 t}} e^{\gamma \frac{|x-y|^2}{t}} dx \leq Ct^{-2} |B^{\Omega}(y; \sqrt{t})|^{-1},$$

and hence, $J_t^{(2)}(y) \leq Ct^{-2} |B^{\Omega}(y; \sqrt{t})|^{-1}$. The desired estimate (3.15) now follows readily. $\square$

Lemma 3.8. *Let Ω be a bounded convex domain in $\mathbb{R}^d$. There exists a constant $C > 0$ independent of $y \in \Omega$ such that*

$$(3.16) \quad \int_{x \in \Omega: |x-y| \geq \sqrt{s}} |\nabla_x^2 p_t(x, y)| dx \leq Ct^{-1} e^{-\gamma s/(4t)}, \quad \forall y \in \Omega, \forall s, t > 0.$$

Proof. Fix $s, t > 0$ and $y \in \Omega$. Then using Cauchy-Schwarz's inequality and estimate (3.15) we obtain

$$\begin{aligned} &\int_{x \in \Omega: |x-y| \geq \sqrt{s}} |\nabla_x^2 p_t(x, y)| dx \\ &\leq \left(\int_{\Omega} |\nabla_x^2 p_t(x, y)|^2 e^{\gamma \frac{|x-y|^2}{t}} dx \right)^{1/2} \left(\int_{|x-y| \geq \sqrt{s}} e^{-\gamma \frac{|x-y|^2}{t}} dx \right)^{1/2} \\ &\leq Ct^{-1} e^{-\gamma s/(4t)}, \end{aligned}$$

where in the last inequality we have used the fact that

$$\int_{x \in \Omega: |x-y| \geq \sqrt{s}} e^{-\gamma \frac{|x-y|^2}{t}} dx \leq C_{\gamma} e^{-\gamma s/(2t)} |B^{\Omega}(y; \sqrt{t})|, \quad \forall s \geq 0, \quad \forall t > 0.$$

This concludes the proof of Lemma 3.8. $\square$

3.2.2.3. *Proof of Theorem 3.3.* Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let ∇^2 denote generic second-order derivatives. From Proposition 3.5 it follows that the operator $T = \nabla^2 \mathcal{L}_N^{-1}$ is bounded on $L^2(\Omega)$. Following an argument of [12], we use the formula

$$(3.17) \quad Tf = \int_0^\infty \nabla^2 e^{-t\mathcal{L}_N} f \, dt,$$

in which the integral converges in the strong operator topology on the space of linear and bounded operators on $L^2(\Omega)$. Indeed, observe that

$$(3.18) \quad \begin{aligned} \int_0^\infty \nabla^2 e^{-t\mathcal{L}_N} f \, dt &= \nabla^2 \mathcal{L}_N^{-1} \int_0^\infty \mathcal{L}_N e^{-t\mathcal{L}_N} f \, dt \\ &= \nabla^2 \mathcal{L}_N^{-1} \int_0^\infty \frac{d}{dt} (-e^{-t\mathcal{L}_N} f) \, dt. \end{aligned}$$

Then use the fact that $\|e^{-t\mathcal{L}_N} f - f\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow 0$ and $\|e^{-t\mathcal{L}_N} f\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow \infty$.

It follows that the associated kernel of T is given by

$$(3.19) \quad \int_0^\infty \nabla_x^2 p_t(x, y) \, dt.$$

To prove that T is of weak type $(1, 1)$, the idea is to make use of Theorem 3.6 above. Consider the composite operator $T(I - e^{-s\mathcal{L}_N})$, $s > 0$, whose associated integral kernel is denoted by $\tilde{K}_s(x, y)$. It follows that

$$(3.20) \quad T(I - e^{-s\mathcal{L}_N}) = \int_0^\infty (\nabla^2 e^{-t\mathcal{L}_N} - \nabla^2 e^{-(s+t)\mathcal{L}_N}) \, dt = \int_0^s \nabla^2 e^{-t\mathcal{L}_N} \, dt,$$

where, in the last step, we have decoupled the integrands and made a change of variables. Thus, by invoking Lemma 3.8,

$$\begin{aligned} \int_{x \in \Omega: |x-y| \geq \sqrt{s}} |\tilde{K}_s(x, y)| \, dx &\leq \int_{x \in \Omega: |x-y| \geq \sqrt{s}} \left| \int_0^s \nabla^2 p_t(x, y) \, dt \right| \, dx \\ &\leq \int_0^s \left(\int_{x \in \Omega, |x-y| \geq \sqrt{s}} |\nabla^2 p_t(x, y)| \, dx \right) \, dt \\ &\leq C \int_0^s t^{-1} e^{-\gamma s/4t} \, dt \leq C' \end{aligned}$$

for some finite, positive constant C' which is independent of $y \in \Omega$ and $t > 0$. With this in hand, and having already established the L^2 -boundedness of T , Theorem 3.6 applies and yields that T is of weak type $(1, 1)$. Hence, by interpolation, T can be extended from $L^2(\Omega) \cap L^p(\Omega)$ to a bounded operator on $L^p(\Omega)$ for all $1 < p \leq 2$. $\square$

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