CONVEX RATIONALLY CONNECTED VARIETIES

R. PANDHARIPANDE

(Communicated by Lev Borisov)

ABSTRACT. We prove that all nonsingular, convex, rationally connected, complete intersections in projective space are homogeneous.

INTRODUCTION

Let X be a nonsingular projective variety over $\mathbb{C}$. A morphism,

$$\mu : \mathbf{P}^1 \rightarrow X,$$

is *unobstructed* if $H^1(\mathbf{P}^1, \mu^*T_X) = 0$. The variety X is *convex* if all morphisms $\mu : \mathbf{P}^1 \rightarrow X$ are unobstructed.

A *rational curve* in X is the image of a morphism

$$\mu : \mathbf{P}^1 \rightarrow X.$$

The variety X is *rationally connected* if all pairs of points of X are connected by rational curves.

Homogeneous spaces $\mathbf{G}/\mathbf{P}$ for connected linear algebraic groups are convex, rationally connected, nonsingular, projective varieties. Convexity is a consequence of the global generation of the tangent bundle of $\mathbf{G}/\mathbf{P}$. Rational connectedness is a consequence of the rationality of $\mathbf{G}/\mathbf{P}$.

The following speculation arose at dinner after an algebraic geometry seminar at Princeton in the fall of 2002.

Speculation. *If X is convex and rationally connected, then X is a homogeneous space.*

The failure of the speculation would perhaps be more interesting than the success.

1. COMPLETE INTERSECTIONS

The only real evidence known to the author is the following result for complete intersections in projective space.

Theorem. *If $X \subset \mathbf{P}^n$ is a convex, rationally connected, nonsingular complete intersection, then X is a homogeneous space.*

Received by the editors May 24, 2011 and, in revised form, September 1, 2011.
 2010 *Mathematics Subject Classification*. Primary 14E08.

Proof. We first consider nonsingular complete intersections of dimension at most 1:

- (i) in dimension 0, only points are rationally connected,
- (ii) in dimension 1, only $\mathbf{P}^1$ is rationally connected.

Hence, the rationally connected complete intersections of dimension at most 1 are simply connected.

Let $X \subset \mathbf{P}^n$ be a *generic* complete intersection of type $(d_1, \dots, d_l)$. Let $d = \sum_{i=1}^l d_i$. By the results of [4], X is rationally connected if and only if $d \leq n$. Moreover, if X is rationally connected, then X must be simply connected: if the dimension of X at least 2, X is simply connected by Lefschetz; see [5].

Let M denote the parameter space of lines in X . M is a nonempty, nonsingular variety of dimension $2n - 2 - d - l$. Nonemptiness can be seen by several methods. For example, the nonvanishing in degree 1 of the 1-point series of the quantum cohomology of X implies that M is nonempty; see [1], [6]. The l equations defining X yield sections s_i of

$$\mathrm{Sym}^{d_i} S^* \rightarrow \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^n),$$

where S is the tautological subbundle over the Grassmannian of lines. Since the bundles $\mathrm{Sym}^{d_i} S^*$ are generated by sections coming from equations and since

$$M \subset \mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^n)$$

is the simultaneous zero locus of $s_1, \dots, s_l$, M is nonsingular for generic X by Bertini. Let

$$\pi : U \rightarrow M$$

denote the universal family of lines over M , and let

$$\nu : U \rightarrow X$$

denote the universal morphism.

Let L be a line on X . If X is convex, the normal bundle N_L of L in X must be semi-positive. If N_L has a negative line summand, then every double cover of L ,

$$\mu : \mathbf{P}^1 \rightarrow L \subset X,$$

is obstructed. Since the degree of N_L is $n - d - 1$, we may assume that $d \leq n - 1$.

Every semi-positive bundle on $\mathbf{P}^1$ is generated by global sections. Hence, if every line L has semi-positive normal bundle, we easily conclude that the morphism ν is smooth and surjects onto X . The fiber of ν over $x \in X$ is the parameter space of lines passing through x .

We now consider the Leray spectral sequence for the fibration ν ; see [2]. The Leray spectral sequence degenerates at the E_2 term,

$$E_2^{pq} = H^p(X, R^q \nu_* \mathbb{C}).$$

Since X is simply connected, all local systems on X are constant. Hence,

$$E_2^{pq} = H^p(X, R^q \nu_* \mathbb{C}) = H^p(X, \mathbb{C}) \otimes H^q(F, \mathbb{C}),$$

where F denotes the fiber of ν .

Let $p_U(t)$, $p_F(t)$, and $p_X(t)$ denote the Poincaré polynomials of the manifolds U , F , and X . We conclude that

$$p_U = p_F \cdot p_X.$$

On the other hand, since U is a locally trivial fibration over M , the polynomial $p_{\mathbf{P}^1}$ must divide p_U . Since

$$p_{\mathbf{P}^1} = 1 + t^2$$

is irreducible over the integers, we find that $1 + t^2$ divides either p_F or p_X .

We have proven the following result. Let $X \subset \mathbf{P}^n$ be a generic complete intersection of type $(d_1, \dots, d_l)$ satisfying $d \leq n - 1$. If every line of X has a semi-positive normal bundle, then either $p_F(i) = 0$ or $p_X(i) = 0$.

Consider the fiber F of ν over x . The dimension of F is $n - 1 - d$. In fact, F is a complete intersection of type

$$(1, 2, 3, \dots, d_1, 1, 2, 3, \dots, d_2, \dots, 1, 2, 3, \dots, d_l)$$

in the projective space $\mathbf{P}^{n-1}$ of lines of $\mathbf{P}^n$ passing through x .

If $p_F(i) = 0$, then the type of F must be one of the three types allowed by the lemma below. If $p_X(i) = 0$, then the type of X must be one of the three allowed by the lemma. Since one of the two polynomial evaluations must vanish, we conclude that the type of X must be either $(1, \dots, 1)$ or $(1, \dots, 1, 2)$. Clearly both are types of homogeneous varieties.

If X is not of one of the above two types, then X must contain a line L for which N_L has a negative line summand. Since X was assumed to be general, every nonsingular complete intersection Y of the type of X must also contain such a line by taking a limit of L .

Therefore, if the type of a nonsingular complete intersection Y is not $(1, \dots, 1)$ or $(1, \dots, 1, 2)$, then Y is not a convex, rationally connected variety. $\square$

The proof of the theorem also shows that homogeneous complete intersections in projective space must be of type $(1, \dots, 1)$ or $(1, \dots, 1, 2)$.

Lemma. *Let $Y \subset \mathbf{P}^n$ be a nonsingular complete intersection of dimension k . Let $p_Y(t)$ be the Poincaré polynomial of Y . If $p_Y(i) = 0$, then one of the following three possibilities holds:*

- (i) *the type of Y is $(1, \dots, 1)$ and k is odd,*
- (ii) *the type of Y is $(1, \dots, 1, 2)$ and k is odd,*
- (iii) *the type of Y is $(1, \dots, 1, 2)$, and $k \equiv 2 \pmod{4}$.*

Proof. Let $Y \subset \mathbf{P}^n$ be a nonsingular complete intersection of dimension k . The cohomology of Y is determined by the Lefschetz isomorphism except in the middle (real) dimension k . The cohomology determined by Lefschetz is of rank 1 in all even (real) dimensions. If k is odd, then

$$p_Y(t) = \sum_{q=0}^k t^{2q} + b_k t^k,$$

where b_k is the k^{th} Betti number. We see that $p_Y(i) = 0$ if and only if $b_k = 0$. If k is even,

$$p_Y(t) = \sum_{q=0}^k t^{2q} + (b_k - 1)t^k.$$

We see that $p_Y(i) = 0$ if and only if $k \equiv 2 \pmod{4}$ and $b_k - 1 = 1$.

Assume $p_Y(i) = 0$. Let $(e_1, \dots, e_k)$ be the type of Y . Let e be the largest element of the type.

Let $Z \subset \mathbf{P}^n$ be a nonsingular projective variety of dimension r . Let $H_d \subset \mathbf{P}^n$ be a general hypersurface of degree d . The dimension,

$$h^{r-1}(Z \cap H_d, \mathbb{C}),$$

is a nondecreasing function of d ; see [3]. Hence, we can bound b_k for Y from below by the middle cohomology b'_k of the complete intersection $Y' \subset \mathbf{P}^n$ of type $(e, 1, \dots, 1)$,

$$b_k \geq b'_k.$$

The variety Y' may then be viewed as a hypersurface of degree e in the smaller projective space $\mathbf{P}^{k+1}$.

For a hypersurface $Y' \subset \mathbf{P}^{k+1}$ of degree e , the middle cohomology b'_k is given by the following formula:

$$b'_k - \delta_k = \frac{(e-1)}{e}((e-1)^{k+1} - (-1)^{k+1}),$$

where δ_k is 1 if k is even and 0 if k is odd. If k is odd,

$$b'_k = \frac{(e-1)}{e}((e-1)^{k+1} - 1).$$

Then, $b'_k > 0$ if $e > 2$. If k is even,

$$b'_k - 1 = \frac{(e-1)}{e}((e-1)^{k+1} + 1).$$

Then, $b_k - 1 > 1$ if $e > 2$. Therefore, we conclude that $e \leq 2$.

If $e = 1$, then case (i) of the lemma is obtained. It is easy to check that k must be odd for $p_Y(i) = 0$ to hold.

Let $e = 2$. If Y is of type $(1, \dots, 1, 2)$, then either case (ii) or (iii) of the lemma is obtained. If k is even,

$$k = 2 \pmod{4}$$

must be satisfied in order for $p_Y(i) = 0$ to hold.

If Y is not of type $(1, \dots, 1, 2)$, then the next largest type of Y must be at least 2. As before, we may bound b_k from below by the middle cohomology b'_k of the complete intersection of type $(2, 2)$ in $\mathbf{P}^{k+2}$. If k is odd, the calculation below shows that

$$b'_k = k + 1 > 0.$$

If k is even, the calculation below shows that

$$b'_k - 1 = k + 3 > 1.$$

In fact, the type of Y *cannot* contain two elements greater than 1 in $p_Y(i) = 0$.

Let $k \geq 0$. The Euler characteristic $\chi_{22}(k)$ of a nonsingular complete intersection of type $(2, 2)$ in $\mathbf{P}^{k+2}$ is

$$\int_{\mathbf{P}^{k+2}} \left(\frac{2H}{1+2H} \right)^2 (1+H)^{k+3} = \sum_{i=0}^k 2^{k+2-i} (-1)^{k-i} (k+1-i) \binom{k+3}{i}.$$

On the other hand, since

$$\begin{aligned} (k+3)(t-1)^{k+2} - (t-1)^{k+3} + (-1)^{k+3} \\ = \sum_{i=0}^{k+2} t^{k+2-i} (-1)^i (k+3-i-t) \binom{k+3}{i}, \end{aligned}$$

we find

$$(-1)^k \chi_{22}(k) = k+2 + (-1)^k (k+2).$$

The formulas for b'_k then follow easily. $\square$

2. HOMOGENEOUS COMPLETE INTERSECTIONS

It is interesting to see how the homogeneous complete intersections survive the above analysis.

First, consider a complete intersection $X \subset \mathbf{P}^n$ of type $(1, \dots, 1)$. Then, F is of dimension $n-1-l$, and X is of dimension $n-l$. Both are complete intersections of hyperplanes. Since one of $n-1-l$ and $n-l$ is odd, exactly one of the conditions $p_F(i) = 0$ or $p_X(i) = 0$ holds by part (i) of the lemma.

Next, consider a complete intersection $X \subset \mathbf{P}^n$ of type $(1, \dots, 1, 2)$, where $l+1 \leq n-1$. Then, F is of dimension $n-1-l-1$, and X is of dimension $n-l$. There are two cases:

- (i) If $n-l-2$ and $n-l$ are odd, then both $p_F(i) = 0$ and $p_X(i) = 0$ by part (ii) of the lemma.
- (ii) If $n-l-2$ and $n-l$ are even, then one of $(n-l-2)/2$ and $(n-l)/2$ is odd. Hence, exactly one of the conditions $p_F(i) = 0$ and $p_X(i) = 0$ holds by part (iii) of the lemma.

REFERENCES

- [1] A. Givental, *Equivariant Gromov-Witten invariants*, Int. Math. Res. Notices **13** (1996), 613-663. MR1408320 (97e:14015)
- [2] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley, 1978. MR507725 (80b:14001)
- [3] N. Katz and R. Pandharipande, *Inequalities related to Lefschetz pencils and integrals of Chern classes*, in Geometric aspects of Dwork theory, A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, F. Loeser, Eds., de Gruyter, 2004, 805-819. MR2099087 (2005j:14008)
- [4] J. Kollár, Y. Miyaoka, and S. Mori, *Rationally connected varieties*, J. Alg. Geom. **1** (1992), 429-448. MR1158625 (93i:14014)
- [5] J. Milnor, *Morse theory*, Princeton University Press, 1963. MR0163331 (29:634)
- [6] R. Pandharipande, *Rational curves on hypersurfaces (after A. Givental)*, Séminaire Bourbaki, 50ème année, 1997-1998, Astérisque **252** (1998), no. 848. MR1685628 (2000e:14094)

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

Current address: Departement Mathematik, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland