A MONOMIAL BASIS FOR THE HOLOMORPHIC FUNCTIONS ON c_0

SEÁN DINEEN AND JORGE MUJICA

(Communicated by Thomas Schlumprecht)

ABSTRACT. For over thirty years it has been known that the monomials form a basis for the n -homogeneous polynomials on certain infinite dimensional Banach spaces. Recently, Defant and Kalton have shown that these are never unconditional. In this article we show that the monomials form a basis for both the holomorphic functions and the holomorphic functions of bounded type on c_0 , both with their natural topologies.

INTRODUCTION

Various authors have considered monomial expansions of polynomials defined on infinite dimensional Banach spaces ([1, 2, 4, 8, 10, 13]), and although it has been shown that they form a Schauder basis in some spaces, so far we do not have analogous results for spaces of holomorphic functions. To show that the monomials form a basis for various spaces of holomorphic functions on c_0 we use three different decompositions: $\mathcal{S}_*$ -absolute decompositions of locally convex spaces (see section 3.3 in [6]), finite dimensional monotone decompositions of a Banach space [4], and a Schauder basis (p. 32 in [3]). We discuss these in section 1. In section 2 we recall the definitions of the different spaces of holomorphic functions and discuss the square order on the monomials. We prove our main result in section 3.

1. LINEAR DECOMPOSITIONS

A sequence of subspaces $\{E_n\}_{n=1}^\infty$ of a locally convex space E is called a decomposition for E if for each $x \in E$ there exists a unique sequence $(x_n)_{n=1}^\infty$, $x_n \in E_n$ for all n , such that

$$x = \sum_{n=1}^{\infty} x_n := \lim_{n \rightarrow \infty} \sum_{j=1}^n x_j.$$

We say that $\{E_n\}_{n=1}^\infty$ is an $\mathcal{S}_*$ -absolute decomposition of E if E admits a fundamental system $\mathcal{N}$ of semi-norms p such that for any sequence of scalars $(\alpha_n)_{n=1}^\infty$ satisfying $\limsup_{n \rightarrow \infty} |\alpha_n|^{1/n} < \infty$ the semi-norm q ,

$$(1) \quad q\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} |\alpha_n| p(x_n),$$

is continuous.

Received by the editors March 4, 2011 and, in revised form, July 5, 2011 and September 6, 2011.

2010 *Mathematics Subject Classification.* Primary 46G20, 32A05.

Key words and phrases. Holomorphic function, Schauder basis, monomial.

©2012 American Mathematical Society
 Reverts to public domain 28 years from publication

This concept coincides with the notion of a global Schauder decomposition given in [7] and [14] and is a variation on $\mathcal{S}$ -absolute decomposition discussed in [6], section 3.3.

If each E_n is a finite dimensional space, then the decomposition is called finite dimensional. If E is a normed linear space with norm $\|\cdot\|$, then the decomposition constant is the constant defined as the infimum of all c such that

$$(2) \quad \left\| \sum_{j=1}^n x_j \right\| \leq c \left\| \sum_{j=1}^m x_j \right\|$$

for all positive integers m and n , $n < m$, and all $x_j \in E_j$. The decomposition constant is always greater than or equal to 1, and if it equals 1 we say that the decomposition is monotone. A renorming generally changes the decomposition constant.

If each E_n is one dimensional and e_n spans E_n , we say that $(e_n)_{n=1}^\infty$ is a Schauder basis for E . In this case there exists for each x a sequence of scalars $(x_n)_{n=1}^\infty$ such that $x = \sum_{n=1}^\infty x_n e_n$, and we use the term basis constant in place of decomposition constant. The linear functional $x \mapsto x_n$ is called the n th coefficient functional and is denoted by e_n^* .

If E has an $\mathcal{S}_*$ -decomposition, $\{E_n\}_{n=1}^\infty$, then it admits, by (1), a fundamental system $\mathcal{N}$ of semi-norms p such that

$$(3) \quad p\left(\sum_{n=1}^\infty x_n\right) = \sum_{n=1}^\infty p(x_n)$$

for all $\sum_{n=1}^\infty x_n \in E$, $x_n \in E_n$ for all n . If each E_n has a Schauder basis $(e_{n,m})_{m=1}^\infty$, then an ordering of $(e_{n,m})_{n,m=1}^\infty$ into a sequence is given by a bijective mapping $\phi: \mathbb{N}^2 \rightarrow \mathbb{N}$. We say that the ordering is compatible if

$$(4) \quad m < \overline{m} \implies \phi(n, m) < \phi(n, \overline{m})$$

for all n , that is, if it induces on each E_n its original order (see Proposition 4.1 in [6]).

Lemma 1. *Let $\phi: \mathbb{N}^2 \rightarrow \mathbb{N}$ denote a compatible ordering. Then for every positive integer j , there exists a finite subset S_j of positive integers and a finite set of positive integers $(k_n(j))_{n \in S_j}$ such that*

$$(5) \quad \{k : 1 \leq k \leq j\} = \bigcup_{n \in S_j} \{\phi(n, m) : 1 \leq m \leq k_n(j)\}.$$

Moreover, if l is a positive integer, then $S_{j+l} = S_j \cup S$ for some finite subset S of $\mathbb{N}$, disjoint from S_j , and $k_n(j) \leq k_n(j+l)$ for all $n \in S_j$.

Proof. We prove this result by induction on j . Since ϕ is surjective there exists a pair of integers (n_1, m_1) such that $\phi(n_1, m_1) = 1$. Let $S_1 = \{n_1\}$. If $m_1 > 1$, then, since ϕ is injective, we have $\phi(n_1, 1) > 1 = \phi(n_1, m_1)$ and this contradicts (4). Hence $m_1 = 1$ and (5) holds when $j = 1$.

Now suppose (5) holds for the positive integer j . This implies, in particular, that

$$(6) \quad \{1, 2, \dots, j\} \subset \bigcup_{n \in S_j} \{\phi(n, m) : m \in \mathbb{N}\}.$$

By surjectivity of ϕ we can find a pair of positive integers (n_{j+1}, m_{j+1}) such that $\phi(n_{j+1}, m_{j+1}) = j+1$. We consider two cases.

If $n_{j+1} \notin S_j$, then, since ϕ is injective, (6) implies that $\phi(n_{j+1}, m) \geq j+1$ for all $m \in \mathbb{N}$. If $m_{j+1} > 1$, then, since $\phi(n_{j+1}, m_{j+1}) = j+1$, the injectivity of ϕ implies $\phi(n_{j+1}, 1) > j+1$. We then have $\phi(n_{j+1}, 1) > j+1 = \phi(n_{j+1}, m_{j+1})$ and this contradicts (4). Hence $m_{j+1} = 1$. Letting $S_{j+1} = S_j \cup \{n_{j+1}\}$, $k_n(j) = k_n(j+1)$ for $n \in S_j$ and $k_{n_{j+1}}(j+1) = 1$ we obtain (5).

If $n_{j+1} \in S_j$, let $S_{j+1} = S_j$. If $m_{j+1} > k_{n_{j+1}}(j) + 1$, then, by (6) and since ϕ is injective,

$$\phi(n_{j+1}, k_{n_{j+1}}(j) + 1) > j+1 = \phi(n_{j+1}, m_{j+1})$$

and this contradicts (4). Hence $m_{j+1} \leq k_{n_{j+1}}(j) + 1$. By (6), $\phi(n_{j+1}, l) \leq j$ for all $l \leq k_{n_{j+1}}(j)$ and, as ϕ is injective, this implies $m_{j+1} > k_{n_{j+1}}(j)$. Hence $m_{j+1} = k_{n_{j+1}}(j) + 1$. If $n \in S_j$, $n \neq n_{j+1}$, let $k_n(j+1) = k_n(j)$ and let $k_{n_{j+1}}(j+1) = k_{n_{j+1}}(j) + 1$. This implies that (5) holds for $j+1$. By induction this completes the proof of (5) and the remainder of the proof follows easily. $\square$

Theorem 1. Suppose E has an $\mathcal{S}_*$ -decomposition, $\{E_n\}_{n=1}^\infty$, with fundamental system $\mathcal{N}$ of semi-norms p satisfying (3) and that each E_n has a Schauder basis $(e_{n,m})_{m=1}^\infty$. Then $(e_{n,m})_{n,m=1}^\infty$, with any compatible ordering, is a basis for E if the basis constants $c_{p,n}$ of (E_n, p) for $(e_{n,m})_{m=1}^\infty$ have at most exponential growth, i.e.,

$$\limsup_{n \rightarrow \infty} c_{p,n}^{1/n} < \infty$$

for $p \in \mathcal{N}$.

Proof. Let $\phi : \mathbb{N}^2 \rightarrow \mathbb{N}$ denote a fixed compatible order on $\mathbb{N}^2$. By the definitions of decomposition and basis we see that $(e_{n,m})_{n,m=1}^\infty$ spans a dense subspace of E and hence it suffices to show it is a basic sequence in E . To show this we apply Theorem 6, p. 298, in [9]. Let p denote a semi-norm on E satisfying (3). By (1), the semi-norm

$$q\left(\sum_{k=1}^\infty x_n\right) := \sum_{k=1}^\infty c_{p,n} p(x_n), x_n \in E_n, \sum_{n=1}^\infty x_n \in E$$

is continuous on E .

Let $(\alpha_k)_{k \in \mathbb{N}}$ denote an arbitrary set of scalars. We now use the notation employed in Lemma 1. If j and l are positive integers, then $S_{j+l} = S_j \cup S$ for some finite subset $S \subset \mathbb{N}$ disjoint from S_j , and $k_n(j) \leq k_n(j+l)$ for all $n \in S_j$. We then have

for all positive integers j and l ,

$$\begin{aligned}
 p\left(\sum_{k=1}^j \alpha_k e_{\phi^{-1}(k)}\right) &= p\left(\sum_{n \in S_j} \left\{ \sum_{m=1}^{k_n(j)} \alpha_{\phi(n,m)} e_{n,m} \right\}\right) \\
 &= \sum_{n \in S_j} p\left(\sum_{m=1}^{k_n(j)} \alpha_{\phi(n,m)} e_{n,m}\right) \\
 &\leq \sum_{n \in S_j} c_{p,n} p\left(\sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m}\right) \\
 &\leq \sum_{n \in S_j} c_{p,n} p\left(\sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m}\right) + c_{n,p} p\left(\sum_{n \in S} \left\{ \sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m} \right\}\right) \\
 &= \sum_{n \in S_{j+l}} c_{p,n} p\left(\sum_{m=1}^{k_n(j+l)} \alpha_{\phi(n,m)} e_{n,m}\right) \\
 &= q\left(\sum_{k=1}^{j+l} \alpha_k e_{\phi^{-1}(k)}\right).
 \end{aligned}$$

This completes the proof. $\square$

Let $c_0 = \{(z_j)_{j=1}^\infty : z_j \in \mathbb{C} \text{ all } j, \lim_{j \rightarrow \infty} z_j = 0\}$ and let $c_0^+ = \{(z_j)_{j=1}^\infty \in c_0 : z_j \geq 0 \text{ for all } j\}$. We denote by $(e_j)_{j=1}^\infty$ the standard unit vector basis for c_0 and let $(e_j^*)_{j=1}^\infty$ denote the dual unit vector basis for $\ell_1 = c_0'$. The polydiscs

$$P_\beta := \{(z_j)_{j=1}^\infty \in c_0 : |z_j| \leq \beta_j \text{ all } j\}$$

form a fundamental system for the compact subsets of c_0 when $\beta = (\beta_j)_{j=1}^\infty$ ranges over c_0^+ .

2. POLYNOMIALS AND HOLOMORPHIC FUNCTIONS

In this section we discuss concepts from infinite dimensional holomorphy and refer to [6] and [11] for details. Our main result concerns holomorphic functions on c_0 .

For each positive integer n and each Banach space X , let $\mathcal{P}(^n X)$ denote the space of continuous n -homogeneous polynomials on X . Endowed with the supremum norm of uniform convergence over the unit ball B of X , B_X , $\mathcal{P}(^n X)$ is a Banach space.

Let $\mathbb{N}^{(\mathbb{N})}$ denote the set of all sequences of nonnegative integers which are eventually zero. If $(m_i)_{i=1}^\infty \in \mathbb{N}^{(\mathbb{N})}$, we call $|m| := \sum_i m_i$ and $l(m) := \sup\{i : m_i \neq 0\}$ the modulus and length of m , respectively, and call the mapping

$$(z_j)_{j=1}^\infty \in c_0 \longrightarrow z^m := z_1^{m_1} \cdots z_s^{m_s} \cdots$$

a monomial (we use the convention $0^0 = 1$). For positive integers n and k let $\mathcal{P}_k(^n c_0)$ denote the subspace of $\mathcal{P}(^n c_0)$ spanned by $\{z^m : l(m) = k, |m| = n\}$. For all n the sequence $\{\mathcal{P}_k(^n c_0)\}_{k=1}^\infty$ is a finite dimensional decomposition of $\mathcal{P}(^n c_0)$

(see [4], and [6], section 4.1). This follows, for $P \in \mathcal{P}^{(n+1)}c_0$, from the identities

$$\begin{aligned} P\left(\sum_{j=1}^{\infty} z_j e_j\right) &= P(z_1 e_1) + \sum_{k=1}^{\infty} \left\{ P\left(\sum_{j=1}^{k+1} z_j e_j\right) - P\left(\sum_{j=1}^k z_j e_j\right) \right\} \\ &= a_1 z_1^{n+1} + \sum_{s=1}^{n+1} a_s z_1^{n+1-s} z_2^s + \sum_{s=1, t \geq 0, s+t < n+1}^{n+1} a_{s,t} z_1^{n-s-t} z_2^t z_3^s + \dots \end{aligned}$$

and

$$Q_{k+1}\left(\sum_{j=1}^{\infty} z_j e_j\right) := P\left(\sum_{j=1}^{k+1} z_j e_j\right) - P\left(\sum_{j=1}^k z_j e_j\right) =: R_{k+1}\left(\sum_{j=1}^{\infty} z_j e_j\right) \cdot z_{k+1},$$

where $Q_{k+1} \in \mathcal{P}_{k+1}^{(n+1)}c_0$ and $R_{k+1} \in \bigoplus_{j=1}^{k+1} \mathcal{P}_j^{(n)}c_0$. Hence

$$Q_{k+1} = R_{k+1} \cdot e_{k+1}^*.$$

We consider different equivalent norms on $P \in \mathcal{P}^{(n)}c_0$, generated by uniform convergence over bounded polydiscs in c_0 . If $(\lambda_j)_{j=1}^{\infty} \in c_0^+$,

$$A := \{(z_j)_{j=1}^{\infty} \in c_0 : |z_j| \leq \lambda_j, j = 1, 2, \dots\}$$

is a bounded polydisc in c_0 , and $Q_k \in \mathcal{P}_k^{(n)}c_0$ for all k , then for all s, t with $s < t$, we have, since $Q_l(\sum_{j=1}^k z_j e_j) = 0$ for all $l > k$,

$$(7) \quad \left\| \sum_{k=1}^s Q_k \right\|_A \leq \left\| \sum_{k=1}^t Q_k \right\|_A,$$

and this implies that $\{\mathcal{P}_k^{(n)}c_0\}_{k=1}^{\infty}$ is a finite dimensional monotone decomposition of $\mathcal{P}^{(n)}c_0$ for all n . Using the identity $Q_{k+1} = R_{k+1} \cdot e_{k+1}^*$ and the fact that holomorphic functions on polydiscs achieve their absolute maxima on the distinguished boundary we see that

$$(8) \quad \|Q_{k+1}\|_A = \lambda_{k+1} \|R_{k+1}\|_A.$$

We now define the square order on the monomials in $\mathcal{P}^{(n)}c_0$. On $c_0' = \ell_1$ we use the sequential order inherited from the standard unit vector basis $(e_j^*)_{j=1}^{\infty}$. The square order on the monomials in $\mathcal{P}^{(n+1)}c_0$ is defined as follows: if $m = (m_i)_{i=1}^{\infty}$ and $m' = (m'_i)_{i=1}^{\infty}$ are in $\mathbb{N}^{(\mathbb{N})}$ and $|m| = |m'|$, then $m < m'$ if either $l(m) < l(m')$ or $l(m) = l(m')$ and for some positive integer $s \leq l(m)$, $m_s < m'_s$ and $m_t = m'_t$ for all $t > s$.

The square order on the monomials appears naturally when we use the finite dimensional decomposition $(\mathcal{P}_k^{(n)}c_0)_{k=0}^{\infty}$. Clearly, if $k < k'$, then the monomials in $\mathcal{P}_k^{(n+1)}c_0$ precede those in $\mathcal{P}_{k'}^{(n+1)}c_0$ and the order within $\mathcal{P}_k^{(n+1)}c_0$ is determined by the order inherited from $\mathcal{P}^{(n)}c_0$. If $m \in \mathbb{N}^{(\mathbb{N})}$ and $l(m) = s$, then there is a unique $\underline{m} \in \mathbb{N}^{(\mathbb{N})}$ such that $z^m = z^{\underline{m}} z_s$ for all $\sum_{j=1}^{\infty} z_j e_j \in c_0$. Note that $|\underline{m}| = |m| - 1$ and $l(\underline{m}) \leq l(m)$. If $m, m' \in \mathbb{N}^{(\mathbb{N})}$, then $m < m'$ if either $l(m) < l(m')$ or $l(m) = l(m')$ and $\underline{m} < \underline{m}'$.

The square order was introduced by Ryan ([13]), and various authors have shown that the monomials of degree n with the square order are a Schauder basis for $\mathcal{P}^{(n)}c_0$. Theorem 2 contains within it yet another proof of this fact, modulo the result of W. Bogdanowicz and A. Pelczyński in 1957 (see [6], p. 81) that polynomials on c_0 are weakly continuous on bounded sets. The order is important as Defant and

Kalton have shown in [2] that when the monomials of degree n for any $n \geq 2$ form an unconditional basis for the space of n -homogeneous polynomials on a Banach space X , endowed with the norm of uniform convergence over the unit ball of X , then X is finite dimensional.

We let $\mathcal{H}(X)$ denote the space of holomorphic functions on the Banach space X and let $\mathcal{H}_b(X)$ denote the subspace of $\mathcal{H}(X)$ consisting of all f bounded on bounded subsets of X . Let $\mathcal{K}(X)$ denote the set of all compact subsets of X . By Proposition 3.18 in [5] the semi-norms

$$p_K\left(\sum_{n=0}^{\infty} P_n\right) := \sum_{n=0}^{\infty} \|P_n\|_K,$$

where $K \in \mathcal{K}(X)$, generate the compact open topology τ_0 on $\mathcal{H}(X)$. When $X = c_0$, Ex. 6, p. 15, in [3] implies that the semi-norms

$$p_{\beta}\left(\sum_{n=0}^{\infty} P_n\right) := \sum_{n=0}^{\infty} \sup\{|P_n(z)| : z = (z_j)_{j=1}^{\infty} \in c_0, |z_j| \leq \beta_j \text{ for all } j\},$$

where $\beta = (\beta_j)_{j=1}^{\infty}$ ranges over c_0^+ , generates $(\mathcal{H}(c_0), \tau_0)$.

For an arbitrary Banach space X , Proposition 4.39 in [5] shows that the τ_{ω} ported topology of Nachbin is generated by

$$p_{K,\alpha}\left(\sum_{n=0}^{\infty} P_n\right) := \sum_{n=0}^{\infty} \|P_n\|_{K+\alpha_n B_X},$$

where $K \in \mathcal{K}(X)$ and $(\alpha_n)_{n=0}^{\infty} \in c_0^+$ are arbitrary.

Similarly, the τ_b topology on $\mathcal{H}_b(X)$ is generated by the semi-norms

$$p\left(\sum_{n=0}^{\infty} P_n\right) := \sum_{n=0}^{\infty} \|P_n\|_{\alpha B_{c_0}},$$

where $\sum_{n=0}^{\infty} P_n \in \mathcal{H}_b(c_0)$ and α ranges over R^+ .

If $X = c_0$, P is a monomial, and $K = \{(z_j)_{j=1}^{\infty} : |z_j| \leq \beta_j \text{ for all } j\}$, where $(\beta_j)_{j=1}^{\infty} \in c_0^+$, then

$$\|P\|_{K+\alpha B_{c_0}} = \{|P(z)| : z = (z_j)_{j=1}^{\infty}, |z_j| \leq \beta_j + \alpha\}.$$

Note that if $[\beta] := \{(z_j)_{j=1}^{\infty} : |z_j| \leq \beta_j \text{ for all } j\}$, where $(\beta_j)_{j=1}^{\infty} \in c_0^+$, then for any $\sum_{j=1}^{n+1} \alpha_j e_j^*$ we have

$$(9) \quad \left\| \sum_{j=1}^n \alpha_j e_j^* \right\|_{[\beta]} = \sum_{j=1}^n |\alpha_j \beta_j| \leq \sum_{j=1}^{n+1} |\alpha_j \beta_j| = \left\| \sum_{j=1}^{n+1} \alpha_j e_j^* \right\|_{[\beta]}.$$

This shows that for any space X with basis $(e_j)_{j=1}^{\infty}$ and closed unit ball $\{\sum_{j=1}^n \alpha_j e_j : |\alpha_j| \leq \beta_j \text{ for all } j\}$ the basis constant for $(e_j^*)_{j=1}^{\infty}$ is 1.

3. A SCHAUDER BASIS FOR $(\mathcal{H}(c_0), \tau_{\omega})$ AND $(\mathcal{H}_b(c_0), \tau_b)$

In this section we let $(P_{n,m})_{m=1}^{\infty}$ denote the monomials of degree n on c_0 endowed with the square order and we suppose that the set of all monomials is given a compatible order.

Theorem 2. *The monomials with a compatible order are a Schauder basis for $(\mathcal{H}(c_0), \tau_{\omega})$, $(\mathcal{H}(c_0), \tau_0)$, and $(\mathcal{H}_b(c_0), \tau_b)$.*

Proof. In view of the fundamental systems of semi-norms described in the previous section it suffices, by Theorem 1, to take an arbitrary bounded polydisc

$$A := \{(z_m)_{m=1}^\infty \in c_0 : |z_m| \leq \lambda_m \text{ all } m\},$$

where $\lambda_m \geq 0$ for all m , and to show that the basis constant, c_n , for $(\mathcal{P}^n(c_0), \|\cdot\|_A)$ satisfies $c_n \leq 3^n$ for all n . We prove this by induction. Different choices of A then prove the required result for the different spaces of holomorphic functions.

The square ordering on $\mathcal{P}^1(c_0) = c'_0 = \ell_1$ is just the standard ordering of the positive integers and, by (9), the basis constant is 1. We now suppose that $c_n \leq 3^n$ and aim to show $c_{n+1} \leq 3^{n+1}$.

Let $(\alpha_m)_{m=1}^\infty$ denote an arbitrary sequence of scalars. Fix positive integers s and t , $s < t$. For some nonnegative integer k we have the expansion

$$(10) \quad \sum_{m=1}^s \alpha_m P_{n+1,m} = \sum_{u=1}^{k+1} Q_{n+1,u},$$

where $Q_{n+1,u} := \sum_{1 \leq m \leq s, l(P_{n+1,m})=u} \alpha_m P_{n+1,m} \in \mathcal{P}_u(n c_0)$ for $1 \leq u \leq k+1$.

Note that each $P_{n+1,m}$ is a monomial of degree $n+1$ and that if $m_1 < m_2$, then

$$l(P_{n+1,m_1}) \leq l(P_{n+1,m_2}).$$

If $s < m \leq t$, then $l(P_{n+1,m}) \geq k+1$ and for some integer $k^* \geq k+1$,

$$\sum_{m=s+1}^t \alpha_m P_{n+1,m} = \sum_{u=k+1}^{k^*} \left\{ \sum_{s < m \leq t, l(P_{n+1,m})=u} \alpha_m P_{n+1,m} \right\}.$$

If

$$Q_{n+1,u}^* = \sum_{s < m \leq t, l(P_{n+1,m})=u} \alpha_m P_{n+1,m}$$

for $k+1 < u \leq k^*$, then with the convention $\sum_{u>k+1}^{k^*} = 0$ when $k+1 = k^*$, we have

$$(11) \quad \sum_{m=s+1}^t \alpha_m P_{n+1,m} = \sum_{m>s, l(P_{n+1,m})=k+1} \alpha_m P_{n+1,m} + \sum_{u>k+1}^{k^*} Q_{n+1,u}^*.$$

If we let

$$(12) \quad Q_{n+1,k+1}^* = Q_{n+1,k+1} + \sum_{s < m \leq t, l(P_{n+1,m})=k+1} \alpha_m P_{n+1,m},$$

then

$$\begin{aligned} \sum_{m=1}^t \alpha_m P_{n+1,m} &= \sum_{m=1}^s \alpha_m P_{n+1,m} + \sum_{m=s+1}^t \alpha_m P_{n+1,m} \\ &= \sum_{u=1}^{k+1} Q_{n+1,u} + \sum_{s < m \leq t, l(P_{n+1,m})=k+1} \alpha_m P_{n+1,m} + \sum_{u>k+1}^{k^*} Q_{n+1,u}^* \\ &= \sum_{u=1}^k Q_{n+1,u} + Q_{n+1,k+1}^* + \sum_{u>k+1}^{k^*} Q_{n+1,u}^* \\ &= \sum_{u=1}^k Q_{n+1,u} + \sum_{u=k+1}^{k^*} Q_{n+1,u}^*. \end{aligned}$$

This identity and (7) imply that

$$(13) \quad \left\| \sum_{u=1}^k Q_{n+1,u} \right\|_A \leq \left\| \sum_{m=1}^t \alpha_m P_{n+1,m} \right\|_A.$$

If $l(P_{n+1,m}) \leq k$ for $m < m_0$ and $l(P_{n+1,m_0}) = k+1$, then, by (10) and (12),

$$(14) \quad Q_{n+1,k+1} = \sum_{m=m_0}^s \alpha_m P_{n+1,m} = e_{k+1}^* \cdot \sum_{m=m_0}^s \alpha_m P_{n,m-m_0+1}$$

and

$$(15) \quad Q_{n+1,k+1}^* = \sum_{m=m_0}^{s^*} \alpha_m P_{n+1,m} = e_{k+1}^* \cdot \sum_{m=m_0}^{s^*} \alpha_m P_{n,m-m_0+1}$$

for some integer s^* , $s \leq s^* \leq t$. Applying in turn (14), induction, (15), and (13), we obtain

$$\begin{aligned} \|Q_{n+1,k+1}\|_A &= \|e_{k+1}^* \cdot \sum_{m=m_0}^s \alpha_m P_{n,m-m_0+1}\|_A \\ &= \|e_{k+1}^*\|_A \cdot \left\| \sum_{m=m_0}^s \alpha_m P_{n,m-m_0+1} \right\|_A \\ &\leq 3^n \|e_{k+1}^*\|_A \cdot \left\| \sum_{m=m_0}^{s^*} \alpha_m P_{n,m-m_0+1} \right\|_A \\ &= 3^n \|e_{k+1}^* \cdot \sum_{m=m_0}^{s^*} \alpha_m P_{n,m-m_0+1}\|_A \\ &= 3^n \left\| \sum_{m=m_0}^{s^*} \alpha_m P_{n+1,m} \right\|_A \\ &= 3^n \|Q_{n+1,k+1}^*\|_A \\ &\leq 3^n \left\| \sum_{u=k+1}^{k^*} Q_{n+1,u}^* \right\|_A \\ &\leq 3^n \left\{ \left\| \sum_{u=1}^k Q_{n+1,u} \right\|_A + \sum_{u=k+1}^{k^*} \|Q_{n+1,u}^*\|_A + \left\| \sum_{u=1}^k Q_{n+1,u} \right\|_A \right\} \\ &\leq 2 \cdot 3^n \left\| \sum_{m=1}^t \alpha_m P_{n+1,m} \right\|_A. \end{aligned}$$

This estimate, together with (7) and (13), implies that

$$\begin{aligned} \left\| \sum_{m=1}^s \alpha_m P_{n+1,m} \right\|_A &= \left\| \sum_{u=1}^{k+1} Q_{n+1,u} \right\|_A \\ &\leq \left\| \sum_{u=1}^k Q_{n+1,u} \right\|_A + \|Q_{n+1,k+1}\|_A \\ &\leq (1 + 2 \cdot 3^n) \left\| \sum_{m=m_0}^t \alpha_m P_{n+1,m} \right\|_A \end{aligned}$$

and hence

$$c_{n+1} \leq 1 + 2 \cdot 3^n \leq 3^{n+1}.$$

This completes the proof. $\square$

Added in proof. The authors should mention that Ryan has shown that the monomials on l_1 , with the square order, form an unconditional basis for $H(l_1)$ with the compact open topology. See [12].

Our result depended heavily on special features possessed by c_0 . Nevertheless, the existence of a monomial basis for the space of holomorphic functions on c_0 facilitates the search for improved estimates, and this should, in turn, make c_0 a more attractive and flexible space in which to find positive results and on which to construct counterexamples.

REFERENCES

- [1] R. Alencar, On reflexivity and basis for $\mathcal{P}(^m E)$, Proc. Royal Irish Acad., Sect A, 85, 2, 1985, 131-138. MR845536 (87i:46101)
- [2] A. Defant, N. Kalton, Unconditionality in spaces of m -homogeneous polynomials, Q. J. Math., 56, 2005, 53-64. MR2124579 (2005k:46107)
- [3] J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, 1984. MR737004 (85i:46020)
- [4] V. Dimant, S. Dineen, Banach subspaces of spaces of holomorphic mappings and related topics, Math. Scand., 83, 1998, 142-160. MR1662092 (99m:46031)
- [5] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Stud., 57, 1981. MR640093 (84b:46050)
- [6] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag Monographs in Mathematics, 1999. MR1705327 (2001a:46043)
- [7] P. Galindo, M. Maestre, P. Rueda, Biduality in spaces of holomorphic functions, Math. Scand., 86(1), 2000, 5-16. MR1738512 (2001i:46062)
- [8] B.C. Grecu, R.A. Ryan, Polynomials on Banach spaces with unconditional bases, Proc. Amer. Math. Soc., 133, 2005, 4, 1083-1091. MR2117209 (2005i:46051)
- [9] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981. MR632257 (83h:46008)
- [10] M. Matos, On holomorphy in Banach spaces and absolute convergence of Fourier series, Port. Math., 45, 4, 1988, 429-450; and 47, 1, 1990, 13. MR982911 (90f:46075); MR1079501 (91j:46054)
- [11] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland Math. Studies, 120, 1986; reprinted by Dover, Mineola, New York, 2010. MR842435 (88d:46084)
- [12] R.A. Ryan, *Holomorphic mappings on l_1* , Trans. Amer. Math. Soc., 302, 1987, 797-811. MR0891648 (88h:46089)

- [13] R.A. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Thesis, Trinity College Dublin, 1980.
- [14] M. Venkova, Global Schauder decompositions of locally convex spaces, *Math. Scand.*, 101, 1, 2007, 65-82. MR2353242 (2008k:46005)

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY COLLEGE DUBLIN, DUBLIN 4, IRELAND

E-mail address: `sean.dineen@ucd.ie`

IMECC-UNICAMP, RUA SERGIO BUARQUE DE HOLANDA 651, 13083-859 CAMPINAS, SP,
BRAZIL

E-mail address: `mujica@ime.unicamp.br`