GENERALIZED ANALYTICITY ON THE N -TORUS

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ABSTRACT. In dimension $N = 2$, let v be a unit vector of irrational slope. With T_2 the 2-torus, say $f \in \mathcal{A}_v^+$ if $f \in L^1(\mathbf{T}_2)$ and $m \cdot v < 0 \Rightarrow \hat{f}(m) = 0$ for m an integral lattice point. Let G_v be the one-parameter group generated by v on T_2 , and let $E \subset G_v$ be a closed and bounded set. Call E a set of uniqueness for $\mathcal{A}_v^+ \cap C(T_2)$ if $f \in \mathcal{A}_v^+ \cap C(T_2)$ and if $f(x) = 0$ for $x \in E$ implies $f \equiv 0$ on T_2 . The following result is established: *A necessary and sufficient condition that E be a set of uniqueness for $\mathcal{A}_v^+ \cap C(T_2)$ is that E is a set of positive linear measure.*

1. INTRODUCTION

We shall operate in real N -dimensional Euclidean space, $\mathbf{R}^N$, $N \geq 2$, and use the following notation:

$$\begin{aligned} x &= (x_1, \dots, x_N), & y &= (y_1, \dots, y_N), \\ \alpha x + \beta y &= (\alpha x_1 + \beta y_1, \dots, \alpha x_N + \beta y_N), \\ x \cdot y &= x_1 y_1 + \dots + x_N y_N, & |x| &= (x \cdot x)^{\frac{1}{2}}. \end{aligned}$$

$T_N = \{x : -\pi \leq x_j < \pi, j = 1, \dots, N\}$ will be called the N -Torus, and $f \in C(T_N)$ will denote a complex-valued function which is continuous on $\mathbf{R}^N$ and periodic of period 2π in each variable. $f \in C^\infty(T_N)$ will be defined in a similar manner, and $f \in L^1(T_N)$ will have its usual meaning. Also, we shall let Λ_N designate the integral lattice points in $\mathbf{R}^N$ and for $f \in L^1(T_N)$, set

$$\hat{f}(m) = (2\pi)^{-N} \int_{T_N} f(x) e^{-im \cdot x} dx$$

for $m \in \Lambda_N$.

With $v = (v_1, \dots, v_N)$ designating a unit vector in $\mathbf{R}^N$, we define

$$(1.1) \quad \mathcal{A}_v^+ = \left\{ f \in L^1(T_N) : m \cdot v < 0 \Rightarrow \hat{f}(m) = 0 \right\}.$$

We set $\mathcal{A}_v = \mathcal{A}_v^+ \cup \mathcal{A}_{-v}^+$ and call $f \in \mathcal{A}_v$ a generalized analytic function.

Next, we define the 1-parameter group G_v as follows:

$$(1.2) \quad G_v = \{x \in T_N : x = tv \bmod 2\pi \text{ in each variable, } -\infty < t < \infty\}.$$

If v is linearly independent with respect to rational coefficients, then G_v is dense on T_N . If v is linearly dependent with respect to rational coefficients, then G_v is not dense on T_N . (We say that v is linearly dependent with respect to rational

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coefficients if there exist rational numbers $r_1, \dots, r_n$ not all zero such that $r_1 v_1 + \dots + r_n v_n = 0$.)

In the former case, the notion that $E \subset G_v$ is a set of positive linear measure is well-defined. In particular, we leave

$$(1.3) \quad E^* = \{t \in \mathbf{R} : \exists x \in E \text{ s.t. } x = tv \bmod 2\pi \text{ in each variable}\}.$$

We say that E is of positive linear measure if E^* is of positive linear measure. We denote this fact by $|E|_1 > 0$.

Also, we say that $E \subset G_v$ is a closed and bounded set on G_v if E^* is a closed and bounded set on the real line. Likewise, we say that $|E|_1 = 0$ if $|E^*|_1 = 0$.

It is our intention to establish the following theorem relating G_v to $\mathcal{A}_v$.

Theorem 1. *Suppose v is linearly independent with respect to rational coefficients, and let $E \subset G_v$ be a set of positive linear measure. Suppose, furthermore, that $f \in \mathcal{A}_v \cap C(T_N)$, $N \geq 2$, and that*

$$(1.4) \quad x \in E \Rightarrow f(x) = 0.$$

Then f is identically zero on T_N .

What is particularly interesting about the above theorem is that it is in a certain sense a best possible result. We illustrate this fact with the following result on T_2 .

Theorem 2. *Let $N = 2$ and suppose v is linearly independent with respect to rational coefficients. Let $E \subset G_v$ be a closed and bounded nonempty set on G_v with $|E|_1 = 0$. Then there exists an $f \in \mathcal{A}_v \cap C(T_N)$ which is not identically zero on T_2 and which satisfies (1.4).*

Neither Theorem 1 nor Theorem 2 is contained in the three classical papers dealing with generalized analytic functions on the N -Torus, namely [1], [4], and [5]. (For a recent comment about these last two papers, see the February 2011 *Notices of the AMS*, [7, p. 276 and p. 279].) Theorem 1 is motivated by [8], the author's previous work on the subject of strictly generalized analytic functions denoted by $\mathcal{B}_v$. $\mathcal{B}_v \subset \mathcal{A}_v$ is a strictly smaller class and is defined as $\mathcal{B}_v = \mathcal{B}_v^+ \cup \mathcal{B}_{-v}^+$, where

$$\mathcal{B}_v^+ = \left\{ f \in L^1(T_N) : \exists \gamma \text{ with } 0 < \gamma < 1 \text{ s.t. } m \cdot v < \gamma |m| \Rightarrow \widehat{f}(m) = 0 \right\}.$$

2. PROOF OF THEOREM 1

To prove Theorem 1, we shall need the following lemma in the left half of the complex plane.

Lemma A. *Let $\zeta = \sigma + it$ where $\sigma \leq 0$ and $-\infty < t < \infty$, and let $E^\diamond$ be a Lebesgue measurable set contained on the imaginary axis with $|E^\diamond|_1 > 0$. Suppose that (i) $g(\sigma + it)$ is analytic for $\sigma < 0$, (ii) $g(\sigma + it)$ is uniformly bounded for $\sigma < 0$, and*

$$(2.1) \quad (iii) \quad \lim_{\sigma \rightarrow 0} g(\sigma + it) = 0 \text{ for } it \in E^\diamond.$$

Then $g(\sigma + it) = 0$ for $\sigma < 0$ and $-\infty < t < \infty$.

To be explicit, by (ii) above, we mean that $\exists M > 0$ such that

$$(2.2) \quad |g(\sigma + it)| \leq M \text{ for } \sigma < 0 \text{ and } -\infty < t < \infty.$$

Lemma A is a version of the well-known F. and M. Riesz theorem for bounded analytic functions on the unit disk (see [2, p. 50]) carried over to the left half-plane. The proof of Lemma A can be found in [3, p. 449].

Continuing with the proof of Theorem 1, we are given an $f \in \mathcal{A}_v \cap C(T_N)$ which satisfies (1.4). With no loss in generality, we can assume from the start that

$$(2.3) \quad f \in \mathcal{A}_v^+.$$

We set for $h > 0$,

$$(2.4) \quad f(x, h) = (2\pi)^{-N} \int_{T_N} f(x - y) P(y, h) dy,$$

where $P(y, h)$ is the Poisson kernel

$$(2.5) \quad P(y, h) = \sum_{m \in \Lambda_N} e^{im \cdot y} e^{-|m|h}.$$

It is well-known that for $h > 0$, $P(y, h) > 0$ for $y \in T_N$. From (2.5), we obtain that

$$(2\pi)^{-N} \int_{T_N} P(y, h) dy = 1 \quad \text{for } h > 0,$$

and consequently from (2.4) that

$$(2.6) \quad |f(x, h)| \leq \|f\|_{L^\infty(T_N)} \quad \text{for } x \in T_N \text{ and } h > 0.$$

We set

$$(2.7) \quad \Lambda^v = \{m \in \Lambda_N : m \cdot v \geq 0\},$$

and see from (2.3)-(2.5) that

$$(2.8) \quad f(x, h) = \sum_{m \in \Lambda^v} \hat{f}(m) e^{im \cdot x} e^{-|m|h}.$$

It is clear that for $h > 0$, $f(x, h) \in C^\infty(T_N)$.

Next, for $t \in \mathbf{R}$, we set $F(it, h) = f(tv, h)$ and observe that

$$(2.9) \quad F(it, h) = \sum_{m \in \Lambda^v} \hat{f}(m) e^{it(m \cdot v)} e^{-|m|h}.$$

From (2.6), we see that

$$(2.10) \quad |F(it, h)| \leq \|f\|_{L^\infty(T_N)} \quad \text{for } t \in \mathbf{R} \text{ and } \forall h > 0.$$

We introduce analyticity into the picture by setting

$$F(\sigma + it, h) = \sum_{m \in \Lambda^v} \hat{f}(m) e^{(\sigma + it)(m \cdot v)} e^{-|m|h}$$

for $\sigma < 0$. Then it is clear that $F(\sigma + it, h)$ is analytic in the open left half-plane, continuous in the closed left half-plane, and

$$(2.11) \quad |F(\sigma + it, h)| \leq \sum_{m \in \Lambda^v} |\hat{f}(m)| e^{-|m|h},$$

$\forall h > 0, \forall \sigma < 0$, and for $t \in \mathbf{R}$.

Next, we note that

$$\sup_{t \in \mathbf{R}} |F(\sigma + it, h)| \leq \sum_{m \in \Lambda^v} |\hat{f}(m)| e^{\sigma(m \cdot v)} e^{-|m|h}$$

$\forall h > 0, \forall \sigma < 0$. It follows from (2.7) and this last inequality that

$$\limsup_{\sigma \rightarrow -\infty} \sup_{t \in \mathbf{R}} |F(\sigma + it, h)| \leq \left| \widehat{f}(0) \right| \leq \|f\|_{L^\infty(T_N)}$$

$\forall h > 0$. Invoking the Phragmen-Lindelöf theorem, [6, p. 257], we see from this last inequality in conjunction with (2.10) and (2.11) that

$$(2.12) \quad |F(\sigma + it, h)| \leq \|f\|_{L^\infty(T_N)}$$

$\forall h > 0, \forall \sigma < 0$, and for $t \in \mathbf{R}$.

Next, we recall the formula

$$e^{i\xi\eta} e^{-|\xi|\rho} = \pi^{-1} \int_{\mathbf{R}} e^{i\xi s} \rho[\rho^2 + |\eta - s|^2]^{-1} ds$$

for $\xi, \eta \in \mathbf{R}$ and $\rho > 0$, which can be found in many places.

Setting $\xi = (m \cdot v)$, $\eta = t$, and $\rho = -\sigma$, gives us

$$(2.13) \quad e^{(\sigma+it)m \cdot v} = \frac{-1}{\pi} \int_{-\infty}^{\infty} e^{i(m \cdot v)s} \frac{\sigma}{\sigma^2 + |t - s|^2} ds.$$

Observing also from (2.9) that for $h > 0$,

$$F(it, h) = \lim_{R \rightarrow \infty} \sum_{|m| \leq R} \widehat{f}(m) e^{it(m \cdot v)} e^{-|m|h} \quad \text{uniformly for } t \in \mathbf{R},$$

we see from the definition of $F(\sigma + it, h)$ and (2.13) that

$$(2.14) \quad F(\sigma + it, h) = \frac{-1}{\pi} \int_{-\infty}^{\infty} F(is, h) \frac{\sigma}{\sigma^2 + |t - s|^2} ds \quad \forall \sigma < 0 \text{ and for } t \in \mathbf{R}.$$

Next, we recall that $F(is, h) = f(sv, h)$ and that $|f(sv, h)| \leq \|f\|_{L^\infty(T_N)}$. Also, since $f \in C(T_N)$, it follows from (2.4) above that

$$\lim_{h \rightarrow 0} f(sv, h) = f(sv) \quad \forall s \in \mathbf{R}.$$

We conclude that

$$\lim_{h \rightarrow 0} \int_{-\infty}^{\infty} F(is, h) \frac{\sigma}{\sigma^2 + |t - s|^2} ds = \int_{-\infty}^{\infty} f(sv) \frac{\sigma}{\sigma^2 + |t - s|^2} ds$$

for $\sigma < 0$ and $t \in \mathbf{R}$.

It follows from (2.14) that $\lim_{h \rightarrow 0} F(\sigma + it, h)$ exists for $\sigma < 0$ and $t \in \mathbf{R}$. We designate this limit by $g(\sigma + it)$. Thus

$$\lim_{h \rightarrow 0} F(\sigma + it, h) = g(\sigma + it).$$

From (2.12) and the notion of normal families, [6, p. 282], we obtain from this last limit that

$$(2.15) \quad g(\sigma + it) \text{ is analytic for } \sigma < 0 \text{ and } t \in \mathbf{R}.$$

Also, we see from these last two limits and (2.14) that

$$(2.16) \quad g(\sigma + it) = \frac{-1}{\pi} \int_{-\infty}^{\infty} f(sv) \frac{\sigma}{\sigma^2 + |t - s|^2} ds$$

for $\sigma < 0$ and $t \in \mathbf{R}$.

Next, from the fact that $\frac{-\sigma}{\sigma^2 + |s|^2}$ is the Poisson kernel for $s \in \mathbf{R}$ and $f(sv)$ is a continuous and bounded function for $s \in \mathbf{R}$, we observe that

$$\lim_{\sigma \rightarrow 0} \frac{-1}{\pi} \int_{-\infty}^{\infty} f(sv) \frac{\sigma}{\sigma^2 + |t-s|^2} ds = f(tv) \quad \forall t \in \mathbf{R}.$$

But then, we obtain from (2.16) that

$$(2.17) \quad \lim_{\sigma \rightarrow 0} g(\sigma + it) = f(tv) \quad \forall t \in \mathbf{R}.$$

Now from (1.4), we have that

$$(2.18) \quad f(tv) = 0 \quad \text{for } t \in E^*,$$

where E^* is defined by (1.3). But E is a set of positive linear measure; so E^* is a set of positive linear measure. We conclude from (2.17), (2.18), and Lemma A that

$$g(\sigma + it) = 0 \quad \forall \sigma < 0 \text{ and } \forall t \in \mathbf{R}.$$

From this last fact joined with (2.17), we obtain in turn that

$$f(tv) = 0 \quad \forall t \in \mathbf{R}.$$

But then $f(x) = 0$ for $x \in G_v$, and G_v is dense in T_N . Since $f \in C(T_N)$, we conclude that f is identically zero on T_N . $\square$

3. PROOF OF THEOREM 2

We are given $v = (v_1, v_2)$, a unit vector with both components different from zero and the ratio v_1/v_2 irrational. With no loss in generality, we can assume $v_1 > 0$. Two cases present themselves: $v_2 > 0$ or $v_2 < 0$. We will deal with the former case. The latter case is handled in a similar manner. Thus we assume

$$(3.1) \quad v_1 > 0, v_2 > 0, \frac{v_2}{v_1} = \alpha, \text{ where } \alpha \text{ is irrational.}$$

Next, we are given $E \subset G_v$ a closed and bounded set of measure zero. What this means is that E^* defined by (1.3) is a closed and bounded set on the real line of Lebesgue measure zero.

We recall $G_v \subset T_2$, where

$$T_2 = \{x = (x_1, x_2) : -\pi \leq x_j < \pi, j = 1, 2\}.$$

At this point, we introduce the notion of a portion of G_v . We say that B is a portion of G_v provided (i) $B \subset G_v$, (ii) B is a half-open line segment in the plane with slope α , (iii) B contains its first point x^f but not its last point x^l , (iv) $x_1^f = -\pi$ or $x_2^f = -\pi$, and (v) $x_1^l = \pi$ or $x_2^l = \pi$.

Since $E \subset G_v$ is a closed and bounded set, there are a finite number of portions of G_v , $\{B_j\}_{j=1}^J$, such that $E \subset \bigcup_{j=1}^J B_j$. In particular,

$$(3.2) \quad E \subset \bigcup_{j=1}^J E \cap B_j.$$

With no loss in generality, we can assume that each $E \cap B_j$ is nonempty, for we can throw out those which are empty and renumber.

So we now have the situation that each of the $E \cap B_j$ is a nonempty set of linear measure zero. We will show that

$$(3.3) \quad \exists f_1 \in \mathcal{A}_v^+ \cap C(T_2) \text{ with } f_1(x) = 0 \quad \forall x \in E \cap B_1,$$

where, in addition, $f_1(x) = 0$ on T_2 only on a set of two-dimensional measure zero. Similar reasoning shows that

$$(3.4) \quad \exists f_j \in \mathcal{A}_v^+ \cap C(T_2) \text{ with } f_j(x) = 0 \quad \forall x \in E \cap B_j, \quad j = 2, \dots, J,$$

where, in addition, $f_j(x) = 0$ on T_2 only on a set of two-dimensional measure zero for $j = 2, \dots, J$.

We then set

$$(3.5) \quad f(x) = \prod_{j=1}^J f_j(x).$$

Because $\mathcal{A}_v^+$ is an algebra, it follows from (3.3) and (3.4) that

$$f \in \mathcal{A}_v^+ \cap C(T_2).$$

Also, it follows from (3.2)-(3.5) that

$$f(x) = 0 \text{ for } x \in E$$

and that

$$f \text{ is not identically zero on } T_2.$$

These last three facts establish the theorem.

It remains to show that (3.3) holds. We now do this.

Since B_1 is a line segment of positive slope α , the first components of the points $x \in B_1$ are all different. We set

$$(3.6) \quad E_P = \{\theta \in [-\pi, \pi) : \theta = x_1, \quad x \in E \cap B_1\}.$$

E_P is the projection of $E \cap B_1$ onto the real line. If π is a limit point of the points in E_P and $-\pi$ is not in E_P , insert $-\pi$ in E_P . Then because of the properties of E , we can view E_P as a set on the unit circle which is closed and of linear measure zero. Hence, we can invoke the theorem of Fatou, [9, p. 276], which states (with T_1 the unit circle):

$$(3.7) \quad \begin{aligned} & \text{(i) } \exists g \in C(T_1) \text{ s.t. } g(\theta) = 0 \text{ for } \theta \in E_P, \\ & \text{(ii) } g(\theta) \neq 0 \text{ for } \theta \in T_1 \setminus E_P, \\ & \text{(iii) } \int_{-\pi}^{\pi} g(\theta) e^{in\theta} d\theta = 0 \text{ for } n \geq 1. \end{aligned}$$

What (iii) states is that the negative Fourier coefficients of g vanish.

We set

$$(3.8) \quad f_1(x) = g(x_1) \quad \text{for } x \in T_2.$$

Then it follows from the fact that $g \in C(T_1)$ that

$$f_1 \in C(T_2).$$

Likewise from (3.6) and (3.7) (i), we obtain that

$$f_1(x) = 0 \text{ for } x \in E \cap B_1.$$

Also, from (3.7) (ii), we have that

$$f_1(x) = 0 \text{ on } T_2 \text{ only on a set of two-dimensional measure zero.}$$

To complete the proof that (3.3) holds, we see from these last three facts all that remains to be shown is that $f_1 \in \mathcal{A}_v^+$. But from (3.7)(iii) and (3.8), we have that

$$(3.9) \quad \widehat{f}(m) \neq 0 \Rightarrow m_2 = 0 \text{ and } m_1 \geq 0;$$

i.e., the only nonzero $\widehat{f}(m)$ occur for integral lattice points that lie on the nonnegative real axis. From (3.1), we see that

$$m \cdot v < 0 \Rightarrow m_1 < 0 \text{ or } m_2 < 0.$$

Therefore, we obtain from (3.9) that

$$m \cdot v < 0 \Rightarrow \widehat{f}(m) = 0.$$

Hence, $f_1 \in \mathcal{A}_v^+$, and the proof of the theorem is complete. $\square$

4. REFEREE'S COMMENTS

These are some interesting comments made by the referee about the proof of Theorem 1.

To summarize heuristically what the author did in Theorem 1 in dimension $N = 2$, we let H^+ be the right half-plane $\operatorname{Re} \zeta > 0$, where $\zeta = \sigma + it$ and

$$v = \left(\frac{1}{\sqrt{1 + \alpha^2}}, \frac{\alpha}{\sqrt{1 + \alpha^2}} \right)$$

and where α is a positive irrational number.

We view T_2 as the set $|z| = 1, |w| = 1$ in $\mathbf{C}^2$. G_v appears as a dense subset of T_2 .

Take $\chi(\zeta) = (e^{-\zeta}, e^{-\alpha\zeta})$, which maps H^+ to $\mathbf{C}^2$. The imaginary axis is mapped to the points $(e^{-it}, e^{-i\alpha t})$ on T_2 , which is G_v .

For each $f \in A_v \cap C(T_2)$, using the Phragmen-Lindelöf theorem, the notion of normal families, and some Fourier analysis, find a function $F(\zeta)$ which is bounded and analytic in H^+ and continuous in the closed half-plane and which has the property that

$$F(it) = f(\chi(it)).$$

Fix $f \in A_v \cap C(T_2)$ which vanishes on $E \subset G_v$, where $|E|_1 > 0$. Then F vanishes on $\chi^{-1}(E)$ which has positive linear measure on the imaginary axis. Therefore by the F. and M. Riesz theorem for the right half-plane, $F(\zeta) = 0$ for $\zeta \in H^+$. Hence, $f = 0$ on G_v . Since G_v is dense in T_2 , $f = 0$ on T_2 . $\square$

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