UNBOUNDED MULTIPLIERS ON OPERATOR SPACES

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ABSTRACT. We introduce unbounded multipliers on operator spaces. These multipliers generalize both regular operators on Hilbert C^* -modules and (bounded) multipliers on operator spaces.

1. INTRODUCTION

Unbounded multipliers play an important role in mathematics, for example in the spectral theorem for unbounded self-adjoint operators. In the context of Hilbert C^* -modules, unbounded multipliers are called regular operators. The definition of such an operator goes back to A. Connes, [Con81], who defines the notion of a self-adjoint regular operator on a C^* -algebra. S. L. Woronowicz defines in [Wor91] general regular operators on C^* -algebras. Regular operators are important in the theory of noncompact quantum groups ([Wor91], [WN92]) and can be used to define unbounded Kasparov modules in Kasparov's bivariant KK -theory ([BJ83]).

In this article we define unbounded multipliers on operator spaces. On a Hilbert C^* -module, equipped with a canonical operator space structure, these multipliers coincide with the regular operators. Moreover, every bounded left adjointable multiplier on an operator space belongs to the class of unbounded multipliers.

We will give several characterizations of unbounded multipliers on operator spaces. For example, it turns out that, roughly, every unbounded multiplier on an operator space is the restriction of a regular operator and that an unbounded multiplier can be characterized using the theory of strongly continuous groups on a Hilbert space. As an application we will generalize a perturbation result for regular operators on Hilbert C^* -modules to operator spaces.

Most of the results of this paper are from [Sch09].

2. PRELIMINARIES

For the basic theory of C_0 -semigroups, we refer the reader to [EN00]. Recall that the generator of a C_0 -semigroup is unique, densely defined and closed.

2.1. Regular operators. We begin by recalling some basic facts about regular operators. For more details, we refer to [Wor91], [Lan95] and [Kus97].

Let E, F be Hilbert C^* -modules over a C^* -algebra $\mathfrak{A}$.

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Definition 2.1. Let $A: D(A) \subseteq E \rightarrow F$ be a densely defined, $\mathfrak{A}$ -linear map where $D(A)$ denotes the domain of A . Then

$$D(A^*) := \{y \in F \mid \exists z_y \in E \forall x \in D(A) : \langle Ax, y \rangle = \langle x, z_y \rangle\}$$

is a submodule of F . For all $y \in D(A^*)$ the element z_y is uniquely defined and denoted by A^*y . We get an $\mathfrak{A}$ -linear, closed map $A^*: D(A^*) \subseteq F \rightarrow E$, called the *adjoint* of A , satisfying $\langle x, A^*y \rangle = \langle Ax, y \rangle$ for all $x \in D(A)$ and $y \in D(A^*)$.

Definition 2.2. A *regular operator* is a densely defined, closed, $\mathfrak{A}$ -linear map $A: D(A) \subseteq E \rightarrow F$ such that A^* is densely defined and $1 + A^*A$ has dense image. The set of regular operators is denoted by $\mathcal{R}(E, F)$. Define $\mathcal{R}(E) := \mathcal{R}(E, E)$.

Regular operators are a generalization of densely defined, closed operators on a Hilbert space H , because the set of regular operators on H is equal to the set of densely defined, closed operators on H . Denote by $\mathbb{B}(E)$ the set of adjointable operators on E . Then

$$(2.1) \quad \mathbb{B}(E) = \{A \in \mathcal{R}(E) \mid D(A) = E\}.$$

If $\mathfrak{A}$ is a unital C^* -algebra, we get $\mathcal{R}(\mathfrak{A}) = \mathfrak{A}$.

A regular operator $A \in \mathcal{R}(E)$ is called *self-adjoint* (resp. *skew-adjoint*) if $A^* = A$ (resp. $A^* = -A$). For every regular operator $A \in \mathcal{R}(E, F)$ we can define the adjointable operator $z_A := A((1 + A^*A)^{-1})^{1/2}$ with $\|z_A\| \leq 1$. We denote by $C(\mathbb{R})$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{C}$.

If $A \in \mathbb{B}(E)$ we already have the notion of a functional calculus and spectrum for A , because $\mathbb{B}(E)$ is a C^* -algebra. For self-adjoint regular operators we have the following functional calculus:

Theorem 2.3 ([Lan95, Theorem 10.9]). *Let $A \in \mathcal{R}(E)$ be self-adjoint. Let ι be the canonical inclusion of $\mathbb{R}$ into $\mathbb{C}$ and $f: \mathbb{R} \rightarrow \mathbb{C}, \lambda \mapsto \lambda(1 + \lambda^2)^{-1/2}$. Then there exists a unique $*$ -homomorphism $\varphi_A: C(\mathbb{R}) \rightarrow \mathcal{R}(E)$ such that $\varphi_A(\iota) = A$ and $\varphi_A(f) = z_A$.*

Woronowicz introduced the following consistent definition:

Definition 2.4. Let $A \in \mathcal{R}(E)$ be self-adjoint. The set

$$\sigma(A) := \{\lambda \in \mathbb{R} \mid \forall f \in \ker(\varphi_A) : f(\lambda) = 0\}$$

is called the *spectrum* of A .

The set $\sigma(A)$ is a closed, nonempty subset of $\mathbb{R}$.

2.2. Stone's theorem for Hilbert C^* -modules. Let E be a Hilbert C^* -module over a C^* -algebra $\mathfrak{A}$. The following two results are generalizations of [HQVK92, Theorem 2.1 and Proposition 2.2] from C^* -algebras to Hilbert C^* -modules; see also [WN92, Theorem 2.3]. With some adjustments the proof of the first statement can be transferred to the Hilbert C^* -module case. For details see [Sch09], Proposition 1.27 and Satz 1.30.

Theorem 2.5. *Let $A \in \mathcal{R}(E)$ be self-adjoint. Let $e_t: \mathbb{R} \rightarrow \mathbb{C}, \lambda \mapsto \exp(i\lambda t)$, and $U_t := \exp(itA) := \varphi_A(e_t)$ for all $t \in \mathbb{R}$. Then $(U_t)_{t \in \mathbb{R}}$ is a C_0 -group on E with generator iA such that U_t is a unitary element of $\mathbb{B}(E)$ for all $t \in \mathbb{R}$.*

Theorem 2.6 (Stone's theorem). *Let $(U_t)_{t \in \mathbb{R}}$ be a C_0 -group on E such that U_t is a unitary element of $\mathbb{B}(E)$ for all $t \in \mathbb{R}$.*

- (i) *There exists a self-adjoint $A \in \mathcal{R}(E)$ such that $U_t = \exp(itA)$ for all $t \in \mathbb{R}$.*
- (ii) *If $(U_t)_{t \in \mathbb{R}}$ is norm continuous, then $A \in \mathbb{B}(E)$.*

Proof. To prove (i) denote by $C^*(\mathbb{R})$ the group C^* -algebra of the locally compact group $\mathbb{R}$ and by $\hat{\mathbb{R}}$ the dual group of $\mathbb{R}$. We have $\hat{\mathbb{R}} \cong \mathbb{R}$. Let $C_0(\mathbb{R})$ (resp. $C_c(\mathbb{R})$) denote the set of continuous functions on $\mathbb{R}$ vanishing at infinity (resp. with compact support). For all $f \in C_c(\mathbb{R})$,

$$\alpha(f) = \int_{\mathbb{R}} f(t) U_t dt$$

defines an operator in $\mathbb{B}(E)$ and the map α can be extended to a nondegenerate $*$ -homomorphism $\hat{\alpha}: C^*(\mathbb{R}) \rightarrow \mathbb{B}(E)$. Moreover, $\hat{\alpha}$ can be extended to a $*$ -homomorphism $\check{\alpha}: \mathcal{M}(C^*(\mathbb{R})) \rightarrow \mathbb{B}(E)$ ([RW98, Proposition C.17]), where $\mathcal{M}(C^*(\mathbb{R}))$ denotes the algebra of double centralizers of the algebra $C^*(\mathbb{R})$.

Let $t \in \mathbb{R}$. Define

$$\lambda_t: L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}), f \mapsto (s \mapsto f(s-t)).$$

Using the translation invariance of the Lebesgue measure, we get $\lambda_t \in \mathcal{M}(L^1(\mathbb{R}))$. By approximating any $f \in C^*(\mathbb{R})$ by elements of $L^1(\mathbb{R})$, we can define the double centralizer $\hat{\lambda}_t \in \mathcal{M}(C^*(\mathbb{R}))$ using λ_t . For all $f \in C_c(\mathbb{R})$ and $x \in E$ we obtain

$$\check{\alpha}(\hat{\lambda}_t)\alpha(f)x = \alpha(\lambda_t f)x = U_t \alpha(f)x$$

and thus

$$(2.2) \quad \check{\alpha}(\hat{\lambda}_t) = U_t.$$

The Fourier transform $\mathcal{F}: L^1(\mathbb{R}) \rightarrow C_0(\hat{\mathbb{R}}) \cong C_0(\mathbb{R})$ can be extended to a $*$ -isomorphism $\hat{\mathcal{F}}: C^*(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ ([RW98, Example C.20]). Denote by $C_b(\mathbb{R})$ the set of bounded continuous functions on $\mathbb{R}$. Because there is an embedding of $C_0(\mathbb{R})$ into $\mathcal{M}(C_0(\mathbb{R})) \cong C_b(\mathbb{R})$, we can regard $\hat{\mathcal{F}}$ as a map from $C^*(\mathbb{R})$ to $\mathcal{M}(C_0(\mathbb{R}))$. Using [Lan95, Proposition 2.5], we can extend this nondegenerate $*$ -homomorphism to a $*$ -homomorphism $\check{\mathcal{F}}: \mathcal{M}(C^*(\mathbb{R})) \rightarrow \mathcal{M}(C_0(\mathbb{R}))$ and conclude

$$(2.3) \quad \check{\mathcal{F}}(\hat{\lambda}_t) = e_t.$$

The nondegenerate $*$ -homomorphism $\beta := \hat{\alpha} \circ \hat{\mathcal{F}}^{-1}: C_0(\mathbb{R}) \rightarrow \mathbb{B}(E)$ extends to a $*$ -homomorphism $\check{\beta}: \mathcal{M}(C_0(\mathbb{R})) \rightarrow \mathbb{B}(E)$. Recall that ι is the canonical inclusion of $\mathbb{R}$ into $\mathbb{C}$. Using $\iota \in C(\mathbb{R}) \cong \mathcal{R}(C_0(\mathbb{R}))$ ([Wor91, Example 2]) and [Lan95, Proposition 10.7], it follows that $A := \beta(\iota)$ is a self-adjoint regular operator on E . Because $\varphi_A(\iota) = A = \beta(\iota)$ we conclude with [Kus97, Result 6.3]: $\varphi_A|_{C_0(\mathbb{R})} = \beta$. With [Lan95, Proposition 2.1] it follows that $\varphi_A|_{C_b(\mathbb{R})} = \check{\beta}$ and $\check{\beta} \circ \check{\mathcal{F}} = \check{\alpha}$. Together with (2.3) and (2.2), we obtain

$$\exp(itA) = \varphi_A(e_t) = \check{\beta}(e_t) = \check{\beta}(\check{\mathcal{F}}(\hat{\lambda}_t)) = \check{\alpha}(\hat{\lambda}_t) = U_t.$$

To prove (ii) we notice that there exists a unique nondegenerate injective $*$ -homomorphism $\psi_A: C_0(\sigma(A)) \rightarrow \mathbb{B}(E)$ such that $\psi_A(\iota|_{\sigma(A)}) = A$ ([Kus97, Theorem 3.4]). Due to [Lan95, Proposition 2.5] the map ψ_A can be extended to an injective $*$ -homomorphism $\overline{\psi}_A: C_b(\sigma(A)) \rightarrow \mathbb{B}(E)$. With [Kus97, Lemma 3.6] we have

$\varphi_A(f) = \overline{\psi_A}(f|_{\sigma(A)})$ for all $f \in C(\mathbb{R})$. Denote by $\mathbf{1}$ the function that is constant 1 on $\mathbb{R}$. We conclude that

$$\|U_t - \text{id}_E\| = \|\varphi_A(e_t - \mathbf{1})\| = \|\overline{\psi_A}((e_t - \mathbf{1})|_{\sigma(A)})\| = \|(e_t - \mathbf{1})|_{\sigma(A)}\|.$$

Because $(U_t)_{t \in \mathbb{R}}$ is norm-continuous, $\sigma(A)$ is bounded. Therefore $\iota|_{\sigma(A)}$ is bounded and we get: $A = \psi_A(\iota|_{\sigma(A)}) \in \mathbb{B}(E)$. $\square$

2.3. Operator spaces. This section discusses basic results from operator space theory that we will need later in the paper (see e.g. [ER00], [Pau02], [Pis03] or [BLM04]).

Let H be a Hilbert space. Denote by $L(H)$ the algebra of linear, continuous operators on H . A (concrete) operator space is a closed subspace X of $L(H)$. Let $M_{m,n}(X)$ denote the space of $m \times n$ matrices with entries in X , and set $M_n(X) := M_{n,n}(X)$. We may view the space $M_n(X)$ as a subspace of $L(H^n)$. Thus this matrix space has a natural norm $\|\cdot\|_n$ for each $n \in \mathbb{N}$. Note that operator spaces can be characterized intrinsically.

Let X, Y be operator spaces and $\alpha: X \rightarrow Y$ be linear. Define

$$\alpha_n := \alpha^{(n)}: M_n(X) \rightarrow M_n(Y), x \mapsto (\alpha(x_{ij}))_{i,j},$$

called the n th *amplification* of α . The map α is called *completely contractive* (resp. *completely isometric*) if α_n is contractive (resp. isometric) for all $n \in \mathbb{N}$. Furthermore, α is called *completely bounded* if $\|\alpha\|_{\text{cb}} := \sup_{n \in \mathbb{N}} \|\alpha_n\| < \infty$.

An operator space X is called *unital* if it has a distinguished element e_X such that there exist a C^* -algebra $\mathfrak{A}$ with unit $e_{\mathfrak{A}}$ and a completely isometric map $\eta: X \rightarrow \mathfrak{A}$ satisfying $\eta(e_X) = e_{\mathfrak{A}}$. Observe that unital operator spaces can be characterized intrinsically ([BN11], [HN12]).

Recall that an operator space X is called *injective* if for every operator space $Y \subseteq Z$ and every completely contractive map $\varphi: Y \rightarrow X$ there is a completely contractive map $\Phi: Z \rightarrow X$ such that $\Phi|_Y = \varphi$. For every operator space X there exists a unique *injective envelope* $(I(X), j_X)$, i.e., a completely isometric map $j_X: X \rightarrow I(X)$ and an injective operator space $I(X)$ such that for every injective subspace Y of $I(X)$ satisfying $j_X(X) \subseteq Y$ we have $Y = I(X)$.

Let $X \subseteq L(H)$ be an operator space. We can embed X into an *operator system*, i.e., a self-adjoint closed subspace of $L(H)$ containing id_H . To accomplish this we form the Paulsen system

$$\mathcal{S}(X) := \begin{pmatrix} \mathbb{C}\text{id}_H & X \\ X^* & \mathbb{C}\text{id}_H \end{pmatrix} \subseteq M_2(L(H)).$$

There exists a completely contractive map $\Phi: M_2(L(H)) \rightarrow M_2(L(H))$ such that the image of Φ is an injective envelope of the operator space $\mathcal{S}(X)$. Moreover, $I(\mathcal{S}(X))$ is a unital C^* -algebra with multiplication $x \cdot y := \Phi(xy)$. Let $p_1 := \text{id}_H \oplus 0$ and $p_2 := 0 \oplus \text{id}_H$ be the canonical projections in $I(\mathcal{S}(X))$. Then p_1 and p_2 are orthogonal and corner-preserving. Therefore, with respect to p_1 and p_2 , we may decompose $I(\mathcal{S}(X))$ as

$$I(\mathcal{S}(X)) = \begin{pmatrix} I_{11}(X) & I_{12}(X) \\ I_{12}(X)^* & I_{22}(X) \end{pmatrix}$$

and have $I_{12}(X) = I(X)$.

The following operator spaces will play a crucial role in this article.

If $\mathfrak{A}$ is a C^* -algebra, then the $*$ -algebra $M_n(\mathfrak{A})$ has a unique norm such that $M_n(\mathfrak{A})$ is a C^* -algebra. With respect to these norms, $\mathfrak{A}$ is an operator space.

Let X be an operator space and $m, n \in \mathbb{N}$. We can regard $M_{m,n}(X)$ as a subspace of $M_p(X)$, where $p := \max\{m, n\}$. Thus $M_{m,n}(X)$ is an operator space. The set $C_n(X) := M_{n,1}(X)$ is called a *column operator space*.

Furthermore, every Hilbert C^* -module E over a C^* -algebra $\mathfrak{A}$ carries the structure of an operator space. For every $n \in \mathbb{N}$ and $x \in M_n(E)$,

$$(2.4) \quad \|x\|_n = \left\| \left(\sum_{k=1}^n \langle x_{ki}, x_{kj} \rangle \right)_{i,j} \right\|^{1/2}$$

defines a norm such that E becomes an operator space.

2.4. Multipliers on operator spaces. Let us recall some facts about multipliers on operator spaces (see e.g. [BLM04]). These multipliers generalize multipliers on C^* -algebras.

Definition 2.7. Let X be an operator space and $T: X \rightarrow X$ be linear.

- (i) T is called a (*left*) *multiplier* on X if there exist a Hilbert space H together with a completely isometric map $\eta: X \rightarrow L(H)$ and $a \in L(H)$ such that $\eta(Tx) = a\eta(x)$ for all $x \in X$. The set of all multipliers on X is denoted by $\mathcal{M}_l(X)$.
- (ii) For all $T \in \mathcal{M}_l(X)$ define the multiplier norm

$$\begin{aligned} \|T\|_{\mathcal{M}_l(X)} &:= \inf \{ \|a\| \mid \text{there exist a Hilbert space } H, \text{ a completely} \\ &\quad \text{isometric map } \eta: X \rightarrow L(H) \text{ and } a \in L(H) \text{ s. t.} \\ &\quad \forall x \in X : \eta(Tx) = a\eta(x) \}. \end{aligned}$$

- (iii) T is called a *left adjointable multiplier* on X if there exist a Hilbert space H , a completely isometric map $\eta: X \rightarrow L(H)$ and a map $S: X \rightarrow X$ such that $\eta(Tx)^*\eta(y) = \eta(x)^*\eta(Sy)$ for all $x, y \in X$. The set of all left adjointable multipliers on X is denoted by $\mathcal{A}_l(X)$.

For any operator space X we have $\mathcal{M}_l(X) \subseteq \mathcal{A}_l(X)$. Moreover, $\mathcal{A}_l(X)$ is a unital C^* -algebra. The multiplier algebra of a C^* -algebra $\mathfrak{A}$ is isomorphic to $\mathcal{A}_l(\mathfrak{A})$. If E is a Hilbert C^* -module, then

$$(2.5) \quad \mathcal{A}_l(E) = \mathbb{B}(E)$$

([BLM04, Corollary 8.4.2]). Define

$$\begin{aligned} \mathcal{I}M_\ell(X) &:= \{a \in I_{11}(X) \mid a \cdot X \subseteq X\} \quad \text{and} \\ \mathcal{I}M_\ell^*(X) &:= \mathcal{I}M_\ell(X) \cap \mathcal{I}M_\ell(X)^*. \end{aligned}$$

Then $\mathcal{I}M_\ell(X)$ (resp. $\mathcal{I}M_\ell^*(X)$) is isometrically isomorphic to $\mathcal{M}_l(X)$ (resp. $\mathcal{A}_l(X)$).

Left multipliers are characterized intrinsically by the following result:

Theorem 2.8 ([BEZ02, Theorem 1.1], [Wer04, Corollary 3.13]). *Let X be an operator space, $T: X \rightarrow X$ be linear and $\lambda > 0$. The following assertions are equivalent:*

- (a) $T \in \mathcal{M}_l(X)$ satisfies $\|T\|_{\mathcal{M}_l(X)} \leq \frac{1}{\lambda}$.

(b) The map

$$\tau_{\lambda T}: C_2(X) \rightarrow C_2(X), \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda T x \\ y \end{pmatrix},$$

is completely contractive.

Proposition 2.9 (cf. [Wer99, Corollary 4.3], [Ble01, p. 308]). *Let X be an operator space and $U: X \rightarrow X$ be linear. The following assertions are equivalent:*

(a) U is a unitary element of $\mathcal{A}_1(X)$.

(b) The map

$$\tau_U: C_2(X) \rightarrow C_2(X), \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} Ux \\ y \end{pmatrix},$$

is a completely isometric isomorphism.

(c) There exists a unitary $u \in \mathcal{IM}_\ell^*(X)$ such that $j(Ux) = u \cdot j(x)$ for all $x \in X$.

3. UNBOUNDED MULTIPLIERS ON OPERATOR SPACES

In this section we will introduce three different definitions of unbounded multipliers on an operator space.

3.1. Unbounded skew-adjoint multipliers. An operator space has less structure than a Hilbert C^* -module E . Therefore it is not obvious how to define an unbounded multiplier on an operator space. We know that the set $\mathbb{B}(E)$ of adjointable operators is isomorphic to the set $\mathcal{A}_1(E)$, where E is equipped with the canonical operator space structure of equation (2.4). The fact that skew-adjoint regular operators can be characterized using C_0 -groups (Theorems 2.6 and 2.5) motivates the following definition:

Definition 3.1. Let X be an operator space.

- (i) An operator $A: D(A) \subseteq X \rightarrow X$ is called a C_0 -left multiplier if A generates a C_0 -semigroup $(T_t)_{t \geq 0}$ on X satisfying $T_t \in \mathcal{M}_l(X)$ for all $t \geq 0$.
- (ii) An operator $A: D(A) \subseteq X \rightarrow X$ is called an *unbounded skew-adjoint multiplier* if A generates a C_0 -group $(U_t)_{t \in \mathbb{R}}$ on X satisfying $U_t \in \mathcal{A}_l(X)$ for all $t \in \mathbb{R}$.
- (iii) The set of all C_0 -left multipliers (resp. unbounded skew-adjoint multipliers) on X is denoted by $\mathcal{M}_l^{C_0}(X)$ (resp. $\mathcal{A}_{l,s}^{C_0}(X)$).

Using the fact that $t \mapsto e^{tA}$ ($A \in L(X)$) defines a norm-continuous semigroup with generator A and [EN00, Proposition I.3.5], we have the following, similar to equation (2.1):

Proposition 3.2. *Let X be an operator space. Then:*

- (i) $\mathcal{A}_{l,s}^{C_0}(X) \subseteq \mathcal{M}_l^{C_0}(X)$.
- (ii) $\mathcal{M}_l(X) = \{A \in \mathcal{M}_l^{C_0}(X) \mid D(A) = X\}$.
- (iii) $\{A \in \mathcal{A}_l(X) \mid A \text{ skew-adjoint}\} = \{A \in \mathcal{A}_{l,s}^{C_0}(X) \mid D(A) = X\}$.

In a unital C^* -algebra $\mathfrak{A}$ the set of (bounded) multipliers is equal to $\mathcal{R}(\mathfrak{A})$. A similar result holds for operator spaces:

Proposition 3.3. *If X is a unital operator space, we have*

- (i) $\mathcal{M}_l(X) = \mathcal{M}_l^{C_0}(X)$ and
- (ii) $\{A \in \mathcal{A}_l(X) \mid A \text{ skew-adjoint}\} = \mathcal{A}_{l,s}^{C_0}(X)$.

Proof. We will only prove (i), because the proof of (ii) is similar to the proof of (i). Let $A \in \mathcal{M}_l^{C_0}(X)$. Due to Proposition 3.2 it suffices to show $D(A) = X$.

Let $(T_t)_{t \geq 0}$ be the C_0 -semigroup generated by A . For every $t \geq 0$ there exists an $a_t \in \mathcal{IM}_\ell(X)$ such that $j_X(T_t x) = a_t \cdot j_X(x)$ for all $x \in X$. There exists a Hilbert space H such that $X \subseteq L(H)$ and $e_X = \text{id}_H$. Because X is unital, we have that $\|S\|_{\text{cb}} = \|S\|_{\mathcal{M}_l(X)}$ for all $S \in \mathcal{M}_l(X)$ ([Ble01, p. 20]) and that the embedding $j_X: X \rightarrow I(X)$ is unital ([BLM04, Corollary 4.2.8]). Together we obtain that

$$\begin{aligned} \|T_t - \text{id}_X\| &\leq \|T_t - \text{id}_X\|_{\text{cb}} = \|T_t - \text{id}_X\|_{\mathcal{M}_l(X)} = \|a_t - e_{I_{11}(X)}\| \\ &= \|a_t \cdot e_{I(X)} - e_{I(X)}\| = \|a_t \cdot j_X(e_X) - j_X(e_X)\| = \|T_t(e_X) - e_X\| \end{aligned}$$

converges to 0 for $t \rightarrow 0$. Thus $(T_t)_{t \geq 0}$ is norm-continuous. We conclude that $D(A) = X$, and the assertion follows with Proposition 3.2. $\square$

Theorem 3.4. *For a Hilbert C^* -module E , we have*

$$\{A \in \mathcal{R}(E) \mid A \text{ skew-adjoint}\} = \mathcal{A}_{l,s}^{C_0}(E).$$

Proof. Let $A \in \mathcal{R}(E)$ be skew-adjoint. Then $h := -iA$ is self-adjoint. Moreover $U_t := \exp(iht)$ defines a C_0 -group on E with $U_t \in \mathbb{B}(E) = \mathcal{A}(E)$ unitary and with generator $ih = A$ due to Theorem 2.5 and equation (2.5); thus $A \in \mathcal{A}_{l,s}^{C_0}(E)$.

The other inclusion follows with Theorem 2.6, Theorem 2.5 and the uniqueness of the generator of a C_0 -group. $\square$

3.2. Unbounded multipliers. In this section we will generalize the definition of regular operators on Hilbert C^* -modules to operator spaces.

Let $A_i: D(A_i) \subseteq X \rightarrow X$ be an operator for all $i \in \{1, \dots, 4\}$. Define

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}: (D(A_1) \cap D(A_3)) \times (D(A_2) \cap D(A_4)) \subseteq C_2(X) \rightarrow C_2(X),$$

$$(x, y) \mapsto (A_1 x + A_2 y, A_3 x + A_4 y).$$

To introduce the notion of an unbounded multiplier we use the fact that $i \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ is a skew-adjoint regular operator on $E \oplus E$ for every $A \in \mathcal{R}(E)$. This motivates the following definition:

Definition 3.5. Let X be an operator space. A map $A: D(A) \subseteq X \rightarrow X$ is called an *unbounded multiplier* if there exists $B: D(B) \subseteq X \rightarrow X$ satisfying $i \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{A}_{l,s}^{C_0}(C_2(X))$. The set of all unbounded multipliers on X is denoted by $\mathcal{A}_l^{C_0}(X)$.

Every unbounded multiplier is densely defined and closed. We will show in Theorem 3.20 that B is unique. If X is a Hilbert C^* -module, we will obtain $B = A^*$ (Theorem 3.8). Therefore B can be viewed as the adjoint of A .

Proposition 3.6. *Let X be an operator space. Then:*

- (i) $\mathcal{A}_l(X) \subseteq \mathcal{A}_l^{C_0}(X)$.
- (ii) If X is unital, we have $\mathcal{A}_l(X) = \mathcal{A}_l^{C_0}(X)$.

Proof. To prove (i) let $A \in \mathcal{A}_l(X)$. Then $\tilde{A} := i \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \in \mathcal{A}_l(C_2(X))$ is skew-adjoint, where A^* denotes the adjoint of A in the C^* -algebra $\mathcal{A}_l(X)$. With Proposition 3.2 we conclude that $\tilde{A} \in \mathcal{A}_{l,s}^{C_0}(C_2(X))$; thus $A \in \mathcal{A}_l^{C_0}(X)$.

Part (ii) is deduced by using Proposition 3.3 and the following lemma. $\square$

The following is straightforward using [BP01, Corollary 1.3]:

Lemma 3.7 ([Sch09, Lemma 3.20]). *Let X be an operator space. Let $A: D(A) \subseteq X \rightarrow X$ and $B: D(B) \subseteq X \rightarrow X$ be operators satisfying $\hat{A} := \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{A}_l(C_2(X))$ and $(\hat{A})^* = \hat{A}$. Then $A, B \in \mathcal{A}_l(X)$ and $B^* = A$.*

The following important theorem shows that on a Hilbert C^* -module unbounded multipliers coincide with regular operators:

Theorem 3.8. *Let E be a Hilbert C^* -module.*

- (i) $\mathcal{A}_l^{C_0}(E) = \mathcal{R}(E)$.
- (ii) *Let $A \in \mathcal{A}_l^{C_0}(E)$, and suppose there exists $B: D(B) \subseteq E \rightarrow E$ such that $i \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{A}_{l,s}^{C_0}(C_2(E))$. Then $A^* = B \in \mathcal{R}(E)$.*

Proof. We first show (ii) and the inclusion ‘ $\subseteq$ ’ of (i). Let $A \in \mathcal{A}_l^{C_0}(E)$ and $\hat{A} := \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. The direct sum $E \oplus E$ is equipped with the inner product

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{E \oplus E} := \langle x_1, x_2 \rangle_E + \langle y_1, y_2 \rangle_E,$$

a Hilbert C^* -module. Moreover $E \oplus E$ is isometrically isomorphic to the column operator space $C_2(E)$ of E viewed as an operator space ([Sch09, Proposition 2.14]).

Both A and B are densely defined and closed. Using Theorem 3.4 we obtain

$$(3.1) \quad i\hat{A} \in \mathcal{A}_{l,s}^{C_0}(C_2(E)) \cong \{R \in \mathcal{R}(E \oplus E) \mid R^* = -R\};$$

thus $\begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix} = (\hat{A})^* = \hat{A} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. We conclude: $B = A^*$. Therefore A^* is densely defined. A simple calculation shows that $1 + A^*A$ has dense image. It follows that A is regular. With [Lan95, Corollary 9.6] we get $A^* \in \mathcal{R}(E)$.

To prove the inclusion ‘ $\supseteq$ ’ of (i) let $A \in \mathcal{R}(E)$. Then $\hat{A} := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ is regular and self-adjoint. With equation (3.1) we obtain $i\hat{A} \in \mathcal{A}_{l,s}^{C_0}(C_2(E))$; thus $A \in \mathcal{A}_l^{C_0}(E)$. $\square$

3.3. Characterizations of unbounded multipliers. Our next goal is to characterize unbounded multipliers. Let us recall the following two definitions:

Definition 3.9. Let H, K be Hilbert spaces. A *ternary ring of operators (TRO)* is a closed subspace Z of $L(H, K)$ such that $ZZ^*Z \subseteq Z$. A *subtriple* of a TRO Z is a closed subspace Y of Z satisfying $YY^*Y \subseteq Y$.

Every TRO Z is a full Hilbert C^* -module over the C^* -algebra Z^*Z . Conversely, every Hilbert C^* -module can be represented faithfully as a TRO. We can view the injective envelope $I(X)$ as a TRO.

Definition 3.10. The triple envelope $\mathcal{T}(X)$ of an operator space X is the smallest subtriple of $I(X)$ containing X .

We have

$$(3.2) \quad \mathcal{T}(X) = \overline{\text{lin}}\{x_1 \cdot x_2^* \cdot x_3 \cdot x_4^* \cdots x_{2n+1} \mid n \geq 0, x_1, \dots, x_{2n+1} \in j(X)\}.$$

The following is straightforward from [Har81, p. 341]:

Lemma 3.11. *Let X be an operator space. There exist a Hilbert space K , a unital injective $*$ -representation $\pi: I(\mathcal{S}(X)) \rightarrow L(K)$ and closed subspaces K_1, K_2 of K such that*

- (i) $[\pi(\mathcal{T}(X))K_1] := \text{lin}\{xy \mid x \in \pi(\mathcal{T}(X)), y \in K_1\}$ is dense in K_2 ,

- (ii) $[\pi(\mathcal{T}(X))K_1^\perp] = \{0\}$ and
- (iii) $[\pi(j(X))K] \subseteq K_2$.

Definition 3.12. The quadruple (π, K, K_1, K_2) , introduced in Lemma 3.11, is called a $\mathcal{T}(X)$ -embedding of X .

In the following theorem it is shown that an unbounded multiplier can be characterized intrinsically (part (b)) or with the help of a C_0 -group on a Hilbert space (part (d)) and that an unbounded multiplier is essentially the restriction of a regular operator on the Hilbert C^* -module $\mathcal{T}(X)$ (part (c)).

Theorem 3.13. Let X be an operator space. Let (π, K, K_1, K_2) be a $\mathcal{T}(C_2(X))$ -embedding of $C_2(X)$, and set $\eta := \pi \circ j_{C_2(X)}: C_2(X) \rightarrow L(K)$. Let $A: D(A) \subseteq X \rightarrow X$ be linear. The following assertions are equivalent:

- (a) $A \in \mathcal{A}_l^{C_0}(X)$.
- (b) There exists a map $B: D(B) \subseteq X \rightarrow X$ such that $i\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ generates a completely contractive C_0 -group on $C_2(C_2(X)) \cong C_4(X)$.
- (c) There exists $B \in \mathcal{R}(\mathcal{T}(X))$ with the following properties:
 - (i) $j \circ A = B \circ j|_{D(A)}$ and
 - (ii) there exist $\lambda, \mu > 0$ such that the restriction of

$$\lambda - i\left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right) \quad \text{resp.} \quad \mu + i\left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$$
 to $j_{C_2(X)}(C_2(X))$ is surjective onto $j_{C_2(X)}(C_2(X))$.
- (d) There exist a map $B: D(B) \subseteq X \rightarrow X$ such that $\tilde{A} := i\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ is densely defined with $\rho(\tilde{A}) \neq \emptyset$ and a C_0 -group $(b_t)_{t \in \mathbb{R}}$ consisting of unitary elements of K_2 such that:
 - (i) $L_{b_t}: \eta(C_2(X)) \rightarrow \eta(C_2(X)), y \mapsto b_t y$, and $L_{b_t^*}$ define C_0 -groups on the space $\eta(C_2(X))$ and
 - (ii) for the generator C of $(L_{b_t})_{t \in \mathbb{R}}$, we have $\eta(D(\tilde{A})) \subseteq D(C)$.

Note that part (c)(i) of the above theorem implies that $j(D(A)) \subseteq D(B)$.

A first step in the proof of the above characterization is the following result:

Theorem 3.14. Let X be an operator space and $A: D(A) \subseteq X \rightarrow X$. Then $A \in \mathcal{A}_{l,s}^{C_0}(X)$ if and only if $A \oplus 0$ generates a completely contractive C_0 -group on $C_2(X)$.

Proof. Let $A \in \mathcal{A}_{l,s}^{C_0}(X)$ and denote by $(T_t)_{t \in \mathbb{R}}$ the C_0 -group generated by A . Due to Proposition 2.9, $(T_t \oplus \text{id}_X)_{t \in \mathbb{R}}$ is a completely contractive C_0 -group, which is clearly generated by $A \oplus 0$.

Conversely, let $(S_t)_{t \in \mathbb{R}}$ be the completely contractive C_0 -group generated by $A \oplus 0$. We conclude that $S_t = T_t \oplus \text{id}_X$, where $(T_t)_{t \in \mathbb{R}}$ is the C_0 -group generated by A . Because S_t is a completely isometric isomorphism for all $t \in \mathbb{R}$ the assertion follows with Proposition 2.9. $\square$

With this result we conclude that (a) and (b) in Theorem 3.13 are equivalent. Thus unbounded multipliers can be characterized using generators of completely contractive C_0 -groups. Such generators are analyzed in the following two theorems, which generalize well-known results from the theory of contractive C_0 -semigroups to the setting of completely contractive (c. c.) C_0 -semigroups on an operator space X . By using the classical result on each matrix level, one can prove the following theorem. For details see [Sch09, Satz 3.33].

Theorem 3.15 (Theorem of Hille-Yosida for c.c. C_0 -semigroups). *Let X be an operator space and $A: D(A) \subseteq X \rightarrow X$ be linear. The following assertions are equivalent:*

- (a) *A is the generator of a completely contractive C_0 -semigroup on X .*
- (b) *A is densely defined and closed, $\mathbb{R}_{>0}$ is a subset of the resolvent set $\rho(A)$ of A and*

$$\|\lambda(\lambda - A)^{-1}\|_{cb} \leq 1 \quad \text{for all } \lambda > 0.$$

An operator $A: D(A) \subseteq X \rightarrow X$ is called *completely dissipative* if A_n is dissipative for all $n \in \mathbb{N}$, i.e.,

$$\|(\lambda - A_n)x\| \geq \lambda\|x\| \quad \text{for all } \lambda > 0 \text{ and } x \in D(A_n).$$

Theorem 3.16 (Theorem of Lumer-Phillips for c.c. C_0 -semigroups). *Let X be an operator space and $A: D(A) \subseteq X \rightarrow X$ be linear and densely defined. Then A generates a completely contractive C_0 -semigroup if and only if A is completely dissipative and there exists a $\lambda > 0$ such that $\lambda - A$ is surjective onto X .*

Proof. The direction ‘ $\Rightarrow$ ’ is a consequence of Theorem 3.15, because

$$\|(\lambda - A)_n x\| \geq \frac{1}{\|(\lambda - A)_n^{-1} x\|} \geq \lambda\|x\|$$

for all $\lambda > 0$, $n \in \mathbb{N}$ and $x \in D(A_n)$.

The other direction follows from the theorem of Lumer-Phillips for contractive C_0 -semigroups ([Wer07, Theorem VII.4.16]). $\square$

Using the two theorems above together with the equivalence of (a) and (b) in Theorem 3.13, we obtain the following theorem. For details see [Sch09, Proposition 3.41].

Theorem 3.17. *Let X be an operator space and $A: D(A) \subseteq X \rightarrow X$ be linear. The following assertions are equivalent:*

- (a) *$A \in \mathcal{A}_l^{C_0}(X)$.*
- (b) *There exists $B: D(B) \subseteq X \rightarrow X$ such that $\tilde{A} := i \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ is densely defined and closed satisfying $\mathbb{R} \setminus \{0\} \subseteq \rho(\tilde{A})$ and*

$$\| |\lambda| (\lambda - (\tilde{A} \oplus 0))^{-1} \|_{cb} \leq 1 \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\},$$

where $\tilde{A} \oplus 0 = \tilde{A} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is an operator on $C_2(C_2(X)) \cong C_4(X)$.

- (c) *There exists $B: D(B) \subseteq X \rightarrow X$ with the following properties:*

- (i) *$\tilde{A} := i \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ is densely defined,*
- (ii) *$\tilde{A} \oplus 0$ and $-\tilde{A} \oplus 0$ are completely dissipative and*
- (iii) *there exist $\lambda, \mu > 0$ such that $\lambda - \tilde{A}$ and $\mu + \tilde{A}$ are surjective onto $C_2(X)$.*

The following result shows that we can lift unbounded multipliers from an operator space X to the ternary envelope $\mathcal{T}(X)$:

Proposition 3.18. *Let X be an operator space and $A \in \mathcal{A}_{l,s}^{C_0}(X)$ (resp. $A \in \mathcal{A}_l^{C_0}(X)$). There exists a unique $B \in \mathcal{A}_{l,s}^{C_0}(\mathcal{T}(X))$ (resp. $B \in \mathcal{A}_l^{C_0}(\mathcal{T}(X))$) such that $j \circ A = B \circ j|_{D(A)}$.*

Proof. The case $A \in \mathcal{A}_l^{C_0}(X)$ follows directly from the assertion for $\mathcal{A}_{l,s}^{C_0}(X)$.

Therefore let $A \in \mathcal{A}_{l,s}^{C_0}(X)$. Let $(T_t)_{t \in \mathbb{R}}$ be the C_0 -group generated by A . There exists $a_t \in \mathcal{IM}_\ell^*(X)$ unitary such that $j(T_t x) = a_t \cdot j(x)$ for all $x \in X$. Define $S_t: \mathcal{T}(X) \rightarrow \mathcal{T}(X)$, $z \mapsto a_t \cdot z$. Then S_t is a unitary element of $\mathcal{A}_l(X)$ for all $t \in \mathbb{R}$. We conclude that

$$(3.3) \quad \|a_t \cdot (j(x) \cdot j(y)^* \cdot j(z)) - j(x) \cdot j(y)^* \cdot j(z)\| \leq \|T_t x - x\| \|j(y)^* \cdot j(z)\| \rightarrow 0$$

for $t \rightarrow 0$ and for all $x, y, z \in X$. Equation (3.2) implies that the set

$$V := \lim \{x_1 \cdot x_2^* \cdot x_3 \cdot x_4^* \cdots x_{2n+1} \mid n \geq 0, x_1, \dots, x_{2n+1} \in j(X)\}$$

is dense in $\mathcal{T}(X)$. It follows that $\lim_{t \rightarrow 0} S_t x = \lim_{t \rightarrow 0} a_t \cdot z = z$ for all $z \in V$ and $I_{11}(\mathcal{T}(X)) \cong I_{11}(X)$ ([Sch09, Proposition A.6]). We obtain

$$\|S_t z\|_{\mathcal{T}(X)} = \|a_t \cdot z\|_{\mathcal{T}(X)} \leq \|a_t\|_{I_{11}(\mathcal{T}(X))} \|z\|_{\mathcal{T}(X)} = \|a_t\|_{I_{11}(X)} \|z\|_{\mathcal{T}(X)}$$

for all $t \in \mathbb{R}$ and $z \in \mathcal{T}(X)$; thus $\|S_t\| \leq \|a_t\| = \|T_t\|_{\mathcal{M}_l(X)} = \|T_t\|$. With the characterization of C_0 -semigroups ([EN00, Proposition I.5.3]) we conclude that $(S_t)_{t \in \mathbb{R}}$ is a C_0 -group.

Let B denote the generator of $(S_t)_{t \in \mathbb{R}}$. Then $B \in \mathcal{A}_{l,s}^{C_0}(\mathcal{T}(X))$ and

$$(3.4) \quad j(Ax) = \lim_{t \rightarrow 0} \frac{a_t \cdot j(x) - j(x)}{t} = B(j(x))$$

for all $x \in D(A)$.

To show that B is unique let $C \in \mathcal{A}_{l,s}^{C_0}(\mathcal{T}(X))$ such that $j \circ A = C \circ j|_{D(A)}$. Define

$$W := W_{j(D(A)), j(X)} := \lim \{x_1 \cdot x_2^* \cdot x_3 \cdot x_4^* \cdots x_{2n+1} \mid n \geq 1, x_1 \in j(D(A)), x_2, \dots, x_{2n+1} \in j(X)\}.$$

With equation (3.4) we obtain $B|_W = C|_W$. Because W is a core for B , i. e., W is dense in $D(B)$ for the graph norm, it follows that $B = \overline{B|_W} = \overline{C|_W} = C$. $\square$

Proposition 3.19. *Let X, Y be operator spaces with $X \subseteq Y$ and $A \in \mathcal{A}_{l,s}^{C_0}(Y)$. Suppose there exist $\lambda, \mu > 0$ such that $\lambda - A|_X$ and $\mu + A|_X$ are surjective onto X . Then $A|_X \in \mathcal{A}_{l,s}^{C_0}(X)$.*

Proof. Due to Theorem 3.14 we have that $A \oplus 0$ generates a completely contractive C_0 -group on $C_2(Y)$. Define

$$A|_X := \{(x, y) \mid x \in D(A) \cap X, y = Ax \in X\}.$$

With the theorem of Lumer-Phillips (Theorem 3.16) $A \oplus 0$ and also $A|_X \oplus 0$ are completely dissipative and generate completely contractive C_0 -groups. We conclude with Theorem 3.14: $A|_X \in \mathcal{A}_{l,s}^{C_0}(X)$. Let $(T_t)_{t \in \mathbb{R}}$ be the C_0 -group generated by A . Because the C_0 -group generated by $A|_X$ is equal to $(T_t|_X)_{t \in \mathbb{R}}$, we obtain with [EN00, Corollary II.2.3]: $A|_X = A|_X$. $\square$

Similar to regular operators, every unbounded multiplier has an adjoint:

Theorem 3.20. *Let X be an operator space and $A \in \mathcal{A}_l^{C_0}(X)$. There exists a unique operator $A^*: D(A^*) \subseteq X \rightarrow X$, called the adjoint of A , such that $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \in \mathcal{A}_{l,s}^{C_0}(C_2(X))$. We have $A^* \in \mathcal{A}_l^{C_0}(X)$ and $(A^*)^* = A$.*

Proof. The existence follows directly from the definition of an unbounded multiplier. To show the uniqueness, let $B_i: D(B_i) \subseteq X \rightarrow X$ ($i \in \{1, 2\}$) be such that $\tilde{B}_i := i \begin{pmatrix} 0 & A \\ B_i & 0 \end{pmatrix} \in \mathcal{A}_{l,s}^{C_0}(C_2(X))$. Due to Proposition 3.18, we find a regular operator $\tilde{C}_i \in \mathcal{R}(\mathcal{T}(C_2(X)))$ satisfying $j \circ \tilde{B}_i = \tilde{C}_i \circ j|_{D(\tilde{B}_i)}$. Let $(T_t^{(i)})_{t \in \mathbb{R}}$ be the C_0 -group generated by $\tilde{B}_i$. There exists $a_t^{(i)} \in \mathcal{IM}_\ell^*(C_2(X))$ such that $j(T_t^{(i)}x) = a_t^{(i)} \cdot j(x)$ for all $x \in C_2(X)$. Note that $\tilde{C}_i$ is the generator of the C_0 -group that is defined by $S_t^{(i)}: \mathcal{T}(C_2(X)) \rightarrow \mathcal{T}(C_2(X)), z \mapsto a_t^{(i)} \cdot z$. For all $k \in \mathbb{N} \cup \{0\}$, $x_1 \in D(\tilde{B}_i)$ and $x_2, \dots, x_{2k+1} \in C_2(X)$ we obtain with $z := j(x_2)^* \cdot j(x_3) \cdot j(x_4)^* \cdots j(x_{2k+1})$,

$$j(\tilde{B}_i(x_1)) \cdot z = \lim_{t \rightarrow 0} \frac{a_t^{(i)} \cdot j(x_1) - j(x_1)}{t} \cdot z = \tilde{C}_i(j(x_1) \cdot z).$$

Set $W_i := W_{D(\tilde{B}_i), C_2(X)}$. Using $\mathcal{T}(C_2(X)) \cong C_2(\mathcal{T}(X))$ ([BLM04, 8.3.12.(4)]), we conclude that $\tilde{C}_i|_{W_i}$ has the same form as $\tilde{B}_i$, namely $i \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Moreover, W_i is a core for $\tilde{C}_i$. Therefore, $\tilde{C}_i$ has the form $i \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Using Theorem 3.8, there exists $\check{C}_i \in \mathcal{R}(\mathcal{T}(X))$ such that

$$\tilde{C}_i = i \begin{pmatrix} 0 & \check{C}_i \\ \check{C}_i^* & 0 \end{pmatrix}.$$

The set $W := W_{j(D(A)), j(X)}$ is a core for $\check{C}_1$ and for $\check{C}_2$. Using $\check{C}_1|_W = \check{C}_2|_W$, we obtain

$$\check{C}_1 = \overline{\check{C}_1|_W} = \overline{\check{C}_2|_W} = \check{C}_2.$$

It follows that $\tilde{C}_1 = \tilde{C}_2$; thus $\tilde{B}_1 = \tilde{B}_2$ and $B_1 = B_2$.

With $\check{C}_i^{**} = \check{C}_i$ ([Lan95, Corollary 9.4]) and $\check{C}_i^* \in \mathcal{R}(\mathcal{T}(X))$ ([Lan95, Corollary 9.6]) we conclude that

$$\hat{C}_i := i \begin{pmatrix} 0 & \check{C}_i^* \\ \check{C}_i & 0 \end{pmatrix} \in \mathcal{R}(\mathcal{T}(X) \oplus \mathcal{T}(X))$$

is skew-adjoint. Using $\mathcal{T}(X) \oplus \mathcal{T}(X) \cong C_2(\mathcal{T}(X))$, we get $\hat{C}_i \in \mathcal{A}_{l,s}^{C_0}(C_2(\mathcal{T}(X)))$. Set $Y := j_{C_2(X)}(C_2(X))$. With the theorem of Lumer-Phillips and Proposition 3.19 it follows that $\hat{C}_i|_Y \in \mathcal{A}_{l,s}^{C_0}(Y)$. We obtain $i \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \in \mathcal{A}_{l,s}^{C_0}(C_2(X))$; thus $A^* \in \mathcal{A}_l^{C_0}(X)$ and $(A^*)^* = A$. $\square$

We can now prove the equivalence of (a) and (c) in Theorem 3.13:

Let $A \in \mathcal{A}_l^{C_0}(X)$. Due to Proposition 3.18 and Theorem 3.8 there exists $B \in \mathcal{A}_l^{C_0}(\mathcal{T}(X)) = \mathcal{R}(\mathcal{T}(X))$ satisfying $j \circ A = B \circ j|_{D(A)}$. Then Theorem 3.20 implies $\pm i \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} = \pm i \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}|_{j_{C_2(X)}(C_2(X))} \in \mathcal{A}_{l,s}^{C_0}(C_2(X))$, and by definition this operator generates a C_0 -semigroup. With the theorem of Lumer-Phillips we obtain part (c)(ii) of Theorem 3.13.

Conversely, suppose there exists $B \in \mathcal{R}(\mathcal{T}(X)) = \mathcal{A}_l^{C_0}(\mathcal{T}(X))$ satisfying (i) and (ii) of (c). With Proposition 3.19 we conclude that $B|_{j(X)} \in \mathcal{A}_l^{C_0}(j(X))$; thus $A \in \mathcal{A}_l^{C_0}(X)$.

3.4. Connection to operators on Hilbert spaces. In this subsection we characterize unbounded skew-adjoint multipliers using C_0 -groups on Hilbert spaces.

Theorem 3.21. *Let X be an operator space. Let (π, K, K_1, K_2) be a $\mathcal{T}(X)$ -embedding of X and let $\eta := \pi \circ j: X \rightarrow L(K)$. Let $A: D(A) \subseteq X \rightarrow X$ be linear.*

The following assertions are equivalent:

- (a) $A \in \mathcal{A}_{l,s}^{C_0}(X)$.
- (b) A is densely defined with $\rho(A) \neq \emptyset$, and there exists a C_0 -group $(b_t)_{t \geq 0}$ of unitary elements in $L(K_2)$ such that:
 - (i) $L_{b_t} : \eta(X) \rightarrow \eta(X), y \mapsto b_t y$ and $L_{b_t^*}$ define C_0 -groups on $\eta(X)$ and
 - (ii) for the generator C of $(L_{b_t})_{t \in \mathbb{R}}$, we have $j(D(A)) \subseteq D(C)$.

Proof. At first we show that (a) implies (b). Let $(T_t)_{t \in \mathbb{R}}$ denote the C_0 -group generated by A . There exists $a_t \in \mathcal{I}M_\ell^*(X)$ such that $j(T_t x) = a_t \cdot j(x)$. Define $b_t := \pi(a_t)|_{K_2}$. Because $[\pi(\mathcal{T}(X))K_1]$ is dense in K_2 , we have $b_t(K_2) \subseteq K_2$. With equation (3.3) it follows for all $z \in \mathcal{T}(X)$ and $\xi \in K_1$ that

$$\|b_t(\pi(z)(\xi)) - \pi(z)(\xi)\| = \|(\pi(a_t)\pi(z) - \pi(z))(\xi)\| \leq \|a_t \cdot z - z\| \|\xi\| \rightarrow 0$$

for $t \rightarrow 0$. Therefore $(b_t)_{t \in \mathbb{R}}$ is a C_0 -group on K_2 . Using the theorem of Hille-Yosida, we obtain that A is densely defined with $\rho(A) \neq \emptyset$. It is straightforward to show (i) and (ii).

To prove the other direction, define $d_t : K \rightarrow K, \xi \mapsto b_t(p_{K_2}(\xi))$, where p_{K_2} denotes the projection from K onto K_2 and $S_t := L_{d_t} : \eta(X) \rightarrow \eta(X)$. Then it can be checked that $T_t := \eta^{-1} \circ S_t \circ \eta$ defines a C_0 -group on X with generator A . $\square$

With the above theorem we obtain the equivalence of (a) and (d) in Theorem 3.13. So we have finished the proof of this theorem.

Proposition 3.22. *Let X be an operator space and $A \in \mathcal{A}_{l,s}^{C_0}(X)$. There exist a Hilbert space K , a completely isometric map $\eta : X \rightarrow L(K)$ and a generator C of a unitary C_0 -group on K such that $\eta(Ax) = C \circ \eta(x)$ for all $x \in D(A)$.*

Proof. We use the notation of Theorem 3.21. Using this result, we find a C_0 -group $(b_t)_{t \in \mathbb{R}}$ with the properties stated in Theorem 3.21. Let $\tilde{C}$ be the generator of this C_0 -group. With Theorem 3.21 we get

$$\eta(Ax)(\xi) = \lim_{t \rightarrow 0} \frac{(b_t \circ \eta(x))(\xi) - \eta(x)(\xi)}{t} = \tilde{C}(\eta(x)(\xi))$$

for all $x \in D(A)$ and $\xi \in K$; thus $\eta(Ax) = \tilde{C} \circ \eta(x)$. The assertion follows from setting $C := \tilde{C} \oplus 0_{K_2^\perp}$. $\square$

3.5. The strict X -topology. If we want to transform a C_0 -group $(b_t)_{t \in \mathbb{R}}$ from a Hilbert space to an operator space X , we have to impose a stronger condition than strong continuity on $(b_t)_{t \in \mathbb{R}}$ (cf. Theorem 3.21), namely

$$\lim_{t \rightarrow 0} b_t \circ y = y \quad \text{for all } y \in \eta(X).$$

In order to formulate this in a more abstract fashion, we introduce the following topology for an operator space $X \subseteq L(H_0, H)$:

Definition 3.23. For all $x \in X$ a seminorm is defined by

$$p_x : L(H) \rightarrow \mathbb{R}, T \mapsto \|T \circ x\|.$$

The locally convex topology on $L(H)$ generated by $(p_x)_{x \in X}$ is called the *strict X -topology*.

The norm topology on $L(H)$ is finer than the strict X -topology on $L(H)$. If X is unital and $H_0 = H$, these two topologies coincide.

The proof of the following proposition is straightforward:

Proposition 3.24. *Let $X \subseteq L(H_0, H)$ be an operator space with $[XH_0]$ dense in H . The strict X -topology on $L(H)$ is finer than the strong topology.*

It is not hard to see that the assumption in the above proposition can always be fulfilled by choosing an appropriate embedding of X .

Proposition 3.25. *Let X be an operator space. Let K be a Hilbert space and $\eta: X \rightarrow L(K)$ a completely isometric map. Let $C: D(C) \subseteq K \rightarrow K$ be the generator of a C_0 -group $(b_t)_{t \in \mathbb{R}}$ of unitary elements in $L(K)$ such that*

- (i) $(b_t)_t$ converges in the strict $\eta(X)$ -topology to id_K for $t \rightarrow 0$ and
- (ii) $b_t \eta(X), b_t^* \eta(X) \subseteq \eta(X)$ for all $t \in \mathbb{R}$.

Then there exists $A \in \mathcal{A}_{l,s}^{C_0}(X)$ such that

$$\eta(Ax)(\xi) = C(\eta(x)(\xi)) \quad \text{for all } x \in D(A), \xi \in K.$$

Proof. The proof is similar to the proof of Proposition 3.22. □

3.6. Perturbation theory. The following result generalizes a perturbation result of Damaville ([Dam04, Proposition 2.1]; see also [Dam07]) from Hilbert C^* -modules to operator spaces:

Theorem 3.26. *Let X be an operator space.*

- (i) *If $A \in \mathcal{M}_l^{C_0}(X)$ and $B \in \mathcal{M}_l(X)$, then $A + B \in \mathcal{M}_l^{C_0}(X)$.*
- (ii) *If $A \in \mathcal{A}_{l,s}^{C_0}(X)$ and $B \in \mathcal{A}_l(X)$ with $B^* = -B$, then $A + B \in \mathcal{A}_{l,s}^{C_0}(X)$.*
- (iii) *If $A \in \mathcal{A}_l^{C_0}(X)$ and $B \in \mathcal{A}_l(X)$, then $A + B \in \mathcal{A}_l^{C_0}(X)$.*

Proof. Part (i) follows directly with [EN00, Theorem III.1.10].

To prove (ii), we notice that there exists $\check{A} \in \mathcal{A}_{l,s}^{C_0}(\mathcal{T}(X)) = \mathcal{R}(\mathcal{T}(X))$ satisfying $j \circ A = \check{A} \circ j|_{D(A)}$ due to Theorem 3.18. There exists $\check{B} \in \mathcal{A}_l(\mathcal{T}(X)) \cong \mathbb{B}(\mathcal{T}(X))$ skew-adjoint satisfying $j \circ B = \check{B} \circ j$. Using [Dam04, Proposition 2.1], we conclude that $\check{A} + \check{B} \in \mathcal{R}(\mathcal{T}(X))$. With Theorem 3.4 we get: $\check{A} + \check{B} \in \mathcal{A}_{l,s}^{C_0}(\mathcal{T}(X))$. Because B and the C_0 -group generated by A leave $j(X)$ invariant, the C_0 -group generated by $\check{A} + \check{B}$ leaves $j(X)$ invariant. We obtain with [Sch09], Proposition 3.53: $(\check{A} + \check{B})|_{j(X)} \in \mathcal{A}_{l,s}^{C_0}(j(X))$; thus $A + B \in \mathcal{A}_{l,s}^{C_0}(X)$.

Part (iii) is a consequence of part (ii). □

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