INTRINSIC VOLUMES AND LINEAR CONTRACTIONS

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ABSTRACT. It is shown that intrinsic volumes of a convex body decrease under linear contractions.

Let $C \subset \mathbb{R}^N$ be a convex body and B_2^N the Euclidean ball in $\mathbb{R}^N$. The Steiner formula expresses the volume of the Minkowski sum $C + \varepsilon B_2^N$ in terms of the intrinsic volumes $V_0, V_1, \dots, V_N$ of C :

$$\text{vol}_N(C + \varepsilon B_2^N) = \sum_{n=0}^N \omega_n V_{N-n}(C) \varepsilon^n.$$

Here $\text{vol}_N(\cdot)$ denotes N -dimensional Lebesgue measure and $\omega_n = \text{vol}_n(B_2^n)$. Of particular interest are V_1 , V_{N-1} and V_N , which are multiples of the mean-width, surface area and volume, respectively. We refer the reader to [5] for background on intrinsic volumes. In addition to their role in convex geometry, intrinsic volumes also appear in connection with Gaussian processes; see, e.g., [9], [10] and the references therein.

The purpose of this note is to prove the following.

Proposition 1.1. *Let $C \subset \mathbb{R}^N$ be a convex body and let S be a linear contraction; i.e., $\|Sx\|_2 \leq \|x\|_2$ for each $x \in \mathbb{R}^N$. Then for $n = 1, \dots, N$,*

$$V_n(SC) \leq V_n(C).$$

The case of V_1 and arbitrary contractions (not necessarily linear) is well-studied [6, Theorem 2 in §5], [1, Theorem 1]; see also [2, p. 177]. Of course for V_N one has $V_N(SC) = |\det(S)| \text{vol}_N(C)$. For other intrinsic volumes, we were unable to find Proposition 1.1 in the literature but noticed that it follows from some results in [4] and thought it was worthwhile to show the details.

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Particularly useful for our purpose is the Gaussian representation of intrinsic volumes, as in [10]; see also [8]. If $\Gamma_{N,n} = [\gamma_{ij}]$ is an $n \times N$ matrix with independent $N(0,1)$ Gaussian entries, then the n -th intrinsic volume of $C \subset \mathbb{R}^N$ is given by

$$V_n(C) = \frac{(2\pi)^{n/2}}{\omega_n n!} \mathbb{E} \operatorname{vol}_n(\Gamma_{N,n} C).$$

As in [4], we say that a function $F : (\mathbb{R}^n)^N \rightarrow \mathbb{R}^+$ satisfies **Groemer's Convexity Condition**, or simply **(GCC)**, if for every $z \in \mathbb{R}^n$ and for every $y_1, \dots, y_N \in z^\perp$ the function $F_Y : \mathbb{R}^N \rightarrow \mathbb{R}^+$ defined by

$$F_Y(t) = F(y_1 + t_1 z, \dots, y_N + t_N z)$$

is even and convex. The latter definition was motivated by isoperimetric-type problems for random convex sets in [3]. In particular, by adapting [3, Lemma 3], it was shown in [4, Proposition 4.1] that for a convex body $C \subset \mathbb{R}^N$, the function $F : (\mathbb{R}^n)^N \rightarrow \mathbb{R}^+$ defined by

$$(1) \quad F(x_1, \dots, x_N) = \operatorname{vol}_n([x_1 \dots x_N]C),$$

where $[x_1 \dots x_N]$ denotes the $n \times N$ matrix with columns $x_1, \dots, x_N$, viewed as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$, satisfies **(GCC)**. The latter property fits well with symmetrization techniques and can be used in various isoperimetric-type problems for the volume of random (and non-random) sets [4, Theorem 1.1].

For our present purpose, we require less than the **(GCC)** condition. In fact, we will use only the following consequence.

Lemma 1.2. *If $F : (\mathbb{R}^n)^N \rightarrow \mathbb{R}^+$ satisfies **(GCC)**, then for any $x_1, \dots, x_N \in \mathbb{R}^n$ and any $1 \leq j \leq N$, the function*

$$(2) \quad \mathbb{R} \ni s \mapsto F(x_1, \dots, sx_j, \dots, x_N)$$

is convex.

The lemma is immediate since the restriction of a convex function to a line is itself convex.

Additionally, we will make use of the following elementary lemma (the proof is given in [4, Lemma 3.7]).

Lemma 1.3. *Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a function such that*

$$\mathbb{R} \ni s \mapsto \rho(sx)$$

is convex for each $x \in \mathbb{R}^n$. If X is a symmetric random vector with values in $\mathbb{R}^n$, then

$$\mathbb{R}^+ \ni s \mapsto \mathbb{E}\rho(sX)$$

is an increasing function.

Here and elsewhere, we use the term “increasing” in the non-strict sense.

Proof of Proposition 1.1. As noted above, the function $F : (\mathbb{R}^n)^N \rightarrow \mathbb{R}^+$ defined according to (1) satisfies **(GCC)**. Let $g_1, \dots, g_N$ denote the columns of the Gaussian random matrix $\Gamma_{N,n}$. If $g_1, \dots, g_N$ are fixed, then

$$\mathbb{R} \ni s \mapsto F(g_1, \dots, g_{j-1}, sg_j, g_{j+1}, \dots, g_N)$$

is convex by Lemma 1.2. Letting $\mathbb{E}_j$ denote expectation with respect to g_j and applying Lemma 1.3, we have that

$$\mathbb{R}^+ \ni s \mapsto \mathbb{E}_j F(g_1, \dots, g_{j-1}, sg_j, g_{j+1}, \dots, g_N)$$

is an increasing function.

Suppose first that S is represented by the $N \times N$ diagonal matrix $S = \text{diag}(1, \dots, 1, s_j, 1, \dots, 1)$, where $s_j \in [0, 1]$ is in the j^{th} -column. Then

$$\mathbb{E}_j F(g_1, \dots, g_{j-1}, s_j g_j, g_{j+1}, \dots, g_N) \leq \mathbb{E}_j F(g_1, \dots, g_{j-1}, g_j, g_{j+1}, \dots, g_N)$$

and hence

$$\begin{aligned} (2\pi)^{-n/2} \omega_n n! V_n(SC) &= \mathbb{E} \text{vol}_n(\Gamma_{N,n} SC) \\ &= \mathbb{E} F(g_1, \dots, g_{j-1}, s_j g_j, g_{j+1}, \dots, g_N) \\ &\leq \mathbb{E} F(g_1, \dots, g_N) \\ &= (2\pi)^{-n/2} \omega_n n! V_n(C). \end{aligned}$$

In the general case, using singular value decomposition, one writes $S = UDV^T$, where D is the diagonal matrix $\text{diag}(s_1, \dots, s_N)$, and U and V are orthogonal. Since S is a contraction, its singular values satisfy $0 \leq s_i \leq 1$ for $i = 1, \dots, N$. To conclude, we use the fact that intrinsic volumes are invariant under orthogonal transformations and apply the latter argument iteratively. $\square$

Remark 1.4. The latter proof uses ideas from [7, Lemma 2.7].

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REFERENCES

1. R. Alexander, *Lipschitzian mappings and total mean curvature of polyhedral surfaces. I*, Trans. Amer. Math. Soc. **288** (1985), no. 2, 661–678. MR776397 (86c:52004)
2. Yu. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 285, Springer-Verlag, Berlin, 1988. Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics. MR936419 (89b:52020)
3. H. Groemer, *On the mean value of the volume of a random polytope in a convex set*, Arch. Math. (Basel) **25** (1974), 86–90. MR0341286 (49:6036)
4. G. Paouris and P. Pivovarov, *A probabilistic take on isoperimetric-type inequalities*, Adv. Math. **230** (2012), 1402–1422. MR2921184
5. R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993. MR1216521 (94d:52007)
6. V. N. Sudakov, *Geometric problems in the theory of infinite-dimensional probability distributions*, Proc. Steklov Inst. Math. (1979), no. 2, i–v, 1–178, Cover-to-cover translation of Trudy Mat. Inst. Steklov **141** (1976). MR530375 (80e:60052)
7. S. J. Szarek, *Spaces with large distance to l_∞^n and random matrices*, Amer. J. Math. **112** (1990), no. 6, 899–942. MR1081810 (91j:46023)
8. B. S. Tsirelson, *A geometric approach to maximum likelihood estimation for an infinite-dimensional Gaussian location. II*, Theory Prob. Appl. **30**, 820–828. Translation of Teor. Veroyatnost. i Primenen. **30** (1985), no. 4, 772–779. MR816291 (87i:62152)
9. R. A. Vitale, *Intrinsic volumes and Gaussian processes*, Adv. in Appl. Probab. **33** (2001), no. 2, 354–364. MR1842297 (2002h:60076)
10. ———, *On the Gaussian representation of intrinsic volumes*, Statist. Probab. Lett. **78** (2008), no. 10, 1246–1249. MR2441470 (2009k:60090)

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