BIALYNICKI-BIRULA DECOMPOSITION OF DELIGNE-MUMFORD STACKS

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ABSTRACT. This short note considers the Białynicki-Birula decomposition of Deligne-Mumford stacks under one-dimensional torus actions and extends a result of Oprea.

1. INTRODUCTION

In this short note, we consider actions of one-dimensional tori on tame Deligne-Mumford stacks which are smooth and proper over an algebraically closed field. We extend a result of Oprea [Opr06] to show that in the aforementioned case, if the stack has a scheme for a coarse moduli space, or if it is toric, then it admits a Białynicki-Birula decomposition and often a corresponding decomposition of cohomology. A history of the result can be found in [Bro05, Theorem 3.2].

2. NOTATION AND TERMINOLOGY

Let k be an algebraically closed field of arbitrary characteristic, fixed in what follows.

In this note, an algebraic stack will be a stack X fibered over (Sch/k) in the étale topology, such that the diagonal mapping $\Delta : X \rightarrow X \times X$ is representable, separated and quasi-compact, and such that there exists a smooth, surjective k -morphism $U \rightarrow X$ from a k -scheme U , which will be called an atlas. Deligne-Mumford stacks are those admitting étale atlases. Proper Deligne-Mumford stacks are those admitting a finite, surjective morphism from a proper k -scheme. Tame Deligne-Mumford stacks are those with linearly reductive geometric stabilizer groups.

An affine fibration is a flat morphism $p : E \rightarrow X$ which is étale locally a trivial bundle of affine spaces. This definition weakens the definition of a vector bundle by relaxing the requirement that the transition functions be linear.

Let T be a one-dimensional torus over k with an isomorphism to $\mathbf{G}_m$. Let the fixed points of an action of T on a stack X be denoted by X^T [Rom05].

If an algebraic stack has the form of a quotient $[X/G]$ of a normal toric variety X over an algebraically closed field of characteristic zero by a subgroup G of the torus of X , then it is called a toric stack [GS11a].

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3. BIALYNICKI-BIRULA DECOMPOSITION OF DELIGNE-MUMFORD STACKS

Oprea proved a form of the Białynicki-Birula decomposition [BB73] for Deligne-Mumford stacks assuming there exists a T -equivariant, affine, étale atlas [Opr06, Proposition 5].

Proposition 3.1 (Oprea). *Let X be a smooth, proper Deligne-Mumford stack over an algebraically closed field k with a T -action that admits a T -equivariant, affine, étale atlas $U \rightarrow X$. Let $F = \coprod_i F_i$ be the decomposition of the fixed substack into connected components. Then X decomposes into disjoint, locally closed, T -equivariant substacks X_i which are T -equivariant affine fibrations over F_i .*

Oprea expected the existence of an atlas to be a general fact [Opr06, Section 2]. Here we show that the desired atlas exists under somewhat general conditions.

Proposition 3.2. *Let X be a tame, irreducible Deligne-Mumford stack, smooth and proper over k , whose generic stabilizer is trivial and whose coarse moduli space is a scheme. Furthermore, let an action of T on X be given such that T acts trivially on its fixed locus. Then there exists a T -equivariant, affine, étale atlas $U \rightarrow X$.*

Remark 3.3. If an algebraic group G acts algebraically on a Deligne-Mumford stack X , then the action descends to the coarse moduli space of X by the universal property of the coarse moduli space of $G \times_k X$.

Remark 3.4. If T does not act trivially on its fixed locus, it can be made to do so by a reparametrization of the action. Since T -invariance is not affected by this change, the decomposition for the new action will be a decomposition for the original action.

Proof. Since X is reduced and irreducible with a scheme for a coarse moduli space, the Keel-Mori theorem [KM97, Proposition 4.2] shows that the coarse moduli space is in fact a normal, proper variety over k . It inherits a T -action by Remark 3.3. Since it is proper, any collection of open, T -invariant neighborhoods containing the fixed point locus covers it and contains a finite subcover. They can be chosen to be affine by Sumihiro's theorem [Sum74, Corollary 2]. So it suffices to find the desired atlas for an arbitrary fixed k -point $x \in X$. To this end, let Y be the pullback of X to an open, affine, T -invariant neighborhood of the image of x in the coarse moduli space.

Consider the frame bundle of Y with total space FY . Each k -point $y \in Y$ lies in the image of an étale, representable morphism from a quotient stack of the form $[U/G]$ for an affine, irreducible scheme U with an action of the stabilizer group G of y and containing a point u fixed by the G -action and mapping to y [KM97]. By the tameness hypothesis, G is linearly reductive. Since Y has trivial generic stabilizer, G acts faithfully on U . Applying [BB73, Theorem 2.4] to U and $T_u U$ shows that G acts faithfully on $T_y Y$. So the total space FY is an algebraic space, and $Y = [FY/\mathbf{GL}_n]$, where n is the dimension of X . But Y has an affine coarse moduli space, so FY is, in fact, an affine scheme (cf. [EHKV01, Remark 4.3]).

The action of T on Y induces an action on TY and hence an action on FY , which is an open, T -invariant substack of $TY^{\oplus n}$. Let $p : FY \rightarrow Y$ be the projection. An atlas will be defined by finding a T -equivariant étale slice $U \hookrightarrow FY$ over the fixed point y .

One may assume that a fixed point $f \in FY$ lies over y by modifying the action as follows. Choose a basis of the tangent space $T_y Y$ which diagonalizes the T -action, i.e., so that $t : (v_1, \dots, v_n) \mapsto (t^{a_1} v_1, \dots, t^{a_n} v_n)$. Then the induced T^n -action on FY can be used to define a twisted T -action on FY by

$$\begin{aligned} T &\rightarrow T^n, \\ t &\mapsto (t^{-a_1}, \dots, t^{-a_n}). \end{aligned}$$

The projection from the frame bundle remains T -equivariant after twisting the action, but now the action fixes the frame f formed by the basis vectors.

The proof finishes by arguing as in [Opr06, Lemma 3]. The torus T acts on the tangent space $T_f FY$ at f , with $T_f FY \rightarrow T_y Y$ surjective and T -equivariant. By the linear reductivity of T , $T_y Y$ may be identified with some T -invariant subspace $N \subset T_y FY$, compatibly with the T -actions. By a theorem of Białynicki-Birula [BB73, Theorem 2.1], there exists a reduced and irreducible, closed, T -invariant subscheme Z of FY containing f as a nonsingular point such that $T_f Z = N$.

Taking U to be the largest open subscheme of Z on which the restriction of p is étale ensures that U is T -invariant. By applying Sumihiro's theorem again, one may shrink U to an affine, T -invariant neighborhood of f whose image in Y contains y . $\square$

Theorem 3.5. *Let X be a tame Deligne-Mumford stack, smooth and proper over k , whose coarse moduli space is a scheme, and let an action of T on X be given. Then X admits a Białynicki-Birula decomposition.*

Proof. One first reduces to the case that X is irreducible by decomposing each irreducible component separately and combining to give a decomposition of all of X . Applying Remark 3.4, one may suppose T acts trivially on its fixed locus. By [Ols07, Proposition 2.1], there exists a rigidification $\overline{X}$ of X which has trivial generic stabilizer. In particular, there is an étale, proper morphism $X \rightarrow \overline{X}$ which forms a G -gerbe for a finite group G . The T -action descends to $\overline{X}$ using the universal property of rigidification, so Proposition 3.2 guarantees the existence of a T -equivariant, affine, étale atlas of $\overline{X}$. Let the substacks $\overline{F}_i$ and $\overline{X}_i$, together with T -equivariant affine fibrations $\overline{X}_i \rightarrow \overline{F}_i$, be defined according to Proposition 3.1.

Let the decomposition of X be defined by pulling back along $X \rightarrow \overline{X}$. Pulling back again by the affine fibration $\overline{X}_i \rightarrow \overline{F}_i$ forms the diagram:

$$(3.1) \quad \begin{array}{ccccccc} \overline{X}_i \times_{\overline{F}_i} F_i & \longrightarrow & F_i & \longrightarrow & X_i & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{X}_i & \longrightarrow & \overline{F}_i & \longrightarrow & \overline{X}_i & \longrightarrow & \overline{X}. \end{array}$$

So $F = \coprod_i F_i$ is a decomposition of the fixed substack into connected components [Rom05]. Furthermore, all morphisms are T -equivariant.

In what follows, let i be fixed. It remains to prove the existence of an affine fibration, and diagram (3.1) shows it will suffice to supply a T -equivariant isomorphism $\overline{X}_i \times_{\overline{F}_i} F_i \rightarrow X_i$ over $\overline{X}_i$. This can be done by specifying such an isomorphism, unique up to a canonical 2-isomorphism, over an étale, T -equivariant atlas of $\overline{X}_i$, and then applying the descent property of the stack $\underline{\mathrm{Hom}}_{\overline{X}_i}(\overline{X}_i \times_{\overline{F}_i} F_i, X_i)$ [Ols06].

First, étale, T -equivariant atlases forming the following pullback will be defined:

$$(3.2) \quad \begin{array}{ccc} W & \longrightarrow & P \\ \downarrow & & \downarrow \\ \overline{X}_i & \longrightarrow & \overline{F}_i. \end{array}$$

There is an $\mathrm{Out}(G)$ -torsor on $\overline{F}_i$ associated to the rigidification gerbe $F_i \rightarrow \overline{F}_i$ whose class α lies in $H^1(\overline{F}_i, \mathrm{Out}(G))$. Base change to the total space of the torsor trivializes α and gives an element of the étale cohomology $\beta \in H^2(\overline{F}_i, Z)$ which classifies the gerbe, where Z is the center of G (cf. [EHKV01, Section 3.1]). Let $P \rightarrow \overline{F}_i$ be an affine, étale atlas that trivializes α , β , and the affine fibration $\overline{X}_i \rightarrow \overline{F}_i$. Also, let $W := P \times_{\overline{F}_i} \overline{X}_i$, as in diagram (3.2).

By the Künneth formula for the algebraic fundamental group [Ray71, Proposition 4.6], the atlas W trivializes the $\mathrm{Out}(G)$ -torsor associated to the gerbe $X_i \rightarrow \overline{X}_i$. Then [Art73, Corollary 2.2] shows that the morphism $P \times \mathbf{A}^n \cong W \rightarrow P$ induces an isomorphism $H^2(P, Z) \xrightarrow{\sim} H^2(W, Z)$, implying that the classifying element of $X_i \times_{\overline{X}_i} W \rightarrow W$ vanishes. The isomorphism $X_i \times_{\overline{X}_i} W \xrightarrow{\sim} F_i \times_{\overline{F}_i} W$ of trivial gerbes can be chosen, uniquely up to a canonical 2-isomorphism, to be the isomorphism over W which extends the identity morphism of $F_i \times_{\overline{F}_i} P$ over P . This follows from the triviality of the affine fibration $W \rightarrow P$ and the Künneth formula. The isomorphism will also be T -equivariant by similar reasoning, since the identity is T -equivariant. $\square$

Remark 3.6. If X is a tame Deligne-Mumford stack, smooth and proper over k , with a projective coarse moduli space and a T -action, then the induced decomposition forms a filtration, and a lemma of Oprea [Opr06, Lemma 6] implies that the Betti numbers of the stack are calculated by the Betti numbers of the fixed points.

In what follows, T may be a torus of arbitrary dimension.

Proposition 3.7. *Let $\mathrm{char} k = 0$, and let X be a normal algebraic space, separated and of finite type over k , with an action of T which gives a dense, open embedding of T in X . Then X is a scheme and hence a toric variety.*

Proof. First, let $k = \mathbf{C}$. The scheme locus of the normalized blowup at the closure of any nondense T -orbit forms a toric variety whose image includes the orbit. The associated fans give, in each T -orbit, a limit point of a $\mathbf{G}_m$ -orbit of $1 \in T \hookrightarrow X$ for a subtorus $\mathbf{G}_m$ of T . Then X is a finite union of T -orbits of such points and hence a scheme [Hau00, Theorem 1].

For general k , one may immediately reduce to the case that k is a subfield of $\mathbf{C}$. The pullback of X to $\mathbf{C}$ is a toric variety [Hau00], so a theorem [GS11b, Theorem 6.1] implies there exists an étale, representable, surjective morphism $p : [U/\mathbf{GL}_n] \rightarrow [X_{\mathbf{C}}/T]$ where U is a quasi-affine scheme over $\mathbf{C}$. Let $L \subset \mathbf{C}$ be a subfield of definition of p which is finitely generated over k , giving a morphism $p_L : [U_L/\mathbf{GL}_n] \rightarrow [X_L/T]$ where U is obtained by pulling back U_L to $\mathbf{C}$. Then $d = \mathrm{tr} \deg L/K < \infty$, and p_L remains étale, representable and surjective. Writing $L = k(V)$ for a d -dimensional affine variety V , one may suppose that U_L with its $\mathbf{GL}_n$ -action is defined over V , realizing p_L as the pullback of a dominant, étale morphism $p_V : [U_V/\mathbf{GL}_n] \rightarrow [X/T] \times V$ to the generic point of V . After excluding points of U lying in the image of the pullback of the relative inertia, p_V becomes

representable. The disjoint union of fibers of p_V over finitely many closed points of V forms an étale, representable, surjective morphism to $[X/T]$. Applying [GS11b] in the reverse direction, one deduces that X is a toric algebraic space and hence a scheme. $\square$

Theorem 3.8. *Assume $\text{char } k = 0$, and let X be a Deligne-Mumford stack, smooth and proper over k , with an action of T which gives a dense, open embedding of T in X . Then the induced action of any one-dimensional subtorus of T induces a Białynicki-Birula decomposition of X .*

Proof. By Remark 3.3, the action of T on X descends to the coarse moduli space. The theorem now follows from the previous two results. $\square$

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REFERENCES

- [Art73] M. Artin, *Morphismes acycliques*, Exp. XV, Théorie des topes et cohomologie étale des schémas. SGA 4, Tome 3, Lecture Notes in Math., Vol. 305, Springer-Verlag, Berlin, 1973. MR0354654 (50:7132)
- [BB73] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497. MR0366940 (51:3186)
- [Bro05] P. Brosnan, *On motivic decompositions arising from the method of Białynicki-Birula*, Invent. Math. **161** (2005), no. 1, 91–111. MR2178658 (2006h:14026)
- [EHKV01] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. MR1844577 (2002f:14002)
- [GS11a] A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans*, arXiv:1107.1906, 2011.
- [GS11b] A. Geraschenko and M. Satriano, *Toric stacks II: Intrinsic characterization of toric stacks*, arXiv:1107.1907, 2011.
- [Hau00] J. Hausen, *Algebraicity criteria for almost homogeneous complex spaces*, Arch. Math. (Basel) **74** (2000), no. 4, 317–320. MR1742909 (2001b:32039)
- [KM97] S. Keel and S. Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213. MR1432041 (97m:14014)
- [Ols06] M. C. Olsson, *Hom-stacks and restriction of scalars*, Duke Math. J. **134** (2006), no. 1, 139–164. MR2239345 (2007f:14002)
- [Ols07] M. Olsson, *A boundedness theorem for Hom-stacks*, Math. Res. Lett. **14** (2007), no. 6, 1009–1021. MR2357471 (2009e:14003)
- [Opr06] D. Oprea, *Tautological classes on the moduli spaces of stable maps to $\mathbb{P}^r$ via torus actions*, Adv. Math. **207** (2006), no. 2, 661–690. MR2271022 (2009d:14008)
- [Ray71] M. Raynaud, *Propriété cohomologique des faisceaux d’ensembles et des faisceaux de groupes non commutatifs*, Exp. XIII, Revêtements étales et groupe fondamental. SGA 1, Lecture Notes in Math., 224, Springer-Verlag, Berlin, 1971. MR0354651 (50:7129)
- [Rom05] M. Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236. MR2125542 (2005m:14005)
- [Sum74] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28. MR0337963 (49:2732)

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