TRACES OF HECKE OPERATORS IN LEVEL 1 AND GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We provide formulas for traces of p^{th} Hecke operators in level 1 in terms of values of finite field ${}_2F_1$ -hypergeometric functions, extending previous work of the author to all odd primes p instead of only those $p \equiv 1 \pmod{12}$. We first give a general level 1 trace formula in terms of the trace of Frobenius on a family of elliptic curves, and then we draw on recent work of Lennon to produce level 1 trace formulas in terms of hypergeometric functions for all primes $p > 3$.

1. INTRODUCTION

In recent years, relationships between traces of Hecke operators and counting points on families of varieties have been explored. For example, Ahlgren and Ono [2] described traces of p^{th} Hecke operators in weight 4 and level 8 in terms of the number of $\mathbb{F}_p$ -points on a Calabi-Yau threefold, while in [1], Ahlgren related traces of Hecke operators in weight 6 and level 4 to counting $\mathbb{F}_p$ -points on the Legendre family of elliptic curves. The level 2 formula (for all weights) was made explicit in terms of the number of $\mathbb{F}_p$ -points on a family of elliptic curves by Frechette, Ono, and Papanikolas in [5]. In [7], the author considered the level 1 case and provided a formula in terms of the number of $\mathbb{F}_p$ -points on a one-parameter family of elliptic curves for primes $p \equiv 1 \pmod{12}$. Most recently, Lennon [12] considered the levels 3 and 9 scenarios. Earlier work of Ihara [10] and Birch [4] gave reason to believe such formulas were possible.

Interestingly, these trace formulas also have a link to finite field hypergeometric functions introduced by Greene in the 1980s [8]. Various authors [3, 5, 12] have used relations between values of Greene's hypergeometric functions and counting $\mathbb{F}_p$ -points on varieties to produce trace formulas in terms of hypergeometric functions. In [7, Thm. 1.2], the author proved an explicit relationship between counting $\mathbb{F}_p$ -points on a one-parameter family of elliptic curves and the values of a particular ${}_2F_1$ function over $\mathbb{F}_p$, which led to a level 1 trace formula in terms of hypergeometric functions. However, this formula was only proved for primes $p \equiv 1 \pmod{12}$. Recently, Lennon [11, Thms. 1.1 and 2.1] has removed this restriction on the congruence class of p to produce formulas that relate $\#E(\mathbb{F}_q)$ to values of a ${}_2F_1$ function over $\mathbb{F}_q$ for any $q = p^e$ where $q \equiv 1 \pmod{12}$.

In this paper, we provide a level 1 trace formula that holds for all $p > 3$. Then, we use Lennon's result to produce formulas for traces of Hecke operators in level 1 in terms of finite field hypergeometric functions.

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2. STATEMENT OF MAIN RESULTS

Let $p > 3$ be prime and let $k \geq 2$ be an even integer. Define $F_k(x, y) = \frac{x^{k-1} - y^{k-1}}{x - y}$. Then letting $x + y = s$ and $xy = p$ gives rise to polynomials $G_k(s, p) = F_k(x, y)$. These polynomials can be written alternatively as

$$(1) \quad G_k(s, p) = \sum_{j=0}^{\frac{k}{2}-1} (-1)^j \binom{k-2-j}{j} p^j s^{k-2j-2}.$$

Throughout, results will depend on the congruence class of $p \pmod{12}$. As such, we set up some notation for various congruence classes of p to be used throughout the remainder of the paper. Whenever $p \equiv 1 \pmod{4}$, we let $a, b \in \mathbb{Z}$ be such that $p = a^2 + b^2$ and $a + bi \equiv 1 \pmod{2+2i}$ in $\mathbb{Z}[i]$. In that case, we define

$$(2) \quad \mu_k(p) = \frac{1}{2}[G_k(2a, p) + G_k(2b, p)].$$

Similarly, whenever $p \equiv 1 \pmod{3}$, we let $c, d \in \mathbb{Z}$ be such that $p = c^2 - cd + d^2$ and $c + d\omega \equiv 2 \pmod{3}$ in $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$. In this case, we define

$$(3) \quad \nu_k(p) = \frac{1}{3}[G_k(c + d, p) + G_k(2c - d, p) + G_k(c - 2d, p)].$$

We consider a one-parameter family of elliptic curves having j -invariant $\frac{1728}{t}$. Specifically, for $t \in \mathbb{F}_p \setminus \{0, 1\}$, we let

$$(4) \quad E_t : y^2 = 4x^3 - \frac{27}{1-t}x - \frac{27}{1-t}.$$

Let $a(t, p)$ denote the trace of the Frobenius endomorphism on E_t . In particular, for $t \neq 0, 1$, we have

$$a(t, p) = p + 1 - \#E_t(\mathbb{F}_p).$$

Let $\Gamma = SL_2(\mathbb{Z})$ and let M_k and S_k , respectively, denote the spaces of modular forms and cusp forms of weight k for Γ . Further, let $\text{Tr}_k(\Gamma, p)$ denote the trace of the Hecke operator $T_k(p)$ on S_k . Our first main result completely classifies the traces of cusp forms in level 1:

Theorem 2.1. *Let $p > 3$ be prime. Then for even $k \geq 4$,*

$$\text{Tr}_k(\Gamma, p) = -1 - \lambda(k, p) - \sum_{t=2}^{p-1} G_k(a(t, p), p),$$

where

$$\lambda(k, p) = \begin{cases} \mu_k(p) + \nu_k(p) & \text{if } p \equiv 1 \pmod{12}, \\ \mu_k(p) + (-p)^{\frac{k}{2}-1} & \text{if } p \equiv 5 \pmod{12}, \\ \nu_k(p) + (-p)^{\frac{k}{2}-1} & \text{if } p \equiv 7 \pmod{12}, \\ 2(-p)^{\frac{k}{2}-1} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Next, we move to results which link these traces of Hecke operators in level 1 with hypergeometric functions over finite fields. We begin with some preliminaries. Let p be a prime and let $q = p^e$. Let $\widehat{\mathbb{F}_q^\times}$ denote the group of all multiplicative characters on $\mathbb{F}_q^\times$. We extend $\chi \in \widehat{\mathbb{F}_q^\times}$ to all of $\mathbb{F}_q$ by setting $\chi(0) = 0$. We let ε

denote the trivial character. For $A, B \in \widehat{\mathbb{F}_p^\times}$, let $J(A, B)$ denote the usual Jacobi symbol and define

$$(5) \quad \binom{A}{B} := \frac{B(-1)}{q} J(A, \overline{B}) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x) \overline{B}(1-x).$$

Greene defined *hypergeometric functions over $\mathbb{F}_q$* in the following way:

Definition 2.2 ([8], Defn. 3.10). If n is a positive integer, $x \in \mathbb{F}_q$, and $A_0, A_1, \dots, A_n, B_1, B_2, \dots, B_n \in \widehat{\mathbb{F}_q^\times}$, then define

$${}_{n+1}F_n \left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ B_1, & B_2, & \dots, & B_n \end{matrix} \middle| x \right)_q := \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A_0 \chi}{\chi} \binom{A_1 \chi}{B_1 \chi} \dots \binom{A_n \chi}{B_n \chi} \chi(x).$$

In [7, Thm. 1.2], the author proved a formula giving an explicit relationship between $a(t, p)$ and a ${}_2F_1$ -hypergeometric function over $\mathbb{F}_p$, but required that $p \equiv 1 \pmod{12}$. In this case, the result of Theorem 2.1 can be rewritten to be in terms of a hypergeometric function over $\mathbb{F}_p$. However, [7] did not address the other classes of primes mod 12. Notice that either $p \equiv 1 \pmod{12}$ or, if not, then $p^2 \equiv 1 \pmod{12}$. With this in mind, for the remainder of the paper we define $q = p^{e(p)}$, where

$$(6) \quad e(p) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12}, \\ 2 & \text{if } p \not\equiv 1 \pmod{12}. \end{cases}$$

We consider the same family of elliptic curves E_t , as defined in (4), but now over $\mathbb{F}_q$, and with $a(t, q) = q + 1 - \#E_t(\mathbb{F}_q)$. Thanks to Lennon's results [11] and an inverse pair given in [13], we can now describe the traces of Hecke operators in level 1 in terms of a ${}_2F_1$ function over $\mathbb{F}_q$ for the other classes of primes mod 12:

Theorem 2.3. Let $p \geq 3$ be prime such that $p = 5, 7, 11 \pmod{12}$ with $q = p^{e(p)}$ and T a generator of $\widehat{\mathbb{F}_q^\times}$. Let $k \geq 4$ be even and $m = \frac{k}{2} - 1$. Define $H_m(x) := \sum_{i=0}^m \binom{m+i}{m-i} x^i$. Then

$$\mathrm{Tr}_k(\Gamma, p) = -1 - \lambda(k, p) - \sum_{t=2}^{p-1} (-p)^m H_m \left(pT^{\frac{q-1}{2}} (2)T^{\frac{q-1}{4}} (1-t) {}_2F_1 \left(T^{\frac{q-1}{12}}, \begin{matrix} T^{\frac{5(q-1)}{12}} \\ \varepsilon \end{matrix} \middle| t \right)_q - 2 \right),$$

where $\lambda(k, p)$ is as in Theorem 2.1.

Our final result is a generalization of [7, Thm. 1.4], giving a recursive formula for traces of Hecke operators in level 1 in terms of hypergeometric functions, now for all primes $p > 3$:

Theorem 2.4. Let $p > 3$ be prime, and $q = p^{e(p)}$. Let $k \geq 4$ be even, and $m = \frac{k}{2} - 1$. Further, let T be a generator of $\widehat{\mathbb{F}_q^\times}$ and $b_i = p^{m-i} \left[\binom{2m}{m-i} - \binom{2m}{m-i-1} \right]$.

Then

$$\begin{aligned} \mathrm{Tr}_{2(m+1)}(\Gamma, p) = & -1 - \lambda(2m+2, p) + b_0(p-2) \\ & - \sum_{i=1}^{m-1} b_i \cdot (\mathrm{Tr}_{2i+2}(\Gamma, p) + 1 + \lambda(2i+2, p)) \\ & - \sum_{t=2}^{p-1} \left(\psi(t, q)_2 F_1 \left(T^{\frac{q-1}{12}}, \begin{matrix} T^{\frac{5(q-1)}{12}} \\ \varepsilon \end{matrix} \middle| t \right)_q + 2p(e(p) - 1) \right)^{\frac{2m}{e(p)}}, \end{aligned}$$

where

$$\psi(t, q) = -qT^{\frac{q-1}{2}}(2)T^{\frac{q-1}{4}}(1-t)$$

and $\lambda(k, p)$ is as in Theorem 2.1.

3. PROOF OF THEOREM 2.1

To prove Theorem 2.1, we begin with Hijikata's version of the Eichler-Selberg trace formula [9]. The statement of this theorem requires some notation. If $d < 0$, $d \equiv 0, 1 \pmod{4}$, let $\mathcal{O}(d)$ denote the unique imaginary quadratic order in $\mathbb{Q}(\sqrt{d})$ having discriminant d . Let $h(d) = h(\mathcal{O}(d))$ be the order of the class group of $\mathcal{O}(d)$, and let $w(d) = w(\mathcal{O}(d))$ be half the cardinality of the unit group of $\mathcal{O}(d)$. We then let $h^*(d) = h(d)/w(d)$. The following theorem is the level 1 formulation of Hijikata's version of the Eichler-Selberg trace formula for any odd prime.

Theorem 3.1. *Let p be an odd prime and $k \geq 2$ be even. Then*

$$\mathrm{Tr}_k(\Gamma, p) = -1 - \frac{1}{2}\beta(p)(-p)^{\frac{k}{2}-1} - \sum_{0 < s < 2\sqrt{p}} G_k(s, p) \sum_{f|\ell} h^* \left(\frac{s^2 - 4p}{f^2} \right) + \delta(k),$$

where

$$\beta(p) = \begin{cases} h^*(-4p) & \text{if } p \equiv 1 \pmod{4}, \\ h^*(-4p) + h^*(-p) & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$\delta(k) = p + 1$ if $k = 2$ and 0 otherwise, and where we classify integers s with $s^2 - 4p < 0$ by some positive integer ℓ and square-free integer m via

$$s^2 - 4p = \begin{cases} \ell^2 m, & 0 > m \equiv 1 \pmod{4}, \\ \ell^2 4m, & 0 > m \equiv 2, 3 \pmod{4}. \end{cases}$$

To link Theorem 3.1 to $a(t, p)$, we need to consider all isomorphism classes of elliptic curves over $\mathbb{F}_p$. If E is any elliptic curve defined over $\mathbb{F}_p$, let $a(E) = p + 1 - \#E(\mathbb{F}_p)$. Additionally, for a perfect field K , we define

$$\mathrm{Ell}_K := \{[E]_K : E \text{ is defined over } K\},$$

where $[E]_K$ denotes the isomorphism class of E over K and $[E_1]_K = [E_2]_K$ if there exists an isomorphism $\beta : E_1 \rightarrow E_2$ over K . We first address the cases $j(E) = 1728$ and $j(E) = 0$.

Lemma 3.2. *Let p be an odd prime. Whenever $p \equiv 1 \pmod{4}$, define $a, b \in \mathbb{Z}$ to be such that $p = a^2 + b^2$ and $a + bi \equiv 1(2 + 2i)$ in $\mathbb{Z}[i]$. Then, for $n \geq 2$ even,*

$$\sum_{\substack{[E]_{\mathbb{F}_p} \in \mathrm{Ell}_{\mathbb{F}_p} \\ j(E)=1728}} a(E)^n = \begin{cases} 2^{n+1}(a^n + b^n) & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. The case $p \equiv 1 \pmod{4}$ was proved by the author in [6, Lemma IV.3.3]. If $p \equiv 3 \pmod{4}$ and $[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p}$, then $\#E(\mathbb{F}_p) = p + 1$, so $a(E) = 0$. $\square$

Lemma 3.3. *Let $p > 3$ be prime. Whenever $p \equiv 1 \pmod{3}$, we let $c, d \in \mathbb{Z}$ be such that $p = c^2 - cd + d^2$ and $c + d\omega \equiv 2 \pmod{3}$ in $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$. Then, for $n \geq 2$ even,*

$$\sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=0}} a(E)^n = \begin{cases} 2[(c+d)^n + (2c-d)^n + (c-2d)^n] & \text{if } p \equiv 1 \pmod{3}, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. The case $p \equiv 1 \pmod{3}$ was proved by the author in [6, Lemma IV.3.5]. If $p \equiv 2 \pmod{3}$ and $[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p}$, then $\#E(\mathbb{F}_p) = p + 1$, so $a(E) = 0$. $\square$

The proof of Theorem 2.1 proceeds along the same line as the proof of the $p \equiv 1 \pmod{12}$ case proved by the author in [7]. In particular, we begin with the following extension of [7, Lemma 5.3].

Lemma 3.4. *Let $p > 3$ be prime. Then for $n \geq 2$ even,*

$$\begin{aligned} \sum_{0 < s < 2\sqrt{p}} s^n \sum_{f|\ell} h\left(\frac{s^2 - 4p}{f^2}\right) &= \sum_{0 < s < 2\sqrt{p}} s^n \sum_{f|\ell} h^*\left(\frac{s^2 - 4p}{f^2}\right) \\ &\quad + \frac{1}{4} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=1728}} a(E)^n + \frac{1}{3} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=0}} a(E)^n. \end{aligned}$$

Proof. The proof for primes $p \equiv 1 \pmod{12}$ is provided in [7, Lemma 5.3]. It can be adapted to hold for all $p > 3$ once one verifies that the following two identities remain true:

$$(7) \quad \sum_{0 < s < 2\sqrt{p}} s^n \sum_{\substack{f|\ell \\ \frac{s^2 - 4p}{f^2} = -4}} 1 = \frac{1}{2} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=1728}} a(E)^n,$$

$$(8) \quad \sum_{0 < s < 2\sqrt{p}} s^n \sum_{\substack{f|\ell \\ \frac{s^2 - 4p}{f^2} = -3}} 1 = \frac{1}{2} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=0}} a(E)^n.$$

First consider (7). The proof given in [7] holds for all primes $p \equiv 1 \pmod{4}$. In light of Lemma 3.2, we must verify that if $p \equiv 3 \pmod{4}$, then

$$\sum_{0 < s < 2\sqrt{p}} s^n \sum_{\substack{f|\ell \\ \frac{s^2 - 4p}{f^2} = -4}} 1 = 0.$$

We verify this by proving that no s, f exist to contribute to the sums. For, suppose $s, f \in \mathbb{Z}$ such that $0 < s < 2\sqrt{p}$ and $\frac{s^2 - 4p}{f^2} = -4$. Then $4|s^2$, so s must be even. Substituting $s = 2r$ and rearranging gives $r^2 + f^2 = p$, which is not possible since $p \equiv 3 \pmod{4}$. This verifies (7) for the remaining primes.

We handle (8) in a similar way. The proof in [7] verifies the equation for $p \equiv 1 \pmod{3}$. Keeping in mind Lemma 3.3, we must prove that if $p \equiv 2 \pmod{3}$, then

$$\sum_{0 < s < 2\sqrt{p}} s^n \sum_{\substack{f|\ell \\ \frac{s^2-4p}{f^2} = -3}} 1 = 0.$$

Suppose then that we have $s, f \in \mathbb{Z}$ such that $0 < s < 2\sqrt{p}$ and $\frac{s^2-4p}{f^2} = -3$. Then $4p = 3f^2 + s^2$ and hence $p \equiv 2 \equiv s^2 \pmod{3}$. However, since $\left(\frac{2}{3}\right) = -1$, this is impossible. This verifies (8) for the remaining primes and completes the proof of the lemma. $\square$

The following proposition generalizes [7, Prop. 5.4] by removing the restriction on the congruence class of $p \pmod{12}$.

Proposition 3.5. *Let $p > 3$ be prime and $n \geq 2$ be even. Then*

$$\sum_{t=2}^{p-1} a(t, p)^n = \sum_{0 < s < 2\sqrt{p}} s^n \sum_{f|\ell} h^* \left(\frac{s^2-4p}{f^2} \right) - \alpha(n, p) - \gamma(n, p),$$

where

$$\alpha(n, p) = \begin{cases} 2^{n-1}(a^n + b^n) & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

$$\gamma(n, p) = \begin{cases} \frac{1}{3}[(c+d)^n + (2c-d)^n + (c-2d)^n] & \text{if } p \equiv 1 \pmod{3}, \\ 0 & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

and a, b, c, d are defined as in the statements of Lemmas 3.2 and 3.3.

Proof. The bulk of the proof the author provides for the $p \equiv 1 \pmod{12}$ case in [7, Prop. 5.4] holds for the other congruence classes of p , so we give an outline here. Since $j(E_t) = \frac{1728}{t}$, we have that

$$\begin{aligned} \sum_{t=2}^{p-1} a(t, p)^n &= \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p}; E/\mathbb{F}_p \\ j(E) \neq 0, 1728}} a(E)^n = \frac{1}{2} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E) \neq 0, 1728}} a(E)^n \\ &= \frac{1}{2} \left[\sum_{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p}} a(E)^n - \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=1728}} a(E)^n - \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=0}} a(E)^n \right]. \end{aligned}$$

Regardless of the congruence class of $p \pmod{12}$, the first sum in the last line above can still be written in terms of class numbers by combining Hasse's theorem with a theorem of Schoof [14, Thm. 4.6]. This results in

$$\begin{aligned} \sum_{t=2}^{p-1} a(t, p)^n &= \sum_{0 < s < 2\sqrt{p}} s^n \sum_{f|\ell} h \left(\frac{s^2-4p}{f^2} \right) - \frac{1}{2} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=1728}} a(E)^n - \frac{1}{2} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=0}} a(E)^n \\ &= \sum_{0 < s < 2\sqrt{p}} s^n \sum_{f|\ell} h^* \left(\frac{s^2-4p}{f^2} \right) - \frac{1}{4} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=1728}} a(E)^n - \frac{1}{6} \sum_{\substack{[E]_{\mathbb{F}_p} \in Ell_{\mathbb{F}_p} \\ j(E)=0}} a(E)^n \end{aligned}$$

by Lemma 3.4. One now applies Lemmas 3.2 and 3.3 in each appropriate congruence class to obtain the result. $\square$

With these tools in place, we now complete the proof of Theorem 2.1.

Proof of Theorem 2.1. The proof proceeds in a similar fashion to the author's proof of the $p \equiv 1 \pmod{12}$ case in [7, Thm. 1.3], with a few modifications. We still begin with an application of Theorem 3.1 and then substitute the definition of $G_k(s, p)$. This gives

$$(9) \quad \text{Tr}_k(\Gamma, p) = -1 - \frac{1}{2}\beta(p)(-p)^{\frac{k}{2}-1} - (-p)^{\frac{k}{2}-1} \sum_{0 < s < 2\sqrt{p}} 1 \sum_{f|\ell} h^* \left(\frac{s^2 - 4p}{f^2} \right) \\ - \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \sum_{0 < s < 2\sqrt{p}} s^{k-2j-2} \sum_{f|\ell} h^* \left(\frac{s^2 - 4p}{f^2} \right).$$

Now, notice that the $k = 2$ case of Theorem 3.1 gives

$$0 = p - \frac{1}{2}\beta(p) - \sum_{0 < s < 2\sqrt{p}} 1 \sum_{f|\ell} h^* \left(\frac{s^2 - 4p}{f^2} \right).$$

Substituting and applying Proposition 3.5 with $n = k - 2j - 2$ gives

$$(10) \quad \text{Tr}_k(\Gamma, p) = -1 + (-p)^{\frac{k}{2}-1} \cdot (-p) - \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \sum_{t=2}^{p-1} a(t, p)^{k-2j-2} \\ - \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \alpha(k-2j-2, p) \\ - \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \gamma(k-2j-2, p).$$

To complete the proof, we distribute the copies of $(-p)^{\frac{k}{2}-1}$ to the three summations in a specific way. Notice that

$$(-p)^{\frac{k}{2}-1}(-p) = -(-p)^{\frac{k}{2}-1}(p-2) - (-p)^{\frac{k}{2}-1} - (-p)^{\frac{k}{2}-1}.$$

First, since $G_2 = 1$, we see that

$$(11) \quad -(-p)^{\frac{k}{2}-1}(p-2) - \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \sum_{t=2}^{p-1} a(t, p)^{k-2j-2} = - \sum_{t=2}^{p-1} G_k(a(t, p), p).$$

A straightforward calculation for each of the congruence classes of $p \pmod{12}$ verifies

$$\lambda(k, p) = (-p)^{\frac{k}{2}-1} + \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \alpha(k-2j-2, p) \\ + (-p)^{\frac{k}{2}-1} + \sum_{j=0}^{\frac{k}{2}-2} (-1)^j \binom{k-2-j}{j} p^j \gamma(k-2j-2, p).$$

$\square$

4. TRACE FORMULAS IN TERMS OF HYPERGEOMETRIC FUNCTIONS

We now prove Theorems 2.3 and 2.4. As mentioned before, one essential tool is a theorem of Lennon, which writes the trace of Frobenius of any elliptic curve in Weierstrass form in terms of a finite field hypergeometric function:

Theorem 4.1 ([11, Thm. 2.1]). *Let $q = p^e$, where $p > 3$ is prime and $q \equiv 1 \pmod{12}$. Let $E : y^2 = x^3 + ax + b$ be an elliptic curve over $\mathbb{F}_q$ in Weierstrass form with $j(E) \neq 0, 1728$. Then the trace of the Frobenius map on E can be expressed as*

$$a(E(\mathbb{F}_q)) = -q \cdot T^{\frac{q-1}{4}} \left(\frac{a^3}{27} \right) \cdot {}_2F_1 \left(T^{\frac{q-1}{12}}, \quad T^{\frac{5(q-1)}{12}} \middle| -\frac{27b^2}{4a^3} \right)_q.$$

We now specify this theorem to our family of curves.

Corollary 4.2. *Let $p > 3$ be prime and $q = p^{e(p)}$, where $e(p)$ is defined as in (6). Then*

$$a(t, q) = -qT^{\frac{q-1}{2}}(2)T^{\frac{q-1}{4}}(1-t){}_2F_1 \left(T^{\frac{q-1}{12}}, \quad T^{\frac{5(q-1)}{12}} \middle| t \right)_\varepsilon.$$

To prove this corollary, we require a transformation law proved by Greene:

Theorem 4.3 ([8, Thm 4.4(i)]). *If $A, B, C \in \widehat{\mathbb{F}_q^\times}$ and $x \in \mathbb{F}_q \setminus \{0, 1\}$, then*

$${}_2F_1 \left(A, \quad B \middle| x \right)_q = A(-1){}_2F_1 \left(A, \quad B \middle| ABC \right)_q (1-x).$$

Proof of Corollary 4.2. After putting E_t into Weierstrass form, we have $a = b = \frac{-27}{4(1-t)}$ in Theorem 4.1. Then $\frac{a^3}{27} = \frac{-3^6}{4^3(1-t)^3}$ and $-\frac{27b^2}{4a^3} = 1-t$. Combining these simplifications with Greene's theorem above gives

$$(12) \quad a(t, q) = -qT^{\frac{q-1}{4}} \left(\frac{-3^6}{4^3(1-t)^3} \right) T^{\frac{q-1}{12}} (-1){}_2F_1 \left(T^{\frac{q-1}{12}}, \quad T^{\frac{5(q-1)}{12}} \middle| t \right)_\varepsilon.$$

Now, using multiplicativity and the fact that T has order $q-1$, we have

$$\begin{aligned} T^{\frac{q-1}{4}} \left(\frac{-3^6}{4^3(1-t)^3} \right) &= T^{\frac{3(q-1)}{4}} \left(\frac{-9}{4(1-t)} \right) = T^{\frac{q-1}{4}} \left(\frac{4(1-t)}{-9} \right) \\ &= T^{\frac{q-1}{2}}(2)T^{\frac{q-1}{4}}(1-t)T^{\frac{q-1}{4}}(-1)T^{\frac{q-1}{2}}(3) \\ &= T^{\frac{q-1}{2}}(2)T^{\frac{q-1}{4}}(1-t)T^{\frac{q-1}{4}}(-1), \end{aligned}$$

since $T^{\frac{q-1}{2}}$ is its own inverse and $q \equiv 1 \pmod{12}$. The proof is completed by making this substitution for $T^{\frac{q-1}{4}} \left(\frac{-3^6}{4^3(1-t)^3} \right)$ into (12) and noting that $T^{\frac{q-1}{4}}(-1)T^{\frac{q-1}{12}}(-1) = T^{\frac{q-1}{3}}(-1) = 1$, since $-1 = (-1)^3$ and $T^{\frac{q-1}{3}}$ has order 3. $\square$

Remark 4.4. If $e(p) = 1$ (i.e. $q = p$), the above corollary precisely matches the author's result [7, Thm. 1.3].

Remark 4.5. Lennon gives another way of writing the trace of Frobenius in terms of a ${}_2F_1$ function in [11, Thm. 1.1], using the j -invariant and discriminant of E . We use Lennon's Theorem 2.1 because it leads to a simpler hypergeometric function in this instance.

We require two more tools to prove our trace theorems. First, note that whenever $e(p) = 2$ (i.e. $q = p^2$), Theorem 4.1 relates $a(t, p^2)$ to a hypergeometric function over $\mathbb{F}_{p^2}$. Even though our trace formula, Theorem 2.1, is in terms of $a(t, p)$, we can still gain new information, since

$$(13) \quad a(t, p)^2 = a(t, p^2) + 2p.$$

The last tool is an inverse pair given in [13]. As in the statement of Theorems 2.3 and 2.4, we let $m = \frac{k}{2} - 1$ and also define $H_m(x) := \sum_{i=0}^m \binom{m+i}{m-i} x^i$. Then, as in [7], notice that

$$(14) \quad G_k(s, p) = (-p)^m H_m\left(\frac{-s^2}{p}\right).$$

Consider the inverse pair [13, p. 67] given by

$$(15) \quad \rho_n(x) = \sum_{k=0}^n \binom{n+k}{n-k} x^k, \quad x^n = \sum_{k=0}^n (-1)^{k+n} \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] \rho_k(x).$$

Applying this to the definition of H_m , we see that

$$x^m = \sum_{i=0}^m (-1)^{i+m} \left[\binom{2m}{m-i} - \binom{2m}{m-i-1} \right] H_i(x).$$

By combining (14) with the choice $x = \frac{-s^2}{p}$, we have

$$(16) \quad s^{2m} = \sum_{i=0}^m b_i G_{2i+2}(s, p),$$

where $b_i = p^{m-i} \left[\binom{2m}{m-i} - \binom{2m}{m-i-1} \right]$. We may now prove Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Recall that in the statement of this theorem, $p \equiv 5, 7, 11 \pmod{12}$, so $q = p^2$. By (14) and (13), we have

$$\begin{aligned} G_k(a(t, p), p) &= (-p)^m H_m\left(\frac{-a(t, p)^2}{p}\right) \\ &= (-p)^m H_m\left(\frac{-a(t, p^2)}{p} - 2\right) \\ &= (-p)^m H_m\left(pT^{\frac{q-1}{2}}(2)T^{\frac{q-1}{4}}(1-t)_2F_1\left(T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \middle| t\right)_q - 2\right), \end{aligned}$$

by Corollary 4.2. Combining this with Theorem 2.1 completes the proof. $\square$

Proof of Theorem 2.4. Recall that $p > 3$ is any prime and $q = p^{e(p)}$, where $e(p)$ is defined as in (6). The proof begins along the same lines as the proof of [7, Thm. 1.4]. Beginning with Theorem 2.1, we have

$$(17) \quad \text{Tr}_{2(m+1)}(\Gamma, p) = -1 - \lambda(2m+2, p) - \sum_{t=2}^{p-1} G_{2m+2}(a(t, p), p).$$

Now, (16) implies that

$$s^{2m} = \sum_{i=0}^m b_i G_{2i+2}(s, p) = G_{2m+2}(s, p) + \sum_{i=0}^{m-1} b_i G_{2i+2}(s, p).$$

We isolate $G_{2m+2}(s, p)$ and take $s = a(t, p)$. Substituting into (17) gives

$$\begin{aligned} \mathrm{Tr}_{2(m+1)}(\Gamma, p) &= -1 - \lambda(2m+2, p) - \sum_{t=2}^{p-1} \left(a(t, p)^{2m} - \sum_{i=0}^{m-1} b_i G_{2i+2}(a(t, p), p) \right) \\ &= -1 - \lambda(2m+2, p) - \sum_{t=2}^{p-1} a(t, p)^{2m} + b_0(p-2) \\ &\quad + \sum_{i=1}^{m-1} b_i \sum_{t=2}^{p-1} G_{2i+2}(a(t, p), p) \\ &= -1 - \lambda(2m+2, p) - \sum_{t=2}^{p-1} a(t, p)^{2m} + b_0(p-2) \\ &\quad - \sum_{i=1}^{m-1} b_i (\mathrm{Tr}_{2i+2}(\Gamma, p) + 1 + \lambda(2i+2, p)), \end{aligned}$$

since $G_2 = 1$ and by again applying Theorem 2.1 for the last equality. To complete the proof, we consider $a(t, p)^{2m}$. If $e(p) = 1$ (i.e. $q = p$), then using either Corollary 4.2 or [7, Thm. 1.2], we have

$$a(t, p)^{2m} = \left(-q T^{\frac{q-1}{2}}(2) T^{\frac{q-1}{4}} (1-t)_2 F_1 \left(T^{\frac{q-1}{12}}, \left. T^{\frac{5(q-1)}{12}} \right|_t \right) \right)_q^{2m},$$

so our result matches [7, Thm. 1.4] in this case.

If $e(p) = 2$ (i.e. $q = p^2$), then (13) gives

$$\begin{aligned} a(t, p)^{2m} &= (a(t, p)^2)^m = (a(t, p^2) + 2p)^m \\ &= \left(-q T^{\frac{q-1}{2}}(2) T^{\frac{q-1}{4}} (1-t)_2 F_1 \left(T^{\frac{q-1}{12}}, \left. T^{\frac{5(q-1)}{12}} \right|_t \right) + 2p \right)_q^m, \end{aligned}$$

by Corollary 4.2.

The final statement of the theorem combines these two cases together, completing our proof. $\square$

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