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A MARCINKIEWICZ MAXIMAL-MULTIPLIER THEOREM

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Abstract. For $r < 2$, we prove the boundedness of a maximal operator formed by applying all multipliers m with $\|m\|_{V^r} \leq 1$ to a given function.

1. Introduction

Given an exponent r and a function f defined on $\mathbb{R}$, consider the r-variation norm

$$\|f\|_{V^r} = \|f\|_{L^\infty} + \sup_{N,\xi_0<\cdots<\xi_N} \left(\sum_{i=1}^{N} |f(\xi_i) - f(\xi_{i-1})|^r \right)^{1/r},$$

where the supremum is over all strictly increasing finite length sequences of real numbers.

The classical Marcinkiewicz multiplier theorem states that if $r = 1$ and a function m is of bounded r-variation uniformly on dyadic shells, then m is an L^p multiplier for $1 < p < \infty$ and

$$\|(m\hat{f})\check{}\|_{L^p} \leq C_{p,r} \sup_{k\in\mathbb{Z}} \|1_{D_k} m\|_{V^r} \|f\|_{L^p}, \tag{1.1}$$

where $D_k = [-2^{k+1}, -2^k) \cup (2^k, 2^{k+1}]$ and $\hat{}, \check{}$ denote the Fourier-transform and its inverse. Later, Coifman, Rubio de Francia, and Semmes [2] (see also [8]) showed that the requirement of bounded 1-variation can be relaxed to allow for functions of bounded 2-variation, and in fact (1.1) holds whenever $r \geq 2$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{r}$.

The estimate [2] does not discriminate between multipliers of bounded 2-variation and those of bounded r-variation where $r < 2$, and so one might ask whether there is anything to be gained by controlling the variation norm of multipliers in the latter range of exponents.

Defining the maximal-multiplier operator

$$\mathcal{M}_r[f](x) = \sup_{m:\|m\|_{V^r}\leq 1} |(m\hat{f})\check{}(x)|, \tag{1.2}$$

where the supremum is over all functions in the V^r unit ball, we have

Theorem 1.1. *Suppose* $1 \leq r < 2$ *and* $r < p < \infty$. *Then*

$$\|\mathcal{M}_r[f]\|_{L^p} \leq C_{p,r} \|f\|_{L^p}. \tag{1.3}$$

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The case $r = 1$ was observed independently by Lacey [4].

Note that in the definition of $\mathcal{M}_r$, each m is required to have finite r-variation on all of $\mathbb{R}$ rather than simply on each dyadic shell as in (1.1). This is necessary for boundedness, as can be seen from the counterexamples of Christ, Grafakos, Honzík and Seeger [1].

Although the maximal operator (1.2) would seem to be fairly strong, we do not yet know of an application for the bound above. We will, however, quickly illustrate a strategy for its use that falls an (important) ϵ short of success. Let Ψ be (say) a Schwartz function, and for each $\xi, x \in \mathbb{R}$ and $k \in \mathbb{Z}$, consider the 2^k-truncated partial Fourier integral

$$\mathcal{S}_k[f](\xi, x) = p.v. \int f(x-t) e^{2\pi i \xi t} \Psi(2^{-k} t) \frac{1}{t}\, dt.$$

It was proven by Demeter, Lacey, Tao, and Thiele [3] that for $q = 2$ and $1 < p < \infty$,

$$\| \sup_{\|g\|_{L^q}=1} \| \sup_k |(\mathcal{S}_k[f](\cdot, x)\hat{g})\check{\ }| \|_{L^q} \|_{L^p_x} \leq C_{p,q} \|f\|_{L^p}. \tag{1.4}$$

If we had the bound

$$\|\mathcal{S}_k[f](\xi, x)\|_{L^p_x(\ell^\infty_k(V^r_\xi))} \leq C_{p,r} \|f\|_{L^p} \tag{1.5}$$

for some $r < 2$, then an application of Theorem 1.1 would give (1.4) for $q > r$ by rather different means than [3]. In fact, one can see by applying the method in Appendix D of [6] that (1.5) holds for $r > 2$ and $p > r'$. Unfortunately, it does fail for $r \leq 2$.

2. Proof of Theorem 1.1

The following lemma was proven in [2]; see also [5].

Lemma 2.1. *Let m be a compactly supported function on $\mathbb{R}$ of bounded r-variation for some $1 \leq r < \infty$. Then for each integer $j \geq 0$, one can find a collection Υ_j of pairwise disjoint subintervals of $\mathbb{R}$ and coefficients $\{b_\upsilon\}_{\upsilon \in \Upsilon_j} \subset \mathbb{R}$ so that $|\Upsilon_j| \leq 2^j$, $|b_\upsilon| \leq 2^{-j/r} \|m\|_{V_r}$, and*

$$m = \sum_{j \geq 0} \sum_{\upsilon \in \Upsilon_j} b_\upsilon 1_\upsilon, \tag{2.1}$$

where the sum in j converges uniformly.

The lemma above was applied in concert with Rubio de Francia's square function estimate [7] to obtain (1.1). Here, we will argue similarly, exploiting the analogy between the Rubio de Francia square function estimate and the variation-norm Carleson theorem.

It was proven in [7] that for $p \geq 2$,

$$\sup_{\mathcal{I}} \| \big(\sum_{I \in \mathcal{I}} |(1_I \hat{f})\check{\ }|^2 \big)^{1/2} \|_{L^p} \leq C_p \|f\|_{L^p},$$

where the supremum above is over all collections of pairwise disjoint subintervals of $\mathbb{R}$. Consider the partial Fourier integral

$$\mathcal{S}[f](\xi, x) = (1_{(\infty, \xi]} \hat{f})\check{\ }(x).$$

It was proven in [6] that for $s > 2$ and $p > s'$,

$$\|\mathcal{S}[f](\xi, x)\|_{L^p_x(V^s_\xi)} \leq C_{p,s}\|f\|_{L^p}$$

or, equivalently,

$$\|\sup_{\mathcal{I}} \big(\sum_{I\in\mathcal{I}} |(1_I\hat{f})\check{}|^s\big)^{1/s}\|_{L^p} \leq C_{p,s}\|f\|_{L^p}. \tag{2.2}$$

Note that by standardizing limiting arguments, taking the supremum in (1.2) to be over all compactly supported m such that $\|m\|_{V^r} \leq 1$ does not change the definition of $\mathcal{M}_r$. Applying the decomposition (2.1), we see that for any compactly supported m with $\|m\|_{V^r} \leq 1$ we have

$$\begin{aligned}|(m\hat{f})\check{}(x)| &\leq \sum_{j\geq 0}\sum_{\upsilon\in\Upsilon_j} |b_\upsilon (1_\upsilon \hat{f})\check{}(x)| \\ &\leq \sum_{j\geq 0}\sup_{\upsilon\in\Upsilon_j} |b_\upsilon||\Upsilon_j|^{\frac{1}{s'}}\big(\sum_{\upsilon\in\Upsilon_j}|(1_\upsilon\hat{f})\check{}(x)|^s\big)^{1/s} \\ &\leq C_{r,s}\sup_{\mathcal{I}}\big(\sum_{I\in\mathcal{I}}|(1_I\hat{f})\check{}(x)|^s\big)^{1/s},\end{aligned}$$

where, for the last inequality, we require $s < r'$.

Provided that $r < 2$ and $p > r$ we can choose an $s < r'$ with $s > 2$ and $p > s'$, giving (2.2) and hence (1.3).

The argument of Lacey [4] for $r = 1$ follows a similar pattern, except with Marcinkiewicz's method in place of [2] and the standard Carleson-Hunt theorem in place of [6].

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