ON APPROXIMATELY CONVEX TAKAGI TYPE FUNCTIONS

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ABSTRACT. Given a nonnegative function $\phi : [0, \frac{1}{2}] \rightarrow \mathbb{R}_+$, we define the Takagi type function $S_\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$S_\phi(x) := \sum_{n=0}^{\infty} 2\phi\left(\frac{1}{2^{n+1}}\right)d_{\mathbb{Z}}(2^n x),$$

where $d_{\mathbb{Z}}(x) := \text{dist}(x, \mathbb{Z}) := \inf\{|x - k| : k \in \mathbb{Z}\}$. The main result of the paper states that if $\phi(0) = 0$ and the mapping $x \mapsto \phi(x)/x$ is concave, then the Takagi type function S_ϕ is approximately Jensen convex in the following sense:

$$S_\phi\left(\frac{x+y}{2}\right) \leq \frac{S_\phi(x) + S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\left(\frac{x-y}{2}\right) \quad (x, y \in \mathbb{R}).$$

Applications to the theory of approximately convex functions are also given.

1. INTRODUCTION

Throughout this paper denote by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, and $\mathbb{N}$ the sets of real numbers, nonnegative real numbers, integers, and positive integers, respectively, and denote by I a nonempty subinterval of $\mathbb{R}$.

Define, for every $x \in \mathbb{R}$,

$$T(x) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n x)}{2^n},$$

where

$$d_{\mathbb{Z}}(x) := \text{dist}(x, \mathbb{Z}) := \inf\{|x - k| : k \in \mathbb{Z}\}.$$

The function T is 1-periodic, continuous and vanishes on $\mathbb{Z}$ and it is a well-known example of a continuous but nowhere differentiable real function, which is usually called “van der Waerden’s function” (cf. [26]). As was discovered by Knopp [12], this function had been constructed (in terms of the dyadic expansion of x) almost 30 years before by T. Takagi in 1903 [25]. For further historical details and remarks, we refer to the papers by Billingsley [2], Cater [5], and Kairies [11].

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The development of the theory of approximately convex functions began with the paper by Hyers and Ulam [6], who in the year 1952 introduced and investigated ε -convex functions. The Hyers and Ulam decomposition theorem of ε -convex functions was later generalized by Páles in [17]. Since then many papers on this subject have been published. An important issue in these papers is to obtain estimations of the bounds which appear for approximately convex functions; see, for example, Cannarsa and Sinestrari [4], Ng and Nikodem [16], Rolewicz [18, 19, 20]. The research and results in this paper are motivated by the fact that Takagi-like functions appear naturally in the investigation of approximate convexity. For details, we recommend the papers by Hájzy [7], Hájzy and Páles [8, 9, 10], Makó and Páles [13, 14], Mureńko, Tabor and Tabor [15], Tabor and Tabor [21, 22], Tabor, Tabor, and Żoldak [24, 23].

Let I be a nonempty real interval of $\mathbb{R}$ and denote $I^* := (I - I) \cap \mathbb{R}_+$. Let $\varphi : \frac{1}{2}I^* \rightarrow \mathbb{R}_+$ be a given function. A function $f : I \rightarrow \mathbb{R}$ is called φ -Jensen convex on I (cf. Makó and Páles [13], Tabor and Tabor [21, 22]) if, for all $x, y \in I$,

$$(1) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \varphi\left(\left|\frac{x-y}{2}\right|\right).$$

If $\varphi(t) = \varepsilon t^p$ ($t \in \mathbb{R}_+$), where ε, p are nonnegative constants, then φ -Jensen convex functions are called (ε, p) -Jensen convex functions (cf. [9]).

For a fixed error function $\varphi : \frac{1}{2}I^* \rightarrow \mathbb{R}_+$, we introduce two Takagi type functions $\mathcal{T}_\varphi : \mathbb{R} \times I^* \rightarrow \mathbb{R}_+$ and $\mathcal{S}_\varphi : \mathbb{R} \times I^* \rightarrow \mathbb{R}_+$ by

$$(2) \quad \begin{aligned} \mathcal{T}_\varphi(t, u) &:= \sum_{n=0}^{\infty} \frac{\varphi(d_{\mathbb{Z}}(2^n t)u)}{2^n}, \\ \mathcal{S}_\varphi(t, u) &:= \sum_{n=0}^{\infty} 2\varphi\left(\frac{u}{2^{n+1}}\right) d_{\mathbb{Z}}(2^n t), \end{aligned} \quad ((t, u) \in \mathbb{R} \times I^*).$$

Note that the first series converges uniformly if φ is bounded; on the other hand, for the uniform convergence of the second series, it is sufficient if $\sum_{n=n_0}^{\infty} \varphi(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$.

The importance of the function $\mathcal{T}_\varphi$ introduced above is enlightened by the following result (cf. [13], [21, 22]), which is a generalization of the celebrated Bernstein–Doetsch theorem [1].

Theorem A. *Let $f : I \rightarrow \mathbb{R}$ be locally bounded from above on I (i.e., f is bounded from above on any compact subset of I) and let $\varphi : \frac{1}{2}I^* \rightarrow \mathbb{R}_+$. Then f is φ -Jensen convex on I ; i.e., (1) holds for all $x, y \in I$ if and only if*

$$(3) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \mathcal{T}_\varphi(t, |x-y|)$$

for all $x, y \in I$ and $t \in [0, 1]$.

The other Takagi type function $\mathcal{S}_\varphi$ was introduced by Tabor and Tabor. Its role and importance in the theory of approximate convexity is shown by the next theorem ([21]).

FIGURE 1

Theorem B. Let $f : I \rightarrow \mathbb{R}$ be locally bounded from above on I (i.e., f is bounded from above on any compact subset of I) and let $\varphi : \frac{1}{2}I^* \rightarrow \mathbb{R}_+$ such that $\sum_{n=n_0}^{\infty} \varphi(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is φ -Jensen convex on I ; i.e., (1) holds for all $x, y \in I$ if and only if

$$(4) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \mathcal{S}_{\varphi}(t, |x-y|)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let $\varepsilon, p \geq 0$ be arbitrary constants. When $\varphi(t) = \varepsilon t^p$ ($t \in \mathbb{R}_+$), the two corollaries below (see [9] and [21]) are immediate consequences of the previous theorems.

For $p > 0$, define the Takagi type functions S_p and T_p by

$$(5) \quad T_p(x) := \sum_{n=0}^{\infty} \frac{(d_{\mathbb{Z}}(2^n x))^p}{2^n}, \quad S_p(x) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n x)}{2^{np+p-1}} \quad (x \in \mathbb{R}).$$

They generalize the classical Takagi function T in two ways, because $T_1 = S_1 = T$ holds obviously. It is less trivial, but it can be proved that $T_2(x) = S_2(x) = x(1-x)$ for $x \in [0, 1]$. Moreover, $T_p \leq S_p$ holds for $p \in [0, 1] \cup [2, \infty[$ and the inequality is reversed for $p \in [1, 2]$. The pictures in Figure 1 demonstrate the comparison between T_p and S_p for $p = 0.5$ and $p = 1.5$, respectively.

Corollary A. *Let $f : I \rightarrow \mathbb{R}$ be locally bounded from above on I and let $\epsilon, p \geq 0$. Then f is (ϵ, p) -Jensen convex on I if and only if*

$$(6) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon T_p(t)|x - y|^p$$

for all $x, y \in I$ and $t \in [0, 1]$.

Corollary B. *Let $f : I \rightarrow \mathbb{R}$ be locally bounded from above on I and let $\epsilon, p \geq 0$. Then f is (ϵ, p) -Jensen convex on I if and only if*

$$(7) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon S_p(t)|x - y|^p$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [3] Boros proved that if $p = 1$ and $t \in [0, 1]$ is fixed, then $S_1(t) = T_1(t) = T(t)$ is the smallest possible. In [22] Tabor and Tabor showed that if $1 \leq p \leq 2$ and $t \in [0, 1]$ is fixed, then $S_p(t)$ is the smallest possible value so that (7) is valid for all (ϵ, p) -Jensen convex functions f on I .

In [13] the authors proved the error terms $\mathcal{T}_\varphi(t, |x - y|)$ in (3) under certain convexity-monotonicity assumptions on the error function φ and also $T_p(t)$ in (6) when $0 < p \leq 1$ are the smallest possible ones. In other words, the following results provide exact upper bound for the convexity-difference of φ -Jensen convex functions defined by

$$(8) \quad C_\varphi(x, y, t) := \sup_{f \in \mathcal{JC}_\varphi(I)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\} \quad (x, y \in I, t \in [0, 1]),$$

where

$$\mathcal{JC}_\varphi(I) := \{f : I \rightarrow \mathbb{R} \mid f \text{ is locally bounded from above and } \varphi\text{-Jensen convex on } I\}.$$

Theorem C. *Let $\varphi : \frac{1}{2}I^* \rightarrow \mathbb{R}_+$ be a differentiable function with $\varphi(0) = 0$ such that φ is nondecreasing, concave and φ' is convex on $\frac{1}{2}I^*$. Then, for all $x, y \in I$ and $t \in [0, 1]$,*

$$C_\varphi(x, y, t) = \mathcal{T}_\varphi(t, |x - y|).$$

Corollary C. *Let $0 < p \leq 1$ and $\varepsilon \geq 0$. Let $\varphi_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by $\varphi_p(t) := \varepsilon t^p$ ($t \in \mathbb{R}_+$). Then, for all $x, y \in I$ and $t \in [0, 1]$,*

$$C_{\varphi_p}(x, y, t) = \varepsilon T_p(t)|x - y|^p.$$

The main purpose of this paper is to prove that the inequality (4) is also sharp under a certain concavity assumption on φ .

To approach and discuss this problem, we introduce the Takagi type function $S_\phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$(9) \quad S_\phi(x) = \sum_{n=0}^{\infty} 2\phi\left(\frac{1}{2^{n+1}}\right)d_{\mathbb{Z}}(2^n x),$$

where $P := \{\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\}$ and $\phi : P \rightarrow \mathbb{R}_+$ is a nonnegative function. Taking $\phi(t) := t^p$ ($t \in \mathbb{R}_+$), we obtain the function S_p defined in (5).

In Theorem 4 and Theorem 9 of Section 2 below, we will prove that for a certain class of functions ϕ , the Takagi type function S_ϕ is approximately Jensen convex. In Section 3, we shall apply this result to obtain the sharpness of the error term in Theorem B.

2. APPROXIMATE CONVEXITY OF TAKAGI TYPE FUNCTIONS

The main results of this section state that under certain assumptions on the function $\phi : P \rightarrow \mathbb{R}$ (where $P := \{\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\}$), S_ϕ is well-defined and approximately Jensen convex in the following sense: For all $x, y \in \mathbb{R}$,

$$S_\phi\left(\frac{x+y}{2}\right) \leq \frac{S_\phi(x) + S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\left(\frac{x-y}{2}\right).$$

First we describe the situation when the definition of S_ϕ is correct.

Lemma 1. *Let $\phi : P \rightarrow \mathbb{R}_+$ be a nonnegative function. Then S_ϕ is well-defined; i.e., the series on the right-hand side of (9) is convergent everywhere if and only if*

$$(10) \quad \sum_{n=1}^{\infty} \phi\left(\frac{1}{2^n}\right) < \infty.$$

Proof. Observe that $d_{\mathbb{Z}}(2^n \frac{1}{3}) = \frac{1}{3}$ for all $n \in \mathbb{N}$. Hence the convergence of the right-hand side of (9) at $x = \frac{1}{3}$ yields that (10) holds.

Conversely, if (10) is satisfied, then, for all $x \in \mathbb{R}$,

$$S_\phi(x) = \sum_{n=0}^{\infty} 2\phi\left(\frac{1}{2^{n+1}}\right)d_{\mathbb{Z}}(2^n x) \leq \sum_{n=0}^{\infty} \phi\left(\frac{1}{2^{n+1}}\right) < \infty,$$

which proves the statement. $\square$

In the sequel, the class of nonnegative functions $\phi : P \rightarrow \mathbb{R}_+$ satisfying the condition (10) will be denoted by $\mathcal{H}$:

$$\mathcal{H} := \left\{ \phi : P \rightarrow \mathbb{R}_+ \mid \sum_{n=1}^{\infty} \phi\left(\frac{1}{2^n}\right) < \infty \right\}.$$

The next theorem, which was discovered by Jacek Tabor and Józef Tabor, has an important role in the proof of our main theorem.

Theorem 2 ([22]). *For every $p \in [1, 2]$ and $x, y \in \mathbb{R}$,*

$$S_p\left(\frac{x+y}{2}\right) \leq \frac{S_p(x) + S_p(y)}{2} + d_{\mathbb{Z}}^p\left(\frac{x-y}{2}\right).$$

In the next result we give a representation of $S_\phi(x)$ as an infinite linear combination of the values $S_p(2^n x)$, $n = 1, 2, \dots$. The particular case when $p = 2$ and $\phi(x) = x^q$ for some $q \in [1, 2]$ was established in [22].

Theorem 3. *Let $\phi \in \mathcal{H}$. Then, for every $p > 0$ and $x \in \mathbb{R}$,*

$$(11) \quad S_\phi(x) = 2^p \phi\left(\frac{1}{2}\right) S_p(x) + \sum_{n=1}^{\infty} \left(2^p \phi\left(\frac{1}{2^{n+1}}\right) - \phi\left(\frac{1}{2^n}\right) \right) S_p(2^n x).$$

Proof. Since $S_p(x) = \sum_{n=0}^{\infty} \frac{1}{2^{np+p-1}} d_{\mathbb{Z}}(2^n x)$, the right-hand side of (11) can be written in the form $\sum_{n=0}^{\infty} a_n d_{\mathbb{Z}}(2^n x)$, where

$$\begin{aligned} a_n &= 2^p \phi\left(\frac{1}{2}\right) \frac{1}{2^{np+p-1}} + \sum_{k=1}^n \frac{1}{2^{(n-k)p+p-1}} \left(2^p \phi\left(\frac{1}{2^{k+1}}\right) - \phi\left(\frac{1}{2^k}\right) \right) \\ &= \phi\left(\frac{1}{2}\right) \frac{1}{2^{np-1}} + \sum_{k=1}^n \left(\frac{1}{2^{(n-k)p-1}} \phi\left(\frac{1}{2^{k+1}}\right) - \frac{1}{2^{(n-k+1)p-1}} \phi\left(\frac{1}{2^k}\right) \right) = 2\phi\left(\frac{1}{2^{n+1}}\right). \end{aligned}$$

Consequently (11) holds. $\square$

An immediate consequence of the previous two theorems is the next result, which states the approximate convexity of S_ϕ .

Theorem 4. *Let $\phi \in \mathcal{H}$ such that, for all $u \in \frac{1}{2}P$, $\phi(2u) \leq 2^p\phi(u)$, where $p \in [1, 2]$ is an arbitrary constant. Then, for all $x, y \in \mathbb{R}$,*

$$(12) \quad S_\phi\left(\frac{x+y}{2}\right) \leq \frac{S_\phi(x) + S_\phi(y)}{2} + \Phi_p\left(\frac{x-y}{2}\right),$$

where $\Phi_p : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(13) \quad \Phi_p(u) := \sum_{n=0}^{\infty} \phi\left(\frac{1}{2^{n+1}}\right) \left(2^p d_{\mathbb{Z}}^p(2^n u) - d_{\mathbb{Z}}^p(2^{n+1} u)\right).$$

Proof. For an arbitrary function $f : \mathbb{R} \rightarrow \mathbb{R}$ define $Jf : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$Jf(x, y) := f\left(\frac{x+y}{2}\right) - \frac{f(x) + f(y)}{2}.$$

Then, by (11), we get

$$JS_\phi(x, y) = 2^p\phi\left(\frac{1}{2}\right)JS_p(x, y) + \sum_{n=1}^{\infty} \left(2^p\phi\left(\frac{1}{2^{n+1}}\right) - \phi\left(\frac{1}{2^n}\right)\right)JS_p(2^n x, 2^n y).$$

By Theorem 2, for all $x, y \in \mathbb{R}$, we have that

$$JS_p(x, y) \leq d_{\mathbb{Z}}\left(\frac{x-y}{2}\right)^p.$$

On the other hand, by the assumption on ϕ , the coefficient $\left(2^p\phi\left(\frac{1}{2^{n+1}}\right) - \phi\left(\frac{1}{2^n}\right)\right)$ is nonnegative; hence for all $x, y \in \mathbb{R}$, we get

$$\begin{aligned} JS_\phi(x, y) &\leq 2^p\phi\left(\frac{1}{2}\right)d_{\mathbb{Z}}^p\left(\frac{x-y}{2}\right) + \sum_{n=1}^{\infty} \left(2^p\phi\left(\frac{1}{2^{n+1}}\right) - \phi\left(\frac{1}{2^n}\right)\right)d_{\mathbb{Z}}^p\left(\frac{2^n(x-y)}{2}\right) \\ &= \sum_{n=0}^{\infty} \phi\left(\frac{1}{2^{n+1}}\right) \left(2^p d_{\mathbb{Z}}^p\left(\frac{2^n(x-y)}{2}\right) - d_{\mathbb{Z}}^p\left(\frac{2^{n+1}(x-y)}{2}\right)\right) = \Phi_p\left(\frac{x-y}{2}\right), \end{aligned}$$

which proves (12). $\square$

In the next proposition we describe a decomposition property of the function Φ_p .

Proposition 5. *Let $\phi \in \mathcal{H}$ and $p \in [1, 2]$. Then, for all $u \in]0, \frac{1}{2}]$,*

$$(14) \quad \Phi_p(u) = \Phi_p\left(\frac{1}{2^{\lfloor \log_2 \frac{1}{u} \rfloor}} - u\right) + \phi\left(\frac{1}{2^{\lfloor \log_2 \frac{1}{u} \rfloor}}\right) \left((2^{\lfloor \log_2 \frac{1}{u} \rfloor} u)^p - (1 - 2^{\lfloor \log_2 \frac{1}{u} \rfloor} u)^p\right).$$

Proof. For $u \in]0, \frac{1}{2}]$, denote $\ell(u) := \lfloor \log_2 \frac{1}{u} \rfloor$. Then by the definition of the integer-part function, we have $\ell(u) - 1 \leq \log_2 \frac{1}{u} - 1 < \ell(u)$; hence

$$2^{\ell(u)-1}u \leq \frac{1}{2} < 2^{\ell(u)}u.$$

Thus, we have

$$d_{\mathbb{Z}}(2^n u) = \begin{cases} 2^n u, & \text{if } n \leq \ell(u) - 1, \\ d_{\mathbb{Z}}\left(2^n \left(\frac{1}{2^{\ell(u)}} - u\right)\right), & \text{if } n \geq \ell(u). \end{cases}$$

It can also be seen that

$$d_{\mathbb{Z}}\left(2^n\left(\frac{1}{2^{\ell(u)}} - u\right)\right) = 2^n\left(\frac{1}{2^{\ell(u)}} - u\right), \quad \text{if } n \leq \ell(u).$$

Using these formulas and applying (13), we get

$$\begin{aligned} \Phi_p(u) &= \sum_{n=0}^{\ell(u)-2} \phi\left(\frac{1}{2^{n+1}}\right) \left(2^p(2^n u)^p - (2^{n+1} u)^p\right) + \phi\left(\frac{1}{2^{\ell(u)}}\right) \left(2^p(2^{\ell(u)-1} u)^p - (1 - 2^{\ell(u)} u)^p\right) \\ &\quad + \sum_{n=\ell(u)}^{\infty} \phi\left(\frac{1}{2^{n+1}}\right) \left(2^p d_{\mathbb{Z}}^p\left(2^n\left(\frac{1}{2^{\ell(u)}} - u\right)\right) - d_{\mathbb{Z}}^p\left(2^{n+1}\left(\frac{1}{2^{\ell(u)}} - u\right)\right)\right) \\ &= \phi\left(\frac{1}{2^{\ell(u)}}\right) \left((2^{\ell(u)} u)^p - (1 - 2^{\ell(u)} u)^p\right) \\ &\quad + \sum_{n=\ell(u)}^{\infty} \phi\left(\frac{1}{2^{n+1}}\right) \left(2^p d_{\mathbb{Z}}^p\left(2^n\left(\frac{1}{2^{\ell(u)}} - u\right)\right) - d_{\mathbb{Z}}^p\left(2^{n+1}\left(\frac{1}{2^{\ell(u)}} - u\right)\right)\right) \\ &= \phi\left(\frac{1}{2^{\ell(u)}}\right) \left((2^{\ell(u)} u)^p - (1 - 2^{\ell(u)} u)^p\right) \\ &\quad + \sum_{n=0}^{\infty} \phi\left(\frac{1}{2^{n+1}}\right) \left(2^p d_{\mathbb{Z}}^p\left(2^n\left(\frac{1}{2^{\ell(u)}} - u\right)\right) - d_{\mathbb{Z}}^p\left(2^{n+1}\left(\frac{1}{2^{\ell(u)}} - u\right)\right)\right) \\ &= \phi\left(\frac{1}{2^{\ell(u)}}\right) \left((2^{\ell(u)} u)^p - (1 - 2^{\ell(u)} u)^p\right) + \Phi_p\left(\frac{1}{2^{\ell(u)}} - u\right), \end{aligned}$$

which means that (14) holds. $\square$

In the next proposition an important class of functions ϕ from $\mathcal{H}$ will be described.

Proposition 6. *Let $\phi : [0, \frac{1}{2}] \rightarrow \mathbb{R}_+$. Assume that $\phi(0) = 0$ and the mapping $x \mapsto \frac{\phi(x)}{x}$ is concave on $[0, \frac{1}{2}]$. Then $\phi|_P \in \mathcal{H}$, the function $x \mapsto \frac{\phi(x)}{x^2}$ is decreasing on $]0, \frac{1}{2}[$ and ϕ is continuous on $[0, \frac{1}{2}[$.*

Proof. By the concavity of the function $x \mapsto \frac{\phi(x)}{x} =: \psi(x)$, there exists an affine function that majorizes ψ ; hence ψ is bounded from above by a constant $C \geq 0$. Therefore,

$$\sum_{n=1}^{\infty} \phi|_P\left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \phi\left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi\left(\frac{1}{2^n}\right) \leq C \sum_{n=1}^{\infty} \frac{1}{2^n} = C,$$

which proves that (10) holds, i.e., $\phi|_P \in \mathcal{H}$.

Let $0 < z < y < x \leq \frac{1}{2}$. By the concavity of $x \mapsto \frac{\phi(x)}{x}$, we get

$$(15) \quad \frac{\phi(y)}{y} \geq \frac{x-y}{x-z} \cdot \frac{\phi(z)}{z} + \frac{y-z}{x-z} \cdot \frac{\phi(x)}{x}.$$

Since ϕ is nonnegative, by (15), we have that

$$\frac{\phi(y)}{y} \geq \frac{y-z}{x-z} \cdot \frac{\phi(x)}{x}.$$

Then, taking the limit $z \rightarrow 0$, we obtain $\frac{\phi(y)}{y^2} \geq \frac{\phi(x)}{x^2}$, which means that the mapping $x \mapsto \frac{\phi(x)}{x^2}$ is decreasing.

Rearranging the inequality in (15), we get

$$\phi(z) \leq z \left(\frac{\phi(y)}{y} \cdot \frac{x-z}{x-y} - \frac{\phi(x)}{x} \cdot \frac{y-z}{x-y} \right).$$

Upon taking the limit $z \rightarrow 0$ again, we get that ϕ is continuous at 0. By the concavity assumption, it follows that ϕ is also continuous on $]0, \frac{1}{2}[$. $\square$

In the next result we show that if ϕ is a cone combination of power functions, then $x \mapsto \frac{\phi(x)}{x}$ is concave.

Proposition 7. *Let μ be a nonnegative bounded Borel measure on $[1, 2]$. Let the error function $\phi_\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by*

$$(16) \quad \phi_\mu(x) := \int_1^2 x^p d\mu(p) \quad (x \in \mathbb{R}_+).$$

Then $x \mapsto \frac{\phi_\mu(x)}{x}$ is concave on $]0, +\infty[$.

Proof. Define $\psi_\mu :]0, +\infty[\rightarrow \mathbb{R}_+$ by

$$\psi_\mu(x) := \frac{\phi_\mu(x)}{x} = \int_1^2 x^{p-1} d\mu(p).$$

Applying standard calculus rules, we have

$$\psi_\mu''(x) = \int_1^2 (p-1)(p-2)x^{p-3} d\mu(p),$$

for all $x \in]0, +\infty[$. By the nonnegativity of the measure and the nonpositivity of the integrand, we get that $\psi_\mu''(x) \leq 0$, if $x > 0$. This implies that ψ_μ is concave on $]0, +\infty[$, which completes the proof. $\square$

The next theorem has an important role in the proof of our subsequent main results.

Theorem 8. *Let $\phi : [0, \frac{1}{2}] \rightarrow \mathbb{R}_+$. Assume that $\phi(0) = 0$ and the mapping $x \mapsto \frac{\phi(x)}{x}$ is concave on $]0, \frac{1}{2}[$. Then, for all $u \in \mathbb{R}$,*

$$(17) \quad \Phi_2(u) \leq \phi \circ d_{\mathbb{Z}}(u).$$

Proof. By the 1-periodicity of Φ_2 , in the proof of (17), we may assume that $|u| \leq \frac{1}{2}$, and since Φ_2 is even, we may also assume that $u > 0$. We show, for every fixed $k \in \mathbb{N}$, that the theorem holds for all $u \in \mathbb{Z}/2^k$. Then the statement follows from the continuity of Φ_2 and ϕ and the denseness of dyadic rational numbers in $\mathbb{R}$.

Using induction on m , we prove that (17) holds for all $u = \frac{m}{2^k}$, where $m \in \{1, \dots, 2^{k-1}\}$. First we consider the case when $m = 1$. Then, by (13),

$$\begin{aligned} \Phi_2\left(\frac{1}{2^k}\right) &= \sum_{n=0}^{k-1} \phi\left(\frac{1}{2^{n+1}}\right) \left(4d_{\mathbb{Z}}^2\left(\frac{2^n}{2^k}\right) - d_{\mathbb{Z}}^2\left(\frac{2^{n+1}}{2^k}\right)\right) \\ &= \sum_{n=0}^{k-2} \phi\left(\frac{1}{2^{n+1}}\right) \left(4\left(\frac{4^n}{4^k}\right) - \frac{4^{n+1}}{4^k}\right) + \phi\left(\frac{1}{2^k}\right) = \phi\left(\frac{1}{2^k}\right), \end{aligned}$$

which shows that in this case (17) holds with equality.

Now let $m \in \{2, \dots, 2^{k-1}\}$ and assume that (17) has been verified for all $u = \frac{j}{2^k}$, where $j \in \{1, \dots, m-1\}$. With the notation $\ell(u) := \lfloor \log_2 \frac{1}{u} \rfloor$, it is easy to see that $\frac{1}{2^{\ell(u)}} - u \in \mathbb{Z}/2^k$ and $0 < \frac{1}{2^{\ell(u)}} - u < u$; thus by the inductive assumption, we have

$$(18) \quad \Phi_2\left(\frac{1}{2^{\ell(u)}} - u\right) \leq \phi\left(\frac{1}{2^{\ell(u)}} - u\right).$$

In view of Proposition 5 with $p = 2$, the identity

$$(19) \quad \Phi_2(u) = \Phi_2\left(\frac{1}{2^{\ell(u)}} - u\right) + \phi\left(\frac{1}{2^{\ell(u)}}\right)(2^{\ell(u)+1}u - 1)$$

holds. By (19) and (18) we have that

$$\Phi_2(u) \leq \phi\left(\frac{1}{2^{\ell(u)}} - u\right) + \phi\left(\frac{1}{2^{\ell(u)}}\right)(2^{\ell(u)+1}u - 1).$$

Thus to prove the assertion of the theorem it is enough to show that

$$(20) \quad \phi\left(\frac{1}{2^{\ell(u)}} - u\right) + \phi\left(\frac{1}{2^{\ell(u)}}\right)(2^{\ell(u)+1}u - 1) \leq \phi(u).$$

Let $\psi(x) = \frac{\phi(x)}{x}(x \in]0, \frac{1}{2}])$. Then, by the concavity of ψ , we get

$$\begin{aligned} \left(\frac{1}{2^{\ell(u)}} - 1\right)\psi\left(\frac{1}{2^{\ell(u)}} - u\right) + \left(2 - \frac{1}{2^{\ell(u)}}\right)\psi\left(\frac{1}{2^{\ell(u)}}\right) \\ \leq \psi\left(\left(\frac{1}{2^{\ell(u)}} - u\right)\left(\frac{1}{2^{\ell(u)}} - 1\right) + \frac{1}{2^{\ell(u)}}\left(2 - \frac{1}{2^{\ell(u)}}\right)\right) = \psi(u), \end{aligned}$$

which is equivalent to (20), completing the proof of (17). $\square$

The main result of this paper is stated in the following theorem.

Theorem 9. *Let $\phi : [0, \frac{1}{2}] \rightarrow \mathbb{R}_+$. Assume that $\phi(0) = 0$ and the mapping $x \mapsto \frac{\phi(x)}{x}$ is concave on $]0, \frac{1}{2}]$. Then S_ϕ is approximately Jensen convex in the following sense: For all $x, y \in \mathbb{R}$,*

$$(21) \quad S_\phi\left(\frac{x+y}{2}\right) \leq \frac{S_\phi(x) + S_\phi(y)}{2} + \phi \circ d_{\mathbb{Z}}\left(\frac{x-y}{2}\right).$$

Proof. For an arbitrary function $f : \mathbb{R} \rightarrow \mathbb{R}$ define $Jf : \mathbb{R}^2 \rightarrow \mathbb{R}$ as in the proof of Theorem 4. By Proposition 6, we have that $\phi|_P \in \mathcal{H}$, which, by Lemma 1, implies that S_ϕ is well-defined. By Proposition 6, we get that $x \mapsto \frac{\phi(x)}{x^2}$ is decreasing, which means, for all $u \in]0, \frac{1}{2}]$, that

$$\phi(2u) \leq 4\phi(u).$$

This means that we can apply Theorem 4 with $p = 2$ for the function S_ϕ . Thus we obtain that for all $x, y \in \mathbb{R}$,

$$(22) \quad JS_\phi(x, y) \leq \Phi_2\left(\frac{x-y}{2}\right).$$

In view of Theorem 8, we also have (17) for all $u \in \mathbb{R}$. Therefore, (22) and (17) imply that (21) holds for all $x, y \in \mathbb{R}$, which means that S_ϕ is approximately Jensen convex in the sense of (21). $\square$

The next result immediately follows from Proposition 7 and the above theorem. The particular case when ϕ is of the form $\phi(t) = \varepsilon t^p$ (where $\varepsilon \geq 0$ and $p \in [1, 2]$ are constants) was discovered by Tabor and Tabor in [22] and was already stated in Theorem 4.

Corollary 10. *Let μ be a nonnegative bounded Borel measure on $[1, 2]$. Let the error function $\phi_\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by (16). Then, for all $x, y \in \mathbb{R}$,*

$$S_{\phi_\mu}\left(\frac{x+y}{2}\right) \leq \frac{S_{\phi_\mu}(x) + S_{\phi_\mu}(y)}{2} + \phi_\mu \circ d_{\mathbb{Z}}\left(\frac{x-y}{2}\right).$$

3. APPLICATIONS TO φ -CONVEXITY

We shall prove that the error term $\mathcal{S}_\varphi(t, |x - y|)$ in (4) under certain assumptions on the error function φ is the smallest possible one. In other words, the next theorem will provide an exact upper bound for the convexity-difference of φ -Jensen convex functions defined by (8).

Theorem 11. *Let $\varphi : \frac{1}{2}I^* \rightarrow \mathbb{R}$ be nondecreasing and assume that $\varphi(0) = 0$ and $x \mapsto \frac{\varphi(x)}{x}$ is concave on $\frac{1}{2}I^*$. Then, for all $x, y \in I$ and $t \in [0, 1]$,*

$$(23) \quad C_\varphi(x, y, t) = \mathcal{S}_\varphi(t, |x - y|);$$

furthermore, for fixed $x, y \in I$, the function $f_{x,y} : I \rightarrow \mathbb{R}$ defined by

$$(24) \quad f_{x,y}(u) = \mathcal{S}_\varphi\left(\frac{y-u}{y-x}, |x-y|\right) \quad (u \in I)$$

is φ -Jensen convex on I and, for all $t \in [0, 1]$,

$$(25) \quad f_{x,y}(tx + (1-t)y) - tf_{x,y}(x) - (1-t)f_{x,y}(y) = \mathcal{S}_\varphi(t, |x - y|).$$

Proof. From the concavity of $x \mapsto \frac{\varphi(x)}{x}$ it follows (as in Proposition 6) that, for some $n_0 \in \mathbb{N}$, the series $\sum_{n=n_0}^{\infty} \varphi(2^{-n})$ is convergent. Hence, by Theorem B, we get that

$$(26) \quad C_\varphi(x, y, t) \leq \mathcal{S}_\varphi(t, |x - y|)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Now let $x, y \in I$ be fixed. In order to show the reversed inequality (which then yields (23)), it suffices to prove that the function $f_{x,y}$ defined in (24) is φ -Jensen convex on I and (25) holds for all $t \in [0, 1]$.

Observe that $f_{x,y}(x) = f_{x,y}(y) = 0$. Hence, by putting $u = tx + (1-t)y$ in (24), the equality (25) follows.

To verify that $f_{x,y}$ is φ -Jensen convex on I , define the function $\phi : [0, \frac{1}{2}] \rightarrow \mathbb{R}_+$ by

$$\phi(r) := \varphi(r|x - y|).$$

Then, we get that $r \mapsto \frac{\phi(r)}{r}$ is concave on $]0, \frac{1}{2}]$. Therefore, by Theorem 9, the Takagi type function S_ϕ satisfies

$$(27) \quad S_\phi\left(\frac{s+t}{2}\right) \leq \frac{S_\phi(s) + S_\phi(t)}{2} + \phi \circ d_{\mathbb{Z}}\left(\frac{s-t}{2}\right),$$

for all $t, s \in \mathbb{R}$. Observe that, for all $t \in \mathbb{R}$,

$$S_\phi(t) = \sum_{n=0}^{\infty} 2\phi\left(\frac{1}{2^{n+1}}\right)d_{\mathbb{Z}}(2^n t) = \sum_{n=0}^{\infty} 2\varphi\left(\frac{|x-y|}{2^{n+1}}\right)d_{\mathbb{Z}}(2^n t) = \mathcal{S}_\varphi(t, |x - y|).$$

On the other hand, using the increasingness of ϕ , for $s, t \in \frac{1}{x-y}(I - y)$, we have

$$\phi\left(d_{\mathbb{Z}}\left(\frac{s-t}{2}\right)\right) \leq \phi\left(\frac{|s-t|}{2}\right) = \varphi\left(\frac{|s-t|}{2}|x-y|\right).$$

Thus, for $s, t \in \frac{1}{x-y}(I - y)$, (27) can be rewritten as

$$(28) \quad \mathcal{S}_\varphi\left(\frac{s+t}{2}, |x-y|\right) \leq \frac{\mathcal{S}_\varphi(s, |x-y|) + \mathcal{S}_\varphi(t, |x-y|)}{2} + \varphi\left(\frac{|s-t|}{2}|x-y|\right).$$

Taking $u, v \in I$ arbitrarily, the values $s = \frac{y-u}{y-x}$ and $t = \frac{y-v}{y-x}$ belong to $\frac{1}{x-y}(I-y)$ and hence (28) reduces to

$$f_{x,y}\left(\frac{u+v}{2}\right) \leq \frac{f_{x,y}(u) + f_{x,y}(v)}{2} + \varphi\left(\frac{|u-v|}{2}\right),$$

which completes the proof of the φ -Jensen convexity of $f_{x,y}$. $\square$

Taking an error function φ which is a combination of power functions of exponents from $[1, 2]$, we obtain the following result.

Theorem 12. *Let I be a nonempty real interval and μ be a nonnegative bounded Borel measure on $[1, 2]$. Define the error function $\varphi_\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by (16). Then, for all $x, y \in I$ and $t \in [0, 1]$,*

$$C_{\varphi_\mu}(x, y, t) = \int_1^2 S_p(t) |x - y|^p d\mu(p),$$

where $S_p : \mathbb{R} \rightarrow \mathbb{R}$ is given by (5).

Proof. It is easy to see that φ_μ is nondecreasing and we also have that $\varphi_\mu(0) = 0$. By Proposition 7 we get that $u \mapsto \frac{\varphi_\mu(u)}{u}$ is concave on $]0, +\infty[$. Thus, Theorem 11 can be applied, and hence, for all $x, y \in I$ and $t \in [0, 1]$,

$$C_{\varphi_\mu}(x, y, t) = S_{\varphi_\mu}(t, |x - y|).$$

On the other hand, for all $x, y \in I$ and $t \in [0, 1]$,

$$\begin{aligned} S_{\varphi_\mu}(t, |x - y|) &= \sum_{n=0}^{\infty} 2\varphi_\mu\left(\frac{|x-y|}{2^{n+1}}\right) d_{\mathbb{Z}}(2^n t) = \sum_{n=0}^{\infty} \int_1^2 2\left(\frac{|x-y|}{2^{n+1}}\right)^p d\mu(p) d_{\mathbb{Z}}(2^n t) \\ &= \int_1^2 \sum_{n=0}^{\infty} \frac{1}{2^{np+p-1}} d_{\mathbb{Z}}(2^n t) |x - y|^p d\mu(p) = \int_1^2 S_p(t) |x - y|^p d\mu(p), \end{aligned}$$

which completes the proof. $\square$

Taking μ as the point measure $\varepsilon\delta_p$ where $p \in [1, 2]$ in the above theorem, it reduces to a result by Jacek and Józef Tabor [22], which states that (7) is sharp, i.e., that the term $\varepsilon S_p(t)$ is the smallest possible.

Corollary 13. *Let I be a nonempty real interval, $p \in [1, 2]$ and $\varepsilon \geq 0$. Define the error function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\varphi(u) := \varepsilon u^p$. Then, for all $x, y \in I$ and $t \in [0, 1]$,*

$$C_\varphi(x, y, t) = \varepsilon S_p(t) |x - y|^p.$$

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