COMPARING THE DISTRIBUTION OF VARIOUS SUPREMA ON TWO-PARAMETER WIENER SPACE

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ABSTRACT. Let $Q = [0, S] \times [0, T]$ and let $C_2(Q)$ be two-parameter Wiener space. In this note we consider the ratios of the probabilities $\mathbb{P}[x(S, T) \geq c]$, $\mathbb{P}[\sup_Q x(s, t) \geq c]$ and $\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]$.

1. INTRODUCTION

Let $Q = [0, S] \times [0, T]$ and let $C_2(Q)$ denote two-parameter Wiener space, i.e., the space of all $\mathbb{R}$ -valued continuous functions on Q satisfying $x(s, 0) = x(0, t) = 0$ for all $(s, t) \in Q$. Then the random variable $x(s, t)$ given by point evaluation at (s, t) defines the canonical two-parameter Wiener process (Brownian sheet).

Let ∂Q denote the boundary of Q , that is, the set $\{(s, t) : s = 0, S \text{ or } t = 0, T\}$, and for $c \geq 0$, let

$$(1) \quad \Phi(c) = \frac{1}{\sqrt{2\pi ST}} \int_c^\infty \exp\left(-\frac{u^2}{2ST}\right) du = \mathbb{P}[x(S, T) \geq c].$$

In this note we study ratios of the probabilities $\mathbb{P}[x(S, T) \geq c]$, $\mathbb{P}[\sup_Q x(s, t) \geq c]$ and $\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]$.

For $D = [0, 1] \times [0, 1]$, consider $C_2(D)$ and put

$$(2) \quad N(c) = \frac{1}{\sqrt{2\pi}} \int_c^\infty \exp\left(-\frac{u^2}{2}\right) du = \mathbb{P}[x(1, 1) \geq c].$$

In [2], Goodman obtained some very good bounds on the unknown probability $\mathbb{P}[\sup_D x(s, t) \geq c]$ by showing that

$$(3) \quad 4 \int_c^\infty sN(s)ds \leq \mathbb{P}\left[\sup_D x(s, t) \geq c\right] \leq 4N(c)$$

and that

$$(4) \quad \lim_{c \rightarrow \infty} \frac{\mathbb{P}[\sup_D x(s, t) \geq c]}{4N(c)} = 1.$$

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In [3], Paranjape and Park showed that

$$(5) \quad \mathbb{P} \left[\sup_{\partial D} x(s, t) \geq c \right] = 3N(c) - e^{4c^2} N(3c).$$

Also see some further comments in [4] about (3) and (5).

2. MAIN RESULTS

In this section we establish our main results, namely that

$$(6) \quad \lim_{c \rightarrow \infty} \frac{\mathbb{P}[x(S, T) \geq c]}{\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]} = \frac{3}{8}$$

and that

$$(7) \quad \lim_{c \rightarrow \infty} \frac{\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]}{\mathbb{P}[\sup_Q x(s, t) \geq c]} = \frac{2}{3}.$$

We first note that by the scaling property of the Brownian sheet, it follows from (3) and (5) that

$$(8) \quad \frac{4}{ST} \int_c^\infty s \Phi(s) ds \leq \mathbb{P} \left[\sup_Q x(s, t) \geq c \right] \leq 4\Phi(c)$$

and that

$$(9) \quad \mathbb{P} \left[\sup_{\partial Q} x(s, t) \geq c \right] = 3\Phi(c) - e^{\frac{4c^2}{ST}} \Phi(3c).$$

2.1. Proof of (6). Using (9) and applying L'Hôpital's Rule thrice, we obtain that

$$\begin{aligned} \lim_{c \rightarrow \infty} \frac{\mathbb{P}[x(S, T) \geq c]}{\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]} &= \lim_{c \rightarrow \infty} \frac{\Phi(c)}{3\Phi(c) - e^{\frac{4c^2}{ST}} \Phi(3c)} \\ &= \lim_{c \rightarrow \infty} \frac{\int_c^\infty e^{-\frac{u^2}{2ST}} du}{3 \int_c^\infty e^{-\frac{u^2}{2ST}} du - e^{\frac{4c^2}{ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du} \\ &= \lim_{c \rightarrow \infty} \frac{STe^{-\frac{c^2}{2ST}}}{8ce^{\frac{4c^2}{ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du} \\ &= \lim_{c \rightarrow \infty} \frac{STe^{-\frac{9c^2}{2ST}}}{8c \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du} \\ &= \lim_{c \rightarrow \infty} \frac{9ce^{-\frac{9c^2}{2ST}}}{24ce^{-\frac{9c^2}{2ST}} - 8 \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du} \\ &= \lim_{c \rightarrow \infty} \frac{\left(\frac{81c^2}{ST} - 9\right) e^{-\frac{9c^2}{2ST}}}{\left(\frac{216c^2}{ST} - 48\right) e^{-\frac{9c^2}{2ST}}} \\ &= \frac{3}{8}. \end{aligned}$$

2.2. **Proof of (7).** First, (8) and (9) together show that

$$(10) \quad \frac{3\Phi(c) - e^{\frac{4c^2}{ST}}\Phi(3c)}{4\Phi(c)} \leq \frac{\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]}{\mathbb{P}[\sup_Q x(s, t) \geq c]} \leq \frac{3\Phi(c) - e^{\frac{4c^2}{ST}}\Phi(3c)}{\frac{4}{ST} \int_c^\infty s\Phi(s)ds}.$$

Now, to prove (7) we first note that from (6) it easily follows that

$$\lim_{c \rightarrow \infty} \frac{3\Phi(c) - e^{\frac{4c^2}{ST}}\Phi(3c)}{4\Phi(c)} = \frac{1}{4} \lim_{c \rightarrow \infty} \frac{\mathbb{P}[\sup_{\partial Q} x(s, t) \geq c]}{\mathbb{P}[x(S, T) \geq c]} = \left(\frac{1}{4}\right) \left(\frac{8}{3}\right) = \frac{2}{3}.$$

For the other inequality, proceed as in the proof of (6) above to obtain that

$$\begin{aligned} \lim_{c \rightarrow \infty} \frac{3\Phi(c) - e^{\frac{4c^2}{ST}}\Phi(3c)}{\frac{4}{ST} \int_c^\infty s\Phi(s)ds} &= \lim_{c \rightarrow \infty} \frac{3 \int_c^\infty e^{-\frac{u^2}{2ST}} du - e^{\frac{4c^2}{ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du}{\frac{4}{ST} \int_c^\infty s \left(\int_s^\infty e^{-\frac{u^2}{2ST}} du \right) ds} \\ &= \lim_{c \rightarrow \infty} \frac{2e^{\frac{4c^2}{ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du}{\int_c^\infty e^{-\frac{u^2}{2ST}} du} \\ &= \lim_{c \rightarrow \infty} \frac{6e^{-\frac{c^2}{2ST}} - \frac{16c}{ST} e^{\frac{4c^2}{ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du}{e^{-\frac{c^2}{2ST}}} \\ &= \lim_{c \rightarrow \infty} \left(6 - \frac{16c}{ST} e^{\frac{9c^2}{2ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du \right) \\ &= \frac{2}{3}, \end{aligned}$$

because (again using L'Hôpital's rule)

$$\lim_{c \rightarrow \infty} \frac{16c}{ST} e^{\frac{9c^2}{2ST}} \int_{3c}^\infty e^{-\frac{u^2}{2ST}} du = \frac{16}{ST} \lim_{c \rightarrow \infty} \frac{\int_{3c}^\infty e^{-\frac{u^2}{2ST}} du}{\frac{e^{-\frac{9c^2}{2ST}}}{c}} = \frac{16}{ST} \lim_{c \rightarrow \infty} \frac{3c^2}{\frac{9c^2}{ST} + 1} = \frac{16}{3}.$$

3. DISCUSSION

We briefly remark on some implications of (6) and (7) for the two-parameter Wiener process. Knowing the behavior of these ratios immediately gives

$$(11) \quad \lim_{c \rightarrow \infty} \frac{\mathbb{P}[\sup_Q x(s, t) \geq c]}{\mathbb{P}[x(S, T) \geq c]} = 4,$$

which is the same as in [2]. Given that $1 = \mathbb{P}[\sup_Q x(s, t) \geq 0] = 2\mathbb{P}[x(S, T) \geq 0]$, it is immediately clear that the two-parameter Wiener process cannot exhibit a “reflection principle” relating its behavior on a rectangle to its behavior at a point; this is also concluded in [1] following a different line of reasoning.

The physical interpretations of (6), (7), and (11) are interesting. For $c = 0$, the ratio in (6) is $\frac{1}{2}$ and decreases toward $\frac{2}{3}$ as $c \rightarrow \infty$. For small c , approximately half of the sample functions whose supremum on the boundary exceeds c will also exceed c at the point (S, T) . As c grows without bound, this proportion will decrease toward three-eighths. Similarly, for $c = 0$, the ratio in (7) is equal to 1 and will decrease as c increases; for large c , approximately two-thirds of the sample functions whose supremum on all of Q exceeds c will exceed c somewhere on ∂Q .

We also remark that in [5], it is asserted that for the standard Wiener random field $W(t)$ (Brownian sheet), the distribution of the maximum of $W(t)$ on the rectangle $R(s) = \{t : 0 \leq t_i \leq s_i\}$ can be obtained from the equation

$$(12) \quad \mathbb{P} \left[\max_{R(s)} W(t) \geq c \right] = 2\mathbb{P} [W(s) \geq c].$$

In fact, (11) shows that (12) cannot be true in general; the authors have recently found that this is also noted in a review of [4] by the author of [5] himself in the 1980 *Mathematical Reviews*.

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