ASYMPTOTIC BEHAVIOR OF DIMENSIONS OF SYZYGIES

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ABSTRACT. Let R be a commutative noetherian local ring and M be a finitely generated R -module of infinite projective dimension. It is well-known that the depths of the syzygy modules of M eventually stabilize to the depth of R . In this paper, we investigate the conditions under which a similar statement can be made regarding dimension. In particular, we show that if R is equidimensional and the Betti numbers of M are eventually non-decreasing, then the dimension of any sufficiently high syzygy module of M coincides with the dimension of R .

INTRODUCTION

Throughout this paper R will denote a commutative noetherian local ring with identity element, unique maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Let M be a finitely generated R -module. The i th Betti number of M is given by $\beta_i(M) := \dim_k(\mathrm{Tor}_i^R(k, M))$. A minimal free resolution of M then has the form

$$\cdots \xrightarrow{\delta_3} R^{\beta_2(M)} \xrightarrow{\delta_2} R^{\beta_1(M)} \xrightarrow{\delta_1} R^{\beta_0(M)} \longrightarrow 0.$$

The i th syzygy module of M is $\Omega_i(M) := \mathrm{Coker}(\delta_{i+1})$. We let $\mathrm{Min}(M)$ denote the set of minimal elements under inclusion of $\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

Our main result is the following, which is part of Theorem 8.

Theorem 1. *Let R be a noetherian local ring and M a finitely generated R -module with eventually non-decreasing Betti numbers. Then for all $i \gg 0$ we have $\mathrm{Min}(\Omega_i(M)) \subseteq \mathrm{Min}(R)$ and $\mathrm{Supp}(\Omega_i(M)) = \mathrm{Supp}(\Omega_{i+2}(M))$.*

An important consequence of Theorem 8 is the following corollary, which follows immediately from Corollary 9.

Corollary 2. *Let R be an equidimensional noetherian local ring and M a finitely generated R -module with eventually non-decreasing Betti numbers. Then the sequence $(\dim(\Omega_i(M)))_{i=0}^{\infty}$ is constant for $i \gg 0$.*

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This raises the following open question.

Question 3. Let R be a noetherian local ring and M a finitely generated R -module. Is $(\dim(\Omega_i(M)))_{i=0}^\infty$ constant for all $i \gg 0$?

This question was also explored in the last section of [5]. In [5, Remark 5.2 (i)] it is noted that if R is unmixed and equidimensional, then $(\dim(\Omega_i(M)))_{i=0}^\infty$ is constant for $i \gg 0$. This is clear since the associated primes of any submodule of $R^{\beta_i(M)}$ are also associated primes of R and are therefore primes of maximal dimension by assumption.

It is worth noting that the asymptotic behavior of the depths of syzygy modules is known. Given M the sequence $(\text{depth}(\Omega_i(M)))_{i=0}^\infty$ is constant for all $i \gg 0$. Let $\text{pd}(M)$ denote the projective dimension of M . In particular if $\text{pd}(M) = \infty$, then $\text{depth}(\Omega_n(M)) \geq \text{depth}(R)$ for $n \geq \max\{0, \text{depth}(R) - \text{depth}(M)\}$, with at most one strict inequality at either $n = 0$ or $n = \text{depth}(R) - \text{depth}(M) + 1$; see [9, Proposition 10] or [2, Proposition 1.2.8]. It follows therefore that if $\text{pd}(M) = \infty$ and R is Cohen-Macaulay, then $\dim(\Omega_n(M)) = \dim(R)$ for $n \gg 0$.

All of our results are for modules whose Betti numbers are eventually non-decreasing. Therefore finding a proof for the following conjecture of L. Avramov would improve our results.

Conjecture 4 ([1]). *The Betti numbers of any finitely generated module over an arbitrary noetherian local ring are eventually non-decreasing.*

There are a plethora of cases for which this conjecture is known to be true. J. Lescot [8, Corollaire 6.5] showed that over a Golod ring which is not a hypersurface any finitely generated module of infinite projective dimension will have eventually increasing Betti numbers. Also L.-C. Sun [11, Corollary] showed that over rings of codepth less than or equal to three and Gorenstein rings of codepth four all finitely generated modules have eventually non-decreasing Betti numbers. Several other interesting cases are also proven in [3], [4] and [12].

Whenever the Betti numbers of a module are eventually strictly increasing it is known that the dimension of a sufficiently high syzygy will have the dimension of the ring. This is clear from the next lemma, which is mentioned without proof in [5, Remark 5.2 (iii)].

RESULTS

We denote the length of an R -module M by $\lambda_R(M)$ or simply $\lambda(M)$ when the ring is unambiguous.

Lemma 5. *Let R be a noetherian local ring and M a finitely generated R -module. If $\beta_i(M) > \beta_{i-1}(M)$ for some $i > 0$, then $\text{Supp}(\Omega_{i+1}(M)) = \text{Spec}(R)$; hence $\dim(\Omega_{i+1}(M)) = \dim(R)$.*

Proof. Given $\mathfrak{q} \in \text{Spec}(R)$ there exists $\mathfrak{p} \in \text{Min}(R)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Localizing the exact sequence

$$0 \rightarrow \Omega_{i+1}^R(M) \rightarrow R^{\beta_i(M)} \rightarrow R^{\beta_{i-1}(M)}$$

at $\mathfrak{p}$ we obtain the following inequalities:

$$\lambda_{R_{\mathfrak{p}}}(\Omega_{i+1}(M)_{\mathfrak{p}}) \geq \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\beta_i(M)}) - \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\beta_{i-1}(M)}) = \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})(\beta_i(M) - \beta_{i-1}(M)) > 0.$$

Thus $\mathfrak{p} \in \text{Supp}(\Omega_{i+1}(M))$; hence $\mathfrak{q} \in \text{Supp}(\Omega_{i+1}(M))$ and the result follows. $\square$

The following lemma is used in the proof of our main result, Theorem 8.

Lemma 6. *Let R be a noetherian local ring and M a finitely generated R -module. For a given $n \in \mathbb{N}$ suppose that $\beta_0(M) \leq \beta_1(M) \leq \dots \leq \beta_{2n-1}(M)$ and that $\text{Supp}(\Omega_{2n}(M)) \neq \text{Spec}(R)$. Then we have the following:*

- (a) $\beta_{2i}(M) = \beta_{2i+1}(M)$ for $i = 0, \dots, n-1$;
- (b) $\text{Supp}(\Omega_{2i+2}(M)) \subseteq \text{Supp}(\Omega_{2i}(M))$ for $i = 0, \dots, n-1$; and
- (c) $\text{Supp}(\Omega_{2n}(M)) \cap \text{Min}(R) = \text{Supp}(M) \cap \text{Min}(R)$.

Proof. Choose $\mathfrak{p} \in \text{Min}(R) \setminus \text{Supp}(\Omega_{2n}(M))$. Localizing part of a minimal free resolution of M at $\mathfrak{p}$, we get an exact sequence of finite-length $R_{\mathfrak{p}}$ -modules of the following form:

$$0 \rightarrow R_{\mathfrak{p}}^{\beta_{2n-1}(M)} \xrightarrow{\varphi_{2n-1}} R_{\mathfrak{p}}^{\beta_{2n-2}(M)} \rightarrow \dots \rightarrow R_{\mathfrak{p}}^{\beta_0(M)} \xrightarrow{\varphi_0} M_{\mathfrak{p}} \rightarrow 0.$$

Since φ_{2n-1} is an injection, $\lambda(R_{\mathfrak{p}}^{\beta_{2n-2}(M)}) \geq \lambda(R_{\mathfrak{p}}^{\beta_{2n-1}(M)})$; hence $\beta_{2n-2}(M) \geq \beta_{2n-1}(M)$. It follows that $\beta_{2n-2}(M) = \beta_{2n-1}(M)$. Since $R_{\mathfrak{p}}$ has finite length it follows that φ_{2n-1} is an isomorphism and φ_{2n-2} is the zero map. By repeating this argument, one sees that φ_{2i+1} is an isomorphism, φ_{2i} is the zero map, and $\beta_{2i}(M) = \beta_{2i+1}(M)$ for each $i = 0, \dots, n-1$. In particular, we have shown (a).

Since φ_0 is the zero map, $M_{\mathfrak{p}} = 0$; hence $\mathfrak{p} \notin \text{Supp}(M)$. It follows that

$$(1) \quad \text{Supp}(M) \cap \text{Min}(R) \subseteq \text{Supp}(\Omega_{2n}(M)) \cap \text{Min}(R).$$

Let $\mathfrak{q} \in \text{Spec}(R) \setminus \text{Supp}(\Omega_{2i}(M))$ for some i with $0 \leq i \leq n-1$. Localizing part of a minimal free resolution of M at $\mathfrak{q}$ we obtain an exact sequence of the following form:

$$0 \rightarrow \Omega_{2i+2}(M)_{\mathfrak{q}} \rightarrow R_{\mathfrak{q}}^{\beta_{2i+1}(M)} \rightarrow R_{\mathfrak{q}}^{\beta_{2i}(M)} \rightarrow 0.$$

Since $\beta_{2i+1}(M) = \beta_{2i}(M)$ it follows that $\Omega_{2i+2}(M)_{\mathfrak{q}} = 0$ and $\mathfrak{q} \notin \text{Supp}(\Omega_{2i+2}(M))$. Thus $\text{Supp}(\Omega_{2i+2}(M)) \subseteq \text{Supp}(\Omega_{2i}(M))$ for $i = 0, \dots, n-1$, which proves (b). Consequently $\text{Supp}(\Omega_{2n}(M)) \cap \text{Min}(R) \subseteq \text{Supp}(M) \cap \text{Min}(R)$. Since (1) provides the reverse containment, (c) is now immediate. $\square$

We make the following fact explicit in order to clarify some of our argumentation.

Fact 7. Let $(R, \mathfrak{m})$ be a noetherian local ring. Let B be an $n \times m$ matrix with entries in R defining a map from R^n to R^m . Applying invertible row and column operations to B , one can obtain a matrix $B' = I_h \oplus A$, where I_h is the $h \times h$ identity matrix for some $h \geq 0$ and A is an $(n-h) \times (m-h)$ matrix with entries in $\mathfrak{m}$.

Theorem 8. *Let R be a noetherian local ring and M a finitely generated R -module with eventually non-decreasing Betti numbers. Then for all $n \gg 0$ we have the following:*

- (a) $\text{Min}(\Omega_n(M)) \subseteq \text{Min}(R)$;
- (b) $\text{Supp}(\Omega_n(M)) = \text{Supp}(\Omega_{n+2i}(M))$ for all $i \geq 0$; and
- (c) if $\text{Supp}(\Omega_n(M)) \neq \text{Spec}(R)$, then $\beta_{n+2i}(M) = \beta_{n+2i+1}(M)$ for all $i \geq 0$.

Proof. We may assume that $\text{pd}(M) = \infty$. By replacing M by a sufficiently high syzygy, one may assume that $\beta_{i+1}(M) \geq \beta_i(M)$ for all $i \geq 0$. Assuming M was replaced by an even (odd) syzygy, if $\text{Supp}(\Omega_{2i}(M)) = \text{Spec}(R)$ for $i \gg 0$, then all of the statements hold for even (odd) syzygies. Therefore we may suppose that there exist infinitely many $i \in \mathbb{N}$ such that $\text{Supp}(\Omega_{2i}(M)) \neq \text{Spec}(R)$.

Since $\text{Min}(R)$ is a finite set we may choose $\mathfrak{p} \in \text{Min}(R)$ so that there are infinitely many $i \in \mathbb{N}$ for which $\mathfrak{p} \notin \text{Supp}(\Omega_{2i}(M))$. For each positive integer c such that $\mathfrak{p} \notin \text{Supp}(\Omega_{2c}(M))$, Lemma 6 implies that we have $\text{Supp}(\Omega_{2i+2}(M)) \subseteq \text{Supp}(\Omega_{2i}(M))$ and $\beta_{2i}(M) = \beta_{2i+1}(M)$ for all $0 \leq i < c$. Since c can be chosen to be arbitrarily large we have $\text{Supp}(\Omega_{2i+2}(M)) \subseteq \text{Supp}(\Omega_{2i}(M))$ and $\beta_{2i}(M) = \beta_{2i+1}(M)$ for all $i \geq 0$. Since closed sets in the Zariski topology satisfy the descending chain condition, it follows that we may choose $m \gg 0$ such that $(\text{Supp}(\Omega_{2m+2i}(M)))_{i=0}^\infty$ is constant, proving (b). Therefore the assumption that there exist infinitely many $i \in \mathbb{N}$ such that $\text{Supp}(\Omega_{2i}(M)) \neq \text{Spec}(R)$ is equivalent to assuming that $\text{Supp}(\Omega_{2m}(M)) \neq \text{Spec}(R)$ and (c) follows.

Therefore it remains to show that $\text{Min}(\Omega_{2i}(M)) \subseteq \text{Min}(R)$ for $i \gg 0$. Choose $\mathfrak{q} \in \text{Min}(\Omega_{2m}(M))$. Let $S := R_{\mathfrak{q}}$, $M_i := (\Omega_{2m+2i}(M))_{\mathfrak{q}}$ for $i \geq 0$ and $\mathfrak{n} := \mathfrak{q}R_{\mathfrak{q}}$. Note that $\mathfrak{n}$ is the maximal ideal for S . For all $i \geq 0$ we obtain a commutative diagram of the form

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M_{i+1} & \longrightarrow & S^{b_i} & \xrightarrow{\alpha_i} & S^{b_i} \longrightarrow M_i \longrightarrow 0 \\ & & & & \searrow \phi_i & & \nearrow \psi_i \\ & & & & & N_i & \end{array}$$

where the top row is exact and $N_i := \text{Im}(\alpha_i)$. If the matrix A_i defining the map $\alpha_i : S^{b_i} \rightarrow S^{b_i}$ has some entries which are units, then by Fact 7 we can reduce this sequence by taking away free summands; hence we may assume that A_i has all of its entries in $\mathfrak{n}$.

Let $H_{\mathfrak{n}}^i(-)$ denote the i th local cohomology functor with respect to $\mathfrak{n}$. For background on local cohomology see [7]. Since M_{i+1} has finite length, $H_{\mathfrak{n}}^0(M_{i+1}) \cong M_{i+1}$ and $H_{\mathfrak{n}}^j(M_{i+1}) = 0$ for all $j > 0$. From the long exact sequence of local cohomology modules associated to the short exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow S^{b_i} \xrightarrow{\phi_i} N_i \longrightarrow 0,$$

we get an exact sequence

$$(3) \quad 0 \longrightarrow M_{i+1} \longrightarrow H_{\mathfrak{n}}^0(S^{b_i}) \longrightarrow H_{\mathfrak{n}}^0(N_i) \longrightarrow 0$$

and isomorphisms $H_{\mathfrak{n}}^j(\phi_i) : H_{\mathfrak{n}}^j(S^{b_i}) \rightarrow H_{\mathfrak{n}}^j(N_i)$ for all $j \geq 1$. Also the exact sequence

$$0 \longrightarrow N_i \xrightarrow{\psi_i} S^{b_i} \longrightarrow M_i \longrightarrow 0$$

yields an exact sequence

$$(4) \quad 0 \longrightarrow H_{\mathfrak{n}}^0(N_i) \longrightarrow H_{\mathfrak{n}}^0(S^{b_i}) \longrightarrow M_i \xrightarrow{\gamma_i} H_{\mathfrak{n}}^1(N_i) \longrightarrow H_{\mathfrak{n}}^1(S^{b_i}) \longrightarrow 0$$

and isomorphisms $H_{\mathfrak{n}}^j(\psi_i) : H_{\mathfrak{n}}^j(N_i) \rightarrow H_{\mathfrak{n}}^j(S^{b_i})$ for all $j \geq 2$. Here we are defining $\gamma_i : M_i \rightarrow H_{\mathfrak{n}}^1(N_i)$ to be the map found in exact sequence (4). By the additivity of length we get the first and third steps in the next display from sequences (4) and (3) respectively:

$$\begin{aligned} \lambda(M_i) &= \lambda(H_{\mathfrak{n}}^0(S^{b_i})) - \lambda(H_{\mathfrak{n}}^0(N_i)) + \lambda(\text{Im}(\gamma_i)) \\ &\geq \lambda(H_{\mathfrak{n}}^0(S^{b_i})) - \lambda(H_{\mathfrak{n}}^0(N_i)) \\ &= \lambda(M_{i+1}). \end{aligned}$$

Since the sequence $(\lambda(M_i))_{i=0}^\infty$ is positive and non-increasing it is eventually constant. Choose $\ell \in \mathbb{N}$ such that $\lambda(M_\ell) = \lambda(M_{\ell+1})$. Then $\lambda(\text{Im}(\gamma_\ell)) = 0$. Therefore γ_ℓ is the zero map. From (4), it follows that $H_{\mathfrak{n}}^1(\psi_\ell) : H_{\mathfrak{n}}^1(N_\ell) \rightarrow H_{\mathfrak{n}}^1(S^{b_\ell})$ is an isomorphism. We have shown that $H_{\mathfrak{n}}^j(\psi_\ell)$ and $H_{\mathfrak{n}}^j(\phi_\ell)$ are isomorphisms for all $j \geq 1$. Using the commutativity of (2) it follows that

$$H_{\mathfrak{n}}^j(\alpha_\ell) = H_{\mathfrak{n}}^j(\psi_\ell) \circ H_{\mathfrak{n}}^j(\phi_\ell) : H_{\mathfrak{n}}^j(S^{b_\ell}) \rightarrow H_{\mathfrak{n}}^j(S^{b_\ell})$$

is an isomorphism for all $j \geq 1$. Since $H_{\mathfrak{n}}^j(-)$ is an S -linear functor the map $H_{\mathfrak{n}}^j(\alpha_\ell)$ is defined by matrix multiplication from the matrix A_ℓ applied to the components of $H_{\mathfrak{n}}^j(S^{b_\ell})$. Since A_ℓ has entries in $\mathfrak{n}$ it must kill socle elements of $H_{\mathfrak{n}}^j(S^{b_\ell})$. Therefore $H_{\mathfrak{n}}^j(S^{b_\ell})$ has no socle elements. Since $H_{\mathfrak{n}}^j(S^{b_\ell})$ is $\mathfrak{n}$ -torsion it follows that $H_{\mathfrak{n}}^j(S^{b_\ell}) = 0$ for all $j \geq 1$. By [7, Theorem 9.3] we get the second equality in the next display:

$$\dim(R_{\mathfrak{q}}) = \dim(S) = \sup\{j \mid H_{\mathfrak{n}}^j(S) \neq 0\} = 0.$$

Thus $\mathfrak{q} \in \text{Min}(R)$; hence $\text{Min}(\Omega_{2i}(M)) \subseteq \text{Min}(R)$ for all $i \gg 0$, and (a) follows. $\square$

Corollary 9. *Let R be a noetherian local ring and M a finitely generated R -module with eventually non-decreasing Betti numbers. Then $(\dim(\Omega_{2i}(M)))_{i=0}^\infty$ and $(\dim(\Omega_{2i+1}(M)))_{i=0}^\infty$ are constant for $i \gg 0$. If $\text{pd}(M) = \infty$, then one sequence stabilizes to $\dim(R)$ and the other sequence stabilizes to $\dim(R/\mathfrak{p})$ for some $\mathfrak{p} \in \text{Min}(R)$.*

Proof. By Theorem 8(b) both sequences are constant for $i \gg 0$. If $\text{pd}(M) = \infty$, then $\text{Min}(\Omega_i(M)) \neq \emptyset$ for all i . Therefore by Theorem 8(a) one sequence will stabilize to $\dim(R/\mathfrak{p})$ and the other to $\dim(R/\mathfrak{q})$ for some $\mathfrak{p}, \mathfrak{q} \in \text{Min}(R)$. Since

$$\text{Supp}(\Omega_{2i}(M)) \cup \text{Supp}(\Omega_{2i+1}(M)) = \text{Spec}(R),$$

it follows that $\dim(\Omega_{2i}(M)) = \dim(R)$ or $\dim(\Omega_{2i+1}(M)) = \dim(R)$. $\square$

Corollary 2 follows immediately. Note that if R is a domain or if $\dim(R) \leq 1$, then R is equidimensional; hence, one can apply Corollary 2.

Remark 10. It should be noted that [5, Remark 5.6] claims that using [5, Proposition 5.5] one can show that if R is equidimensional and Conjecture 4 is true, then $\dim(\Omega_n(M))$ is constant for $n \gg 0$. However, [5, Proposition 5.5] requires the assumption that $\dim(R) \geq 2$. Therefore although the conclusions of [5, Remark 5.6] are correct, the justification given for these conclusions is invalid. One should note that the justification uses a localization argument, so it is invalid in every positive dimension, not just dimension 1.

We now turn our attention to determining how quickly $(\text{Supp}(\Omega_{2i}(M)))_{i=0}^\infty$ stabilizes once the Betti numbers of M become non-decreasing.

Lemma 11. *Let R be a noetherian local ring and M a finitely generated R -module. If $\beta_i(M) = \beta_{i+1}(M)$ for some $i > 0$, then $\text{Supp}(\Omega_i(M)) = \text{Supp}(\Omega_{i+2}(M))$.*

Suppose $\beta_0(M) = \beta_1(M)$. Then we have the following:

- (a) *If $\text{Supp}(M) \setminus \text{Supp}(\Omega_2(M)) \neq \emptyset$, then M is not a first syzygy.*
- (b) *If $\mathfrak{p} \in \text{Min}(M) \setminus \text{Supp}(\Omega_2(M))$, then $\text{height}(\mathfrak{p}) = 1$.*

Proof. Suppose $\beta_0(M) = \beta_1(M)$ and $\text{Supp}(M) \setminus \text{Supp}(\Omega_2(M)) \neq \emptyset$. Consider the exact sequence

$$0 \rightarrow \Omega_2(M) \rightarrow R^{\beta_1(M)} \rightarrow R^{\beta_0(M)} \rightarrow M \rightarrow 0.$$

Choose $\mathfrak{p} \in \text{Min}(M) \setminus \text{Supp}(\Omega_2(M))$. Since $M_{\mathfrak{p}}$ has finite length as an $R_{\mathfrak{p}}$ -module, the complex

$$0 \rightarrow R_{\mathfrak{p}}^{\beta_1(M)} \rightarrow R_{\mathfrak{p}}^{\beta_0(M)} \rightarrow 0$$

has non-zero finite length homology. By the New Intersection Theorem [10] it follows that $\dim(R_{\mathfrak{p}}) \leq 1$.

Fact 7 implies that there exists a minimal $R_{\mathfrak{p}}$ -free resolution of $M_{\mathfrak{p}}$ of the form

$$0 \rightarrow R_{\mathfrak{p}}^n \rightarrow R_{\mathfrak{p}}^n \rightarrow M_{\mathfrak{p}} \rightarrow 0$$

for some $n > 0$. Therefore $1 \geq \dim(R_{\mathfrak{p}}) \geq \text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 1$; hence $\text{height}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) = 1$, $\text{depth}(M_{\mathfrak{p}}) = 0$ and $\text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = \text{depth}(M_{\mathfrak{p}}) + \text{pd}(M_{\mathfrak{p}}) = 1$.

Assume that $M = \Omega_1(L)$ for some R -module L . We will obtain a contradiction. Since $M_{\mathfrak{p}}$ is finite length and $\dim(R_{\mathfrak{p}}) = 1$, it follows that $M_{\mathfrak{p}}$ has no $R_{\mathfrak{p}}$ -free summands. Therefore $M_{\mathfrak{p}} = \Omega_1^{R_{\mathfrak{p}}}(L_{\mathfrak{p}})$. Since $M_{\mathfrak{p}}$ is a first syzygy,

$$0 = \text{depth}(M_{\mathfrak{p}}) \geq \min\{1, \text{depth}(R_{\mathfrak{p}})\} = 1.$$

This is a contradiction; hence M is not a first syzygy.

Now suppose that $\beta_i(M) = \beta_{i+1}(M)$ for some $i > 0$. Since $\Omega_i(M)$ is a first syzygy of $\Omega_{i-1}(M)$ it follows that $\text{Supp}(\Omega_i(M)) \subseteq \text{Supp}(\Omega_{i+2}(M))$. By Lemma 6 we get the opposite inclusion and the result follows. $\square$

The following is an example where we have $\beta_1(M) = \beta_0(M)$ and $\text{Supp}(M) \not\subseteq \text{Supp}(\Omega_2(M))$.

Example 12. Let $S = k[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. Let $R = S_{\mathfrak{m}}/yzS_{\mathfrak{m}}$ and let $M = R/xyR$. The complex

$$\cdots \xrightarrow{z} R \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{xy} R \longrightarrow 0$$

is a minimal free resolution of M . We have $\Omega_2(M) \cong zR \cong R/(y)$. The prime ideal $\mathfrak{p} = (x, z)$ of height 1 is in $\text{Supp}_R(M)$, but it is not in $\text{Supp}_R(\Omega_2(M))$.

Proposition 13. *Let R be a noetherian local ring and M a finitely generated R -module with non-decreasing Betti numbers. Then either $\text{Supp}(\Omega_{2i}(M))$ is constant for all $i \geq 1$ or there exists $n \geq 1$ such that $\text{Supp}(\Omega_{2i}(M)) = \text{Spec}(R)$ for all $i > n$ and $\text{Supp}(\Omega_{2j}(M))$ is constant for $1 \leq j \leq n$.*

Proof. Suppose $\text{Supp}(\Omega_{2i}(M)) \neq \text{Spec}(R)$ for some $i \geq 2$. By Lemma 6 it follows that $\beta_{2j}(M) = \beta_{2j+1}(M)$ for all j with $0 \leq j < i$. From Lemma 11 we get that $\text{Supp}(\Omega_{2j}(M))$ is constant for $1 \leq j \leq i$.

Now suppose there exists n such that $\text{Supp}(\Omega_{2n+2}(M)) = \text{Spec}(R)$. Assume that $\text{Supp}(\Omega_{2n+2i}(M)) \neq \text{Spec}(R)$ for some $i > 0$. Then by Lemma 6(c) it follows that

$$\text{Supp}(\Omega_{2n+2i}(M)) \cap \text{Min}(R) = \text{Supp}(\Omega_{2n+2}(M)) \cap \text{Min}(R) = \text{Min}(R).$$

Thus $\text{Supp}(\Omega_{2n+2i}(M)) = \text{Spec}(R)$, a contradiction. Therefore $\text{Supp}(\Omega_{2n+2i}(M)) = \text{Spec}(R)$ for all $i > 0$ and the result follows. $\square$

We conclude with a few key examples. The following example is due to Hamid Rahmati and can be found in [5].

Example 14. Let $R = k[[x, y]]/(x^2, xy)$ and $M = R/(y)$. A minimal free resolution of M has the form

$$\cdots \xrightarrow{\begin{bmatrix} x & y & 0 \\ 0 & 0 & x \end{bmatrix}} R^2 \xrightarrow{[x, y]} R \xrightarrow{x} R \xrightarrow{y} R \longrightarrow 0.$$

Also $\dim(M) = \dim(\Omega_2(M)) = 0$ and $\dim(\Omega_i(M)) = 1 = \dim(R)$ for $i \neq 0, 2$.

In the following example we construct a module such that the support of each of its odd syzygies is equal to $\operatorname{Spec}(R)$ while the support of each of its even syzygies is not.

Example 15. Let $R = [a, b, c, d, e]/(ade - bce)$. Let M be the cokernel of the first map in the following matrix factorization:

$$\cdots \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} de & -be \\ -ce & ae \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} R^2 \cdots$$

Then $\operatorname{Supp}(\Omega_{2i+1}(M)) = \operatorname{Spec}(R)$ and $\operatorname{Supp}(\Omega_{2i}(M)) = \operatorname{Supp}(R/(ad - bc)) \neq \operatorname{Spec}(R)$ for all $i \geq 0$.

The following example is due to Craig Huneke and can be found in [5].

Example 16. Let $S = \mathbb{Q}[x, y, z, u, v]$ and let $I \subseteq S$ be the ideal

$$I = \langle x^2, xz, z^2, xu, zv, u^2, v^2, zu + xv + uv, yu, yv, yx - zu, yz - xv \rangle.$$

Let $R = S/I$, which is a 1-dimensional ring of depth 0. A computation using Macaulay2 [6] yields that y is a parameter, $(0 : y) = (u, v, z^2)$ and $(y) = (0 :_R (0 :_R y))$. Let M be the cokernel of the rightmost map in the following exact complex:

$$\cdots \longrightarrow R^3 \xrightarrow{\begin{bmatrix} u & v & z^2 \end{bmatrix}} R \xrightarrow{y} R \xrightarrow{\begin{bmatrix} u \\ v \\ z^2 \end{bmatrix}} R^3 \cdots$$

Then the first and third syzygy modules of M are $R/(y)$ and $(0 : y)$ respectively. These are both modules of finite length since y is a parameter, but all other syzygies have dimension 1.

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REFERENCES

- [1] L. L. Avramov, *Local algebra and rational homotopy*, Homotopie algébrique et algèbre locale; Luminy, 1982 (J.-M. Lemaire, J.-C. Thomas, eds.), Astérisque **113-114**, Soc. Math. France, Paris, 1984, 15-43. MR749041 (85j:55021)
- [2] L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Prog. Math. **166**, Birkhäuser, Basel, 1998, 1-118. MR1648664 (99m:13022)
- [3] L. L. Avramov, V. N. Gasharov, I. V. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. **86** (1997), 67-114. MR1608565 (99c:13033)
- [4] S. Choi, *Betti numbers and the integral closure of an ideal*, Math. Scand. **66** (1990), 173-184. MR1075135 (91m:13007)
- [5] A. Crabbe, D. Katz, J. Striuli, E. Theodorescu, *Hilbert-Samuel polynomials for the contravariant extension functor*, Nagoya Math. J. **198** (2010), 1-22. MR2666575 (2011g:13033)

- [6] D. R. Grayson, M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, <http://www.math.uiuc.edu/Macaulay2/>
- [7] S. B. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh, and U. Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, **87**, American Mathematical Society, Providence, RI, 2007. MR2355715 (2009a:13025)
- [8] J. Lescot, *Séries de Poincaré et modules inertes*, J. Algebra **132** (1990), 22-49. MR1060830 (91k:13010)
- [9] S. Okiyama, *A local ring is CM if and only if its residue field has a CM syzygy*, Tokyo J. Math. **14** (1991), 489-500. MR1138183 (92m:13017)
- [10] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sci., Paris Sér. I **304** (1987), 177-180. MR880574 (89b:14008)
- [11] L.-C. Sun, *Growth of Betti numbers of modules over rings of small embedding codimension or small linkage number*, J. Pure Appl. Algebra **96** (1994), 57-71. MR1297441 (95j:13014)
- [12] L.-C. Sun, *Growth of Betti numbers of modules over generalized Golod rings*, J. Algebra **199** (1998), 88-93. MR1489355 (98i:13023)

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