ROBUSTLY NON-HYPERBOLIC TRANSITIVE ENDOMORPHISMS ON $\mathbb{T}^2$

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ABSTRACT. We prove that for any regular endomorphism f on a 2-torus $\mathbb{T}^2$ which is not one to one, there is a regular map g homotopic to f such that g is C^1 robustly non-hyperbolic transitive. We also introduce interesting blender phenomena (a fat horseshoe) of 2-dimensional endomorphisms, which play an important role in our construction of some examples.

1. INTRODUCTION

The topic on robustly transitive diffeomorphisms attracts many people's attention. In the past several decades, people made much important progress on it. On one hand, Shub [Sh2], Mañé [M1], Bonatti and Díaz [BD] gave several classes of robustly non-hyperbolic transitive diffeomorphisms respectively. However, the known robustly transitive examples are still very few. On the other hand, in [DPU] and [BDP] it is proved that robustly transitive diffeomorphisms have some weak hyperbolicity: they have dominated splittings and their extremal bundles are volume hyperbolic. We think that they should have more dynamical consequences. For example, under some conditions their extremal bundles may be integrable. Also, there is other progress on it. For more details, readers can refer to the survey [HHU].

It is well known that C^1 robustly transitive diffeomorphisms on any compact smooth surface are Anosov [M2]. However, Sumi [Su] gave a C^1 robustly non-hyperbolic transitive regular endomorphism on $\mathbb{T}^2$ in the homotopic class of f_A , which is induced by $A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$. Let $End^*(\mathbb{T}^2)$ be the set of all regular endomorphisms on $\mathbb{T}^2$ (C^1 endomorphisms and determinants of their derivatives $\neq 0$) which are not one to one. In this paper, we construct robustly non-hyperbolic transitive regular endomorphisms of $\mathbb{T}^2$ in all homotopic classes of $End^*(\mathbb{T}^2)$.

Theorem 1.1. *For any $f \in End^*(\mathbb{T}^2)$, there exists $g \in End^*(\mathbb{T}^2)$ such that g is homotopic to f and robustly non-hyperbolic transitive.*

The homotopic classes of $End^*(\mathbb{T}^2)$ can be divided into two kinds: the one containing hyperbolic linear endomorphisms and the other containing non-hyperbolic linear endomorphisms. For the first kind, robustly non-hyperbolic transitive endomorphisms are derived Anosov endomorphisms ([Su]). For the other kind, we

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introduce a blender of 2-dimensional endomorphisms (Definition 3.1) to construct robustly non-hyperbolic transitive endomorphisms.

In [ABD], Abdenur, Bonatti and Díaz brought up the following problem:

Conjecture 1. *There is a residual subset $\mathcal{R}$ of $\text{Diff}^1(M)$ such that for any $f \in \mathcal{R}$, if the non-wandering set $\Omega(f)$ has non-empty interior, then f is transitive.*

The conjecture is folklore when the non-wandering set is hyperbolic. It is also proved for diffeomorphisms on any compact surface in [ABCD]. But, the answer is negative for 2-dimensional endomorphisms by the following proposition, which is given in [BKRU, Proposition 7.1].

Proposition 1.2. *There is an open subset $\mathcal{U}$ of $\text{End}^*(\mathbb{T}^2)$ such that every $f \in \mathcal{U}$ has a hyperbolic attractor ($\neq \mathbb{T}^2$) containing non-empty interior.*

In our paper, it is a direct application of a blender of endomorphisms. The term blender (of diffeomorphisms) was first introduced by Bonatti and Díaz ([BD]) in order to construct new robustly non-hyperbolic transitive diffeomorphisms. A blender of endomorphisms also plays a similar role in our construction of examples. It is extracted from the following linear model. Let $f : C = [-1, 1]^2 \rightarrow \mathbb{R}^2$,

$$f(x_1, x_2) = \begin{cases} (kx_1, \lambda x_2), & x_1 \in [-\frac{1}{k}, \frac{1}{k}], \\ (kx_1 - 3, \lambda x_2 + t), & x_1 \in [\frac{2}{k}, \frac{4}{k}], \end{cases}$$

where $t > 0$, $\lambda \in (\frac{1}{2}, 1)$ satisfying $1 - \lambda < t < \lambda$, and $k > 4$. The picture of (C, f) looks like a fat horseshoe. Note that $f(([-\frac{1}{k}, \frac{1}{k}] \cup [\frac{2}{k}, \frac{4}{k}]) \times [-1, 1]) \supset [-1, 1] \times [0, 1]$. By simple calculation, it is not difficult to show that the closure of the unstable manifold of $(0, 0)$ contains the open set $(-1, 1) \times (0, t)$. In fact, such property is C^1 -persistent (Proposition 3.2). This model is motivated by an example in [D]. The example is as follows: $f : C \times [-1, 1] \rightarrow \mathbb{R}^2 \times \mathbb{R}$,

$$f(x_1, x_2, x_3) = \begin{cases} (kx_1, \lambda x_2, \frac{1}{k}x_3), & x_1 \in [-\frac{1}{k}, \frac{1}{k}], \\ (kx_1 - 3, \lambda x_2 + t, \frac{1}{k}x_3 + \frac{1}{2}), & x_1 \in [\frac{2}{k}, \frac{4}{k}], \end{cases}$$

where t, λ and k are the same as those in the above example. It is a linear model of Bonatti and Díaz's blender in [BD]. Their blender has the following property: the projection of the closure of an unstable manifold of $(0, 0, 0)$ into the plane $\mathbb{R}^2 \times \{0\}$ contains a non-empty interior of $\mathbb{R}^2 \times \{0\}$ (Lemma 1.8 in [BD]).

In some sense, the linear model of our blender is a special case of the endomorphisms $A_{\lambda, \tau}$ on the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$: $A_{\lambda, \tau}(\theta, t) = (k\theta, \lambda t + \tau(\theta))$, where k is a positive integer, $\lambda \in (0, 1)$ and τ is a continuous function. Tsujii [T] first considered these from a probabilistic perspective. Later, Bamón, Kiwi, Rivera-Letelier and Urzúa [BKRU] also studied their topological properties. Also, these endomorphisms on a cylinder can be regarded as a generalization of fat baker's transformations ([AY]).

Here, we give some description on the construction of robustly non-hyperbolic transitive endomorphisms in the homotopic classes of $\text{End}^*(\mathbb{T}^2)$, which contains non-hyperbolic linear endomorphisms. We (C^1) perturb a non-hyperbolic linear endomorphism to get a blender and, at the same time, control the perturbed dynamics with some stable global properties to make it robustly transitive (Example 3.6). This is just the principle of constructing robustly non-hyperbolic transitive diffeomorphisms in [BD].

In the end, we make some comparisons between regular endomorphisms and diffeomorphisms. Of course, some local arguments of diffeomorphisms, e.g., λ -lemma and the shadowing lemma of hyperbolic sets ([L]), are still valid for regular endomorphisms. But hyperbolic sets are generally not stable (see Remark 3.5). Also, the normal hyperbolic theory is not always valid for endomorphisms. For example, the set $S^1 \times \{0\}$ of f_0 in the proof of Proposition 1.2 is normally contracting and a hyperbolic attractor, but a small perturbation of f_0 may have an attractor with non-empty interior. But in the normally expanding case, Berger [Be] proved some positive results.

It is natural to ask “do robustly transitive endomorphisms have any weak hyperbolicity?” Since they can have sources, we cannot expect them to have dominated splitting. Fortunately, the ergodic closing lemma ([Mo]) is still valid for regular endomorphisms. Then it is not difficult to show that “they (not one to one) are volume expanding.” For details, the reader can refer to [LP], which also gives a sufficient condition for a transitive endomorphism to be robustly transitive.

2. DEFINITIONS AND NOTATION

In this paper, *manifolds* are C^∞ smooth compact Riemannian manifolds without boundary, and *maps* are C^1 smooth maps.

Let M be a compact Riemannian manifold, $\|\cdot\|$ be a Riemannian metric on M , and $End(M)$ be the set of all (C^1) regular endomorphisms on M equipped with the usual C^1 topology. Note that $End(M)$ contains diffeomorphisms. Given $f \in End(M)$, denote by Df the derivative operator of f . Since the determinant of Df is not equal to zero, f is a covering map.

Let E and F be two sub-bundles of TM satisfying $E \oplus F = TM$. Define two cone fields $\mathcal{C}_\alpha^E$ and $\mathcal{C}_\alpha^F$ on TM :

$$\begin{aligned}\mathcal{C}_\alpha^E(z) &= \{v \in T_z M \mid v = (v^E, v^F), \|v^F\| < \alpha \|v^E\|\}, \\ \mathcal{C}_\alpha^F(z) &= \{v \in T_z M \mid v = (v^E, v^F), \|v^E\| < \alpha \|v^F\|\}.\end{aligned}$$

Definition 2.1. An invariant subset Λ ($f(\Lambda) = \Lambda$) of f is said to be hyperbolic if $T\Lambda$ has the bundle splitting $T_\Lambda M = E \oplus F$ and there is $\alpha > \alpha' > 0$ and $\lambda > 1$ such that, for any $x \in \Lambda$ and any unit vector $v \in \mathcal{C}_\alpha^F(x)$,

$$Df(\mathcal{C}_\alpha^F(x)) \subset \mathcal{C}_{\alpha'}^F(f(x)), \quad \|Df(v)\| > \lambda,$$

and such that for any $x \in \Lambda$, any unit vector $v \in \mathcal{C}_\alpha^E(f(x))$, and the diffeomorphism g on a neighborhood of $f(x)$ satisfying that gf is the identity on a neighborhood of x ,

$$Dg(\mathcal{C}_\alpha^E(f(x))) \subset \mathcal{C}_{\alpha'}^E(x), \quad \|Dg(v)\| > \lambda.$$

We say that f is *Anosov* if f is hyperbolic on M . Also, f is said to be an *expanding* map if it is an Anosov map without contracting direction ($E = 0$).

Suppose p is a hyperbolic fixed point of f . Then there is a neighborhood V of p and a neighborhood $\mathcal{U}$ of f in $End(M)$ such that for any $g \in \mathcal{U}$, g has the unique hyperbolic fixed point p_g in V (the continuation of $p_f = p$), p_g has a local C^1 stable (*unstable*) manifold $W_{loc}^s(p_g, g)$ ($W_{loc}^u(p_g, g)$) of a uniform size, and the local stable

and unstable manifolds of p_g continuously vary with g . Let the *stable manifold* of p be

$$W^s(p, f) = \bigcup_{n=0}^{+\infty} f^{-n}(W_{loc}^s(p, f)) \quad (= \{x \in M \mid \lim_{n \rightarrow \infty} d(f^n(x), p) = 0\})$$

and the *unstable manifold* of p be

$$W^u(p, f) = \bigcup_{n=0}^{+\infty} f^n(W_{loc}^u(p, f)).$$

Remark 2.2. Generally, the stable and unstable manifolds are not injective submanifolds. The definition of hyperbolicity is equivalent to the one in [P]. Let $\{x_n\}_{n=-\infty}^{+\infty}$ be an orbit of f . The Df -invariant expanding space

$$E^u(\{x_n\}) = \bigcap_{n=0}^{\infty} Df^n(\mathcal{C}_{\alpha}^F(x_{-n}))$$

depends upon the choice of negative orbits. So, Przytycki [P] considered local unstable manifolds of orbits, which have similar properties with those in the classic hyperbolic theory of diffeomorphisms. The unstable manifold of a hyperbolic fixed point p defined here is just the unstable manifold of the orbit $(\dots, p, p, \dots)$ in [P].

A compact invariant set Λ of f is said to be *transitive* if there is a positive orbit $\{f^n x \mid n \geq 0\}$ dense in Λ . It is equivalent to one of the following conditions: (i) for any non-empty open subset U of Λ , $\bigcup_{n=0}^{\infty} f^n(U) = \Lambda$; (ii) for any non-empty open subset U of Λ , $\bigcup_{n=0}^{\infty} f^{-n}(U) = \Lambda$.

A compact invariant set Λ is said to be a *hyperbolic attractor* if $\Lambda \neq M$ is hyperbolic and transitive and if there is an open subset U of M satisfying $f(\overline{U}) \subset U$ and $\bigcap_{n=0}^{\infty} f^n(\overline{U}) = \Lambda$. The corresponding open set U is said to be a *trapping region* of the attractor Λ .

We say that f is *robustly non-hyperbolic transitive* if there is a neighborhood $\mathcal{U}(f)$ of f in $\text{End}(M)$ such that every $g \in \mathcal{U}(f)$ is transitive and not Anosov.

For a matrix A , denote all eigenvalues of A by $\sigma(A)$. A map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is said to be a τ -*periodic map* if $f(x+v) = f(x)$ for any $x \in \mathbb{R}^2, v = (i\tau, j\tau), (i, j) \in \mathbb{Z}^2$. Let P be a 1-periodic map on $\mathbb{R}^2$ and $A \in GL(2, \mathbb{Z})$. Then $F = A + P$ induces a unique endomorphism f on $\mathbb{T}^2$ satisfying $\pi F = f\pi$, where π is the standard covering map from $\mathbb{R}^2$ to $\mathbb{T}^2$. The endomorphism on $\mathbb{T}^2$ induced by a linear map $A \in GL(2, \mathbb{Z})$ is denoted by f_A . For any endomorphism f on $\mathbb{T}^2$, there is $A \in GL(2, \mathbb{Z})$ and a lifting $F = A + P$ (P is 1-periodic) satisfying $\pi F = f\pi$ (see [AP]). Note that f is homotopic to f_A in the space of all continuous endomorphisms on $\mathbb{T}^2$. Then according to the linear representative f_A , we divide the homotopic classes of $\text{End}^*(\mathbb{T}^2)$ into three types:

- (i) $\sigma(A) = \{\lambda_1, \lambda_2\}, |\lambda_1| < 1 < |\lambda_2|$.
- (ii) $\sigma(A) = \{\lambda_1, \lambda_2\}, |\lambda_1| = 1, |\lambda_2| = k > 1$.
- (iii) $\sigma(A) = \{\lambda_1, \lambda_2\}, 1 < |\lambda_1| \leq |\lambda_2|$.

The robustly non-hyperbolic transitive regular endomorphisms on $\mathbb{T}^2$ given in [Su] belong to the first type. Following his construction and proof, we can get the same conclusion for all homotopic classes of $\text{End}^*(\mathbb{T}^2)$ of type (i). To prove Theorem 1.1, we will need to deal only with type (ii) (Proposition 3.7) and type (iii) (Proposition 4.1).

3. A BLENDER OF ENDOMORPHISMS

In this section, we give the definition of a blender of endomorphisms and use blender phenomena to construct two examples on $\mathbb{T}^2$: (1) robustly hyperbolic attractors with non-empty interior; (2) robustly non-hyperbolic transitive regular endomorphisms of type (ii).

Let N^u, N^{cs} be two smooth Riemannian manifolds, $M = N^u \times N^{cs}$. Then $TM = T(N^u) \oplus T(N^{cs})$. Also, let $\mathcal{C}_\alpha^u$ and $\mathcal{C}_\alpha^{cs}$ be two cone fields around the subbundles $T(N^u)$ and $T(N^{cs})$ respectively. A $\mathcal{C}_\alpha^u$ ($\mathcal{C}_\alpha^{cs}$) curve γ is a smooth curve on M satisfying the tangent space $T\gamma \subset \mathcal{C}_\alpha^u$ ($T\gamma \subset \mathcal{C}_\alpha^{cs}$). In this section, $N^u = N^{cs}$ will be an interval of $\mathbb{R}$ or S^1 .

Let $C = [-1, 1]^u \times [-1, 1]^{cs}$, $u = cs = 1$. A smooth $\mathcal{C}_\alpha^u$ curve $\gamma \subset [-1, 1] \times (-1, 1)$ is said to be a u -curve of C if $\gamma \cap \{\pm 1\} \times [-1, 1] = \partial\gamma$.

Definition 3.1. Let $f : C \rightarrow \mathbb{R}^2$ be a C^1 map. (C, f) is said to be a blender if there exist two compact subsets A_1 and A_2 of $(-1, 1)^2$ such that:

- 1) $f|_{A_i}$ is a diffeomorphism onto its image, which is denoted by $f_i, i = 1, 2$.
- 2) f is hyperbolic on $A_1 \cup A_2$ (the cone fields $\mathcal{C}^u$ and $\mathcal{C}^{cs}$ are defined in $\mathbb{R}^2$). More precisely, there is $0 < \alpha' < \alpha$ and $\lambda > 1$ such that, for any $x \in f_i(A_i)$ and any unit vector $v \in \mathcal{C}_\alpha^{cs}(x)$, one has

$$Df_i^{-1}(\mathcal{C}_\alpha^{cs}(x)) \subset \mathcal{C}_{\alpha'}^{cs}(f_i^{-1}x), \quad \|Df_i^{-1}(v)\| > \lambda, \quad i = 1, 2,$$

and for any $x \in A_1 \cup A_2$ and any unit vector $v \in \mathcal{C}_\alpha^{cs}(x)$, one has

$$Df(\mathcal{C}_\alpha^u(x)) \subset \mathcal{C}_\alpha^u(fx), \quad \|Df(v)\| > \lambda.$$

- 3) There are two u -curves γ_1 and γ_2 of C such that $f(\gamma_1 \cap A_i) \supset \gamma_i, i = 1, 2$, and γ_2 is strictly above γ_1 .
- 4) Let V_1 be the closed subset of C between γ_1 and γ_2 , and let V_2 be the closed subset of C not below the curve γ_2 . Also, let A'_i be the closed subset of A_i not below the curve $\gamma_1, i = 1, 2$. $f(A'_i)$ contains $V_i, i = 1, 2$.

Now we check that the C^1 map $f : [-1, 1]^2 \rightarrow \mathbb{R}^2$ in the introduction is a blender. Take $0 < a < 1$ satisfying that $1 - \lambda a < t < \lambda a$. Let $A_1 = [-\frac{1}{k}, \frac{1}{k}] \times [-a, a]$ and $A_2 = [\frac{2}{k}, \frac{4}{k}] \times [-a, a]$. It is not difficult to show that $f|_{A_1 \cup A_2}$ satisfies all conditions of a blender.

Note that $f(\gamma_1 \cap A_1) \supset \gamma_1$ and f uniformly expands on the curve $f_1^{-1}(\gamma_1) (\subset \gamma_1)$. Then f has a hyperbolic fixed saddle $p \in \gamma_1$ and $\gamma_1 \cup \gamma_2$ is contained in some positive iteration of f on a local unstable manifold of p . For convenience, γ_1 is also said to be a local unstable manifold of p . Now we show two properties of a blender: the closure of the unstable manifold of p contains a non-empty interior, and a blender phenomenon is C^1 persistent.

Proposition 3.2. *If (C, f) is a blender, then:*

- (1) $\overline{W^u(p, f)}$ contains the subset of C above the local unstable manifold γ_1 of p ;
- (2) there exists a neighborhood $\mathcal{U}(f)$ of f in $C^1(C, \mathbb{R}^2)$ such that for any $g \in \mathcal{U}(f)$, (C, g) is a blender.

Proof. (1) Suppose on the contrary that there exists a $\mathcal{C}_\alpha^{cs}$ curve γ^0 in the region $V_1 \cup V_2$ such that $\gamma^0 \cap \overline{W^u(p, f)} = \emptyset$. Then $f^{-n}(\gamma^0) \cap \gamma_2 = \emptyset$ for any integer $n \geq 0$. Suppose $\gamma^0 \subset V_i (\subset f(A'_i)), i = 1$ or 2 . Then the $\mathcal{C}_\alpha^{cs}$ curve

$\gamma^1 = f_i^{-1}(\gamma^0) \subset A'_i \subset V_1 \cup V_2$ and $\ell(\gamma^1) > \lambda\ell(\gamma^0)$. Inductively, $f^{-n}(\gamma^0)$ contains a C_α^{cs} curve γ^n for any integer $n > 0$, and the length of the curve γ^n increases to infinity as $n \rightarrow +\infty$. A contradiction!

- (2) Note that $\gamma_1 \cup \gamma_2$ intersects transversely with $\{\pm 1\} \times [-1, 1]$ and $\gamma_1 \cup \gamma_2$ is contained in the finite-time iteration of f on the local unstable manifold of the hyperbolic fixed point p . Then there exist two closed neighborhoods B_1 and B_2 of A_1 and A_2 , respectively, and a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, one has that

- 1) $g|_{B_1 \cup B_2}$ satisfies conditions 1), 2) and 3) of a blender. Let $\sigma(g)$ be the continuation of σ , $\sigma = \gamma_1, \gamma_2, V_1, V_2$, and $A_i(g) = B_i, i = 1, 2$. Also, the definition of $A'_i(g)$ and g_i is parallel with A'_i and f_i respectively, $i = 1, 2$.

- 2) $g(A_i(g)) \supset V_i(g)$ and $g(A'_i(g))$ intersects with the interior of $V_i(g)$.

Note that for any $g \in \mathcal{U}$, $g_i^{-1}(g(A_i(g)) \cap \{\pm 1\} \times [-1, 1])$ consists of two disjoint C_α^{cs} curves, $i = 1, 2$. Combining this with $g_i^{-1}(\gamma_i(g)) \subset \gamma_1(g)$, $g(A_i(g)) \supset V_i(g)$ and $g(A'_i(g))$ intersects with the interior of $V_i(g)$, one has that $A'_i(g)$ contains $\partial(g_i^{-1}(V_i(g)))$, $i = 1, 2$. Therefore, $g(A'_i(g)) \supset V_i(g)$, $i = 1, 2$. $\square$

Remark 3.3. The blender phenomenon only relies on the map $f|_{A_1 \cup A_2}$, and the hyperbolicity on C_α^u ($Df|_{A_1 \cup A_2}$ preserves C_α^u and uniformly expands over C_α^u) is not necessary. We only demand that γ_1 is a finite-time iteration of f on a local unstable manifold of p and that $\gamma_1 \cup \gamma_2$ intersects transversely with $\{\pm 1\} \times [-1, 1]$ exactly in $\partial\gamma_1 \cup \partial\gamma_2$.

In this section, let $\mathbb{T}^2 = [-\frac{1}{2}, \frac{1}{2}]^2 / \mathbb{Z}^2 = S^1 \times S^1$. Here, we give an example of robustly hyperbolic attractors as an application of a blender. For the reader's convenience, we repeat its statement.

Proposition 1.2. *There is an open subset $\mathcal{U}$ of $\text{End}^*(\mathbb{T}^2)$ such that every $f \in \mathcal{U}$ has a hyperbolic attractor ($\neq \mathbb{T}^2$) containing non-empty interior.*

To prove the proposition, we need a lemma on transitivity.

Lemma 3.4. *Suppose Λ is a hyperbolic attractor of a regular endomorphism f on a manifold M and U is a trapping region of Λ . Then there is a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, g has a hyperbolic attractor $\Lambda_g = \bigcap_{n=0}^{+\infty} g^n(U)$.*

This can be deduced by the following facts:

- 1) three properties of hyperbolic orbit spaces: expansivity (Proposition 2.4 in [L]), shadowing property (Proposition 2.6 in [L]) and ε -stability of hyperbolic orbit spaces (Theorem 2.7a in [L]);
- 2) the local maximality of attractors;
- 3) for any compact invariant subset Δ of a regular endomorphism g on a manifold M , (Δ, g) is transitive iff $(\widetilde{\Delta}, \sigma_g)$ is transitive.

Here, the orbit space (inverse limit space) $\widetilde{\Delta}$ of an invariant subset Δ of g is the set $\{\tilde{x} = \{x_i\}_{-\infty}^{+\infty} | x_i \in \Delta, g(x_i) = x_{i+1}, i \in \mathbb{Z}\}$. Also, $g|_\Delta$ induces the left shift operator $\sigma_g : \widetilde{\Delta} \rightarrow \widetilde{\Delta}$ which satisfies $\pi\sigma_g = g\pi$, where π is the projection to its zeroth coordinate. Since the proof of the lemma is basic, we only give a sketch of it. The proof of the last fact is trivial. By the first two facts, we can deduce that there is a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, $(\widetilde{\Lambda_g}, \sigma_g)$ is topologically

conjugate to $(\widetilde{\Lambda}_f, \sigma_f)$. Its proof parallels that of diffeomorphisms. By the last fact, we see that (Λ_g, g) is an attractor.

Proof of Proposition 1.2. The construction is simple: just take a hyperbolic attractor with a blender. Let $k > 5$ be an integer, $I_1 = [-\frac{1}{4k}, \frac{1}{4k}]$, $I_2 = [\frac{3}{4k}, \frac{5}{4k}]$, $t > 0$ and $\frac{1}{2} < \lambda < 1$ satisfying $t < \frac{1-\lambda}{3}$. Also, take a smooth 1-periodic function α satisfying that $\alpha(I_1) = 0$, $\alpha(I_2) = 1$ and $0 \leq \alpha \leq 1$. Construct regular endomorphisms f_t on $\mathbb{T}^2$:

$$f_t(x_1, x_2) = (kx_1, \lambda x_2 + \alpha(x_1)t), \quad (x_1, x_2) \in S^1 \times [-\frac{1}{3}, \frac{1}{3}].$$

Since $0 \leq \alpha \leq 1$ and $0 < t < \frac{1-\lambda}{3}$, $f_t(S^1 \times [-\frac{1}{3}, \frac{1}{3}]) \subset S^1 \times [-\frac{\lambda}{3}, \frac{\lambda}{3} + t] \subset S^1 \times (-\frac{1}{3}, \frac{1}{3})$. Additionally, note that $\Lambda_{f_0} = \bigcap_{n=0}^{\infty} f_0^n(S^1 \times [-\frac{1}{3}, \frac{1}{3}])$ is a hyperbolic attractor of f_0 . So, by the above lemma, there is a neighborhood $\mathcal{U}_1$ of f_0 such that for any $g \in \mathcal{U}_1$, the compact invariant set $\Lambda_g = \bigcap_{n=0}^{\infty} g^n(S^1 \times [-\frac{1}{3}, \frac{1}{3}])$ is a hyperbolic attractor. Suppose $f = f_t \in \mathcal{U}_1$, and let $p_f = (0, 0)$. Take $a > 0$ satisfying $a - \lambda a < t < \lambda a$. Let $C = [-\frac{1}{4}, \frac{1}{4}] \times [-a, a]$ and $A_i = I_i \times [-a, a]$, $i = 1, 2$. We regard $f|_{A_1 \cup A_2}$ as a map from $A_1 \cup A_2$ to $[-\frac{1}{2}, \frac{1}{2}]^2$:

$$f(x_1, x_2) = \begin{cases} (kx_1, \lambda x_2), & (x_1, x_2) \in A_1, \\ (kx_1 - 1, \lambda x_2 + t), & (x_1, x_2) \in A_2. \end{cases}$$

So (C, f) is a blender. Consequently, by the properties of a blender, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of f such that for any $g \in \mathcal{U}$, the hyperbolic attractor $\Lambda_g \supset \overline{W^u(p_g, g)}$ ($g(W^u(p_g, g)) = W^u(p_g, g)$) contains a non-empty interior. $\square$

Remark 3.5. Note that the hyperbolic attractor Λ_{f_0} is not (structurally) stable. Generally, hyperbolic sets of endomorphisms are not stable. In [P], Przytycki pointed out the astonishing fact that “for any non-expanding and non-injective Anosov endomorphism, there are uncountable Anosov endomorphisms contained in any neighborhood of the Anosov endomorphism such that any two of them are not topologically conjugated.” The complexity of Anosov endomorphisms is mainly caused by the fact that the expanding direction relies on negative orbits. In [AH], Aoki and Hiraide classified some special Anosov endomorphisms on $\mathbb{T}^n$ by topological conjugation.

Now we construct robustly non-hyperbolic transitive examples by the phenomenon of a blender. For the convenience of the reader, we first give a sketch of the construction in the homotopic class of f_A , $A = \text{diag}(k, 1)$, $k > 5$.

Example 3.6. For simplicity, in this example we identify $\mathbb{T}^2 = \mathbb{R}^2/[-1, 1]^2$. But we still use the identification $\mathbb{T}^2 = \mathbb{R}^2/[-1/2, 1/2]^2$ elsewhere.

Let t, a, λ satisfy $0 < 1 - \lambda \ll 1 - a \ll t \ll 1$, and $\varepsilon = \frac{1}{100k}$. Also, let $A_0 = [-\frac{1}{2k} - \varepsilon, \frac{1}{2k} + \varepsilon] \times [-a, a]$, $A_6 = [\frac{3}{2k} - \varepsilon, \frac{5}{2k} + \varepsilon] \times [-a, a]$, and $B_6 = [\frac{3}{2k} - \varepsilon, \frac{5}{2k} + \varepsilon] \times S^1$, $C = [-\frac{1}{2}, \frac{1}{2}] \times [-a, a]$. Take a C^1 -perturbation f of f_A satisfying the following conditions:

$$\text{i) } f(x_1, x_2) = \begin{cases} (kx_1, \lambda x_2), & (x_1, x_2) \in A_0, \\ (kx_1 - 2, \lambda x_2 + t), & (x_1, x_2) \in A_6, \end{cases}$$

ii) for fixed t and a , $f|_{B_6}$ is C^1 -close to the map $(f_A + t)|_{B_6}$.

Here, we only give an explanation as to why f is robustly transitive. Since (C, f) is a blender, $\overline{W^u(p, f)}$ ($p = (0, 0)$) contains $[-\frac{1}{2}, \frac{1}{2}] \times [\frac{a}{2}, a]$. Note that $f|_{B_6}$ is nearly a

rotation R_t in the direction $(0, 1)$ for fixed t and a , and f is C^1 close to f_A . Then we deduce that $\overline{W^u(p, f)}$ ($p=(0,0)$) contains $[-\frac{1}{2}, \frac{1}{2}] \times S^1$. By the uniform expansion on C_α^u and the Df -invariance of C_α^u , we can see that $\overline{W^u(p, f)} = \mathbb{T}^2$. For any C_α^u curve γ , by the uniform expansion on C_α^u and the Df -invariance of C_α^u again, we see that there is a positive integer m such that $f^m(\gamma)$ runs through B_6 in the direction $(1,0)$. Again, note that $f|_{B_6}$ is nearly a rotation R_t in the direction $(0, 1)$ for fixed t and a . We can see that $\bigcup_{n=0}^\infty f^n(\gamma) \cap W_{loc}^{cs}(p, f) \neq \emptyset$. So, $W^s(p, f)$ is dense in $\mathbb{T}^2$. Furthermore, by the λ -lemma, we get the transitivity of f . Note that the properties in our analysis are all robust. Therefore, f is robustly transitive.

Proposition 3.7. *Suppose $A \in GL(2, \mathbb{Z})$, $\sigma(A) = \{\lambda_1, \lambda_2\}$, $|\lambda_1| = k > 1$, $|\lambda_2| = 1$. Then there exists an open set $\mathcal{U}$ of $End^*(\mathbb{T}^2)$ such that $f_A \in \overline{\mathcal{U}}$ and every $g \in \mathcal{U}$ is transitive and not Anosov.*

Proof. Let $\alpha = (a, b)$ satisfy $\alpha A = \lambda_1 \alpha$, where a and b are two integers and co-prime. We can take another integer vector β such that $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in SL(2, \mathbb{Z})$. Let $P = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} A = \begin{pmatrix} \lambda_1 \alpha \\ t_1 \alpha + t_2 \beta \end{pmatrix}$, where t_1 and t_2 are two rational numbers, i.e., $PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ t_1 & t_2 \end{pmatrix}$. Note that $t_2 = \lambda_2$ and t_1 is an integer. Without loss of generality, let $A = \begin{pmatrix} \lambda_1 & 0 \\ t_1 & \lambda_2 \end{pmatrix}$. Take $Q = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ t_1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$; then $AQ = QB$. Also, let $\tau = \frac{1}{|\lambda_1 - \lambda_2|}$.

Claim 3.8. There exists $F \in \text{Diff}^1(\mathbb{R}^2)$ arbitrarily C^1 -close to B such that $F - B$ is a τ -periodic map and the endomorphism f on $\mathbb{T}^2$ induced by F is robustly transitive and has a hyperbolic saddle and a hyperbolic source.

We first deduce the proposition by the claim, postponing its proof. Let $G = QFQ^{-1} = A + Q(F - B)Q^{-1}$. Since $F - B$ is τ -periodic, $Q(F - B)Q^{-1}$ is 1-periodic. Then G induces a map $g \in End^*(\mathbb{T}^2)$. Let $\mathcal{U}(g)$ be a small enough neighborhood of g . For any $h \in \mathcal{U}(g)$, h can be uniquely lifted to $H \in \text{Diff}^1(\mathbb{R}^2)$ satisfying $d(H, G) = d(h, g)$. Since $Q^{-1}HQ = Q^{-1}(H - G)Q + F$, $Q^{-1}HQ$ induces a map h' C^1 -close to f . Then h' is transitive and $hf_Q = f_Qh'$. Since f_Q is a surjection, h is transitive for any $h \in \mathcal{U}(g)$. Since it has a saddle and a source, h is not Anosov. $\square$

Proof of the claim. Let $\delta = \frac{1}{k^5}$, $\varepsilon = \frac{1}{k^{100}}$, $I = [-\delta, \delta]$, $I_i = [-\frac{\delta+\varepsilon}{k^i} + \frac{1}{k^i}, \frac{\delta+\varepsilon}{k^i} + \frac{1}{k^i}]$, $1 \leq i \leq 6$, and $I_0 = [-\frac{\delta+\varepsilon}{k^6}, \frac{\delta+\varepsilon}{k^6}]$. Also, take t, a, λ satisfying $0 < 1 - \lambda \ll 1 - a \ll t \ll \tau$, which will be precisely stated below.

If $\tau = \frac{1}{k+1}$, by simple calculation we see that $[-\delta, -\delta + \tau) \supset I \cup (I_1 - \tau) \cup \bigcup_{i=2}^5 I_i$ ($I_1 - \tau$ denotes the set $\{x - \tau | x \in I_1\}$) and $(I \cup \bigcup_{i=2}^5 I_i) \cap (I_1 - \tau) = \emptyset$. For the case $\tau = \frac{1}{k-1}$, we can get a similar property. Then we can take a τ -periodic smooth function α_1 satisfying that $\alpha_1(I \cup \bigcup_{i=2}^5 I_i) = 0$ and $\alpha_1(I_1) = t$. Also, take a smooth bump function α_2 satisfying that $\alpha_2([-\frac{a\tau}{2}, \frac{a\tau}{2}]) = 1$ and $\alpha_2(x) = 0$ for any $|x| > \frac{\tau}{2}$. Now construct a map $\Phi : \mathbb{R} \times [-\frac{\tau}{2}, \frac{\tau}{2}] \rightarrow \mathbb{R}^2$:

$$\Phi(x_1, x_2) = (0, (\lambda^{\frac{1}{6}} - 1)\lambda_2 x_2 \alpha_2(x_2) + \alpha_1(x_1)).$$

It can be regarded as a τ -periodic endomorphism on $\mathbb{R}^2$. Let $B_6 = I_6 \times S^1$. As $t \rightarrow 0$ and $\lambda \rightarrow 1$ (fix a and α_2), Φ is C^1 close to 0 and $\Phi|_{B_6}$ is C^1 close to the map $\phi : \phi(x_1, x_2) = (0, t)$ for fixed t and a .

Let $F = \Phi + B$, $f \in \text{End}^*(\mathbb{T}^2)$ be induced by F . Then,

$$f(x_1, x_2) = \begin{cases} (\lambda_1 x_1, \lambda^{\frac{1}{6}} \lambda_2 x_2), & (x_1, x_2) \in (I \cup \bigcup_{i=2}^5 I_i) \times [-\frac{a\tau}{2}, \frac{a\tau}{2}], \\ (\lambda_1 x_1, \lambda^{\frac{1}{6}} \lambda_2 x_2 + t), & (x_1, x_2) \in I_1 \times [-\frac{a\tau}{2}, \frac{a\tau}{2}]. \end{cases}$$

A C_β^u curve $\gamma \subset [-\delta, \delta] \times S^1$ is said to be a (u, β) -curve of $[-\delta, \delta] \times S^1$ if $\partial(\gamma) \subset \partial([-\delta, \delta] \times S^1)$. Similarly, we define (u, β) -curves of B_6 . Now we precisely determine t, a and λ . Choose t, a, λ such that

$$\frac{\tau}{2} - \lambda \frac{a\tau}{2} < t < \lambda \frac{a\tau}{2}$$

(i.e, $t < \frac{\tau}{8} (< \lambda \frac{a\tau}{2}), a > 1 - \frac{2t}{\tau}, \lambda > (\frac{\tau}{2} - t)/(\frac{a\tau}{2})$), $a\tau > 8t$, f has a hyperbolic source in $\{0\} \times [0, \frac{\tau}{2}]$, and there is a neighborhood $\mathcal{U}_1$ of f satisfying the following four conditions:

- (1) There exists $\alpha > 0$ such that for any $g \in \mathcal{U}_1$, any $x \in \mathbb{T}^2$ and any unit vector $v \in C_\alpha^u(x)$, one has

$$Dg(C_\alpha^u(x)) \subset C_\alpha^u(gx), \quad \|Dg(v)\| > 1,$$

and for any (u, α) -curve γ of B_6 , $g^6(\gamma)$ contains a (u, α) -curve of $[-\delta, \delta] \times S^1$.

- (2) For any $g \in \mathcal{U}_1$ and any $(x_1, x_2) \in B_6$,

$$\frac{t}{2} < g_2^6(x_1, x_2) - x_2 < 2t$$

where $g^6 = (g_1^6, g_2^6)$.

- (3) Let $\alpha_0 = \frac{t}{4}$. For any $g \in \mathcal{U}_1$, any $\beta > \frac{\alpha_0}{k}$ and any $C_{k\beta}^u$ curve $\gamma \subset B_6$, $g^6(\gamma)$ is a C_β^u curve.

- (4) For any $g \in \mathcal{U}_1$, g is not Anosov.

Let $C = I \times [-\frac{\tau}{2}, \frac{\tau}{2}]$ and $A_i = I_i \times [-\frac{a\tau}{2}, \frac{a\tau}{2}]$, $i = 0, 6$. Note that

$$f^6(x_1, x_2) = \begin{cases} (k^6 x_1, \lambda x_2), & (x_1, x_2) \in A_0, \\ (k^6 x_1 - 1, \lambda x_2 + t), & (x_1, x_2) \in A_6. \end{cases}$$

We regard $f^6|_{A_0 \cup A_6}$ as a map from $A_0 \cup A_6$ to $[-\frac{1}{2}, \frac{1}{2}]^2$. Since $\frac{\tau}{2} - \lambda \frac{a\tau}{2} < t < \lambda \frac{a\tau}{2}$, (C, f^6) is a blender. Let $p_f = (0, 0)$. Since $W_{loc}^u(p_f, f) \supset [-\delta, \delta] \times \{0\}$, by the properties of a blender, there exists a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of f such that for any $g \in \mathcal{U}_2$,

$$\overline{W^u(p_g, g^6)} \supset [-\delta, \delta] \times [\frac{\tau}{4}, \frac{\tau}{2}] \supset [-\delta, \delta] \times [\frac{\tau}{4}, \frac{\tau}{4} + 2t] \quad (\tau > 8t).$$

By conditions (1) and (2), we see that for any interval $[a, b]$ satisfying $b - a \geq 2t$,

$$g^6(I_6 \times [a, b]) \supset [-\delta, \delta] \times [a + 2t, b + \frac{t}{2}] \supset [-\delta, \delta] \times [b, b + \frac{t}{2}].$$

Then $\overline{W^u(p_g, g^6)} \supset [-\delta, \delta] \times S^1$. Note that f uniformly expands over C_α^u and preserves C_α^u (condition (1)). We deduce that there is a positive integer m such that $g^m([-\delta, \delta] \times S^1) = \mathbb{T}^2$. Consequently, $\overline{W^u(p_g, g)} = \mathbb{T}^2$ for any $g \in \mathcal{U}_2$.

By condition (1) again, we see that for any $g \in \mathcal{U}_2$ and any C_α^u curve $\gamma \subset \mathbb{T}^2$, there exists a positive integer m such that $g^m(\gamma)$ contains a (u, α) -curve γ_1 of B_6 . Note that for any (u, α) -curve γ' of B_6 , $g^6(\gamma')$ contains a (u, α) -curve of $[-\delta, \delta] \times S^1$.

and the cone angle of the curve $g^6(\gamma') \cap [-\delta, \delta] \times S^1$ uniformly decreases unless γ' is a $\mathcal{C}_{\alpha_0}^u$ curve (conditions (1) and (3)). So there is another positive integer m_1 such that $g^{m_1}(\gamma_1)$ contains a (u, α_0) -curve γ_2 of $[-\delta, \delta] \times S^1$. By conditions (2) and (3), we deduce that every $\mathcal{C}_{\alpha_0}^u$ curve $\gamma' = \{(x_1, \gamma_2'(x_1)) | x_1 \in [-\delta, \delta]\}$ contains a $\mathcal{C}_{\alpha_0}^u$ curve $\gamma'' = \{(x_1, \gamma_2''(x_1)) | x_1 \in [-\delta, \delta]\}$ satisfying that

$$\begin{aligned} \min\{\gamma_2''(x) - \gamma_2'(x) | x \in [-\delta, \delta]\} &> \frac{t}{2} - \alpha_0 = \frac{t}{4}, \\ \max\{\gamma_2''(x) - \gamma_2'(x) | x \in [-\delta, \delta]\} &< 2t + \alpha_0 < 3t. \end{aligned}$$

Additionally, $W_{loc}^s(p_f, f) \supset \{0\} \times [-\frac{a\tau}{2}, \frac{a\tau}{2}]$ and $a\tau > 3t$. Therefore, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_2$ of f such that for any $g \in \mathcal{U}$, there is a positive integer m_2 satisfying $g^{m_2}(\gamma_2) \cap W_{loc}^s(p_g, g) \neq \emptyset$, i.e., $\overline{W^s(p_g, g)} = \mathbb{T}^2$. By the λ -lemma, we see that g is transitive for any $g \in \mathcal{U}$. $\square$

4. EXPANDING ENDOMORPHISM

In this section, we give robustly non-hyperbolic transitive endomorphisms on $\mathbb{T}^2$ in all homotopic classes containing expanding maps.

Let M be a smooth Riemannian manifold, f be a regular endomorphism on M , and let X be a subset of M , $r > 0$, and $B(X, r) = \{x \in M | d(x, X) < r\}$. The minimal norm of Df on X is $m(Df|_X) = \inf\{\frac{\|Df(v)\|}{\|v\|} | v \in T_X M\}$, and $m(Df|_M) = m(Df)$. The determinant of Df is $|\det(Df)| = \inf\{|\det(Df_x)| | x \in M\}$. Let $\delta(f) = \sup\{r | \forall x \in M, f \text{ is diffeomorphic from } B(x, r) \text{ to its image}\}$.

Let $A \in GL(2, \mathbb{Z})$, whose moduli of eigenvalues are both larger than 1.

Proposition 4.1. *There exists $g \in \text{End}^*(\mathbb{T}^2)$ satisfying that g is homotopic to f_A and is robustly non-hyperbolic transitive.*

The above proposition is easily deduced by the following two lemmas.

Lemma 4.2. *Let $p = (0, 0)$. For any $0 < \mu < 1$ and any neighborhood U of p , there exists g homotopic to f_A having the following properties:*

- i) p is a hyperbolic saddle of g , $g|_{M \setminus U} = f_A|_{M \setminus U}$ and $\delta(g) > \frac{\delta(f_A)}{2}$,
- ii) $|\det(Dg)| > 1$ and $m(Dg) > \mu$.

Lemma 4.3. *Let f be a regular endomorphism on a compact connected Riemannian manifold M . Suppose $|\det(Df)| > 1$, $m(Df) = \mu$, $\delta(f) = 7\delta$, and there is an open subset U of M such that*

- i) $m(Df|_{M \setminus U}) = \lambda > 1$ and $\text{diam}(U) = r < \frac{\lambda - \sqrt{\lambda}}{\lambda} \delta$,
- ii) *there is an integer $k > 0$ satisfying $(\sqrt{\lambda})^k > 7$ and $\lambda \mu^{k-1} = \lambda_1 > 1$.*

Then for any open subset V of M , there exists an integer $m > 0$ such that $f^m(V) = M$. Consequently, f is transitive.

Remark 4.4. Lemma 4.2 gives a specially derived Anosov map. Both Lemma 4.2 and Proposition 4.1 are valid for any expanding endomorphism on $\mathbb{T}^n$. In [Sh1], Shub proved that “any expanding endomorphism on any compact manifold is transitive,” and the non-trivial fact that “any expanding endomorphism on $\mathbb{T}^n$ is homotopic to and topologically conjugate to a linear expanding endomorphisms.”

To show Lemma 4.3, we need the following simple lemma.

Lemma 4.5. *Let f be a regular endomorphism on a compact Riemannian manifold M . Then:*

- (i) *Suppose V is an open subset of M and $m(Df|_V) = \mu$. Then for any $B(x, \tau) \subset V$, $f(B(x, \tau)) \supset B(fx, \mu\tau)$.*
- (ii) *Suppose there is an open subset U of M such that $m(Df|_{M \setminus U}) = \mu > \mu'$ and $\text{diam}(U) = r$. Then for any $\tau > \frac{\mu}{\mu - \mu'}r$, $f(B(x, \tau)) \supset B(fx, \mu'\tau)$. Also, for any $x, y \in M$ satisfying $\frac{\mu}{\mu - \mu'}r < d(x, y) < \delta(f)$, $d(fx, fy) > \mu'd(x, y)$.*

Proof. The proof of (i) is quite easy. We only prove (ii). For any smooth curve $\gamma : [a, b] \rightarrow M$, let $t_1 = \inf\{t | \gamma(t) \in U\}$ and $t_2 = \sup\{t | \gamma(t) \in U\}$. Then $d(\gamma(t_1), \gamma(t_2)) \leq r$ and $\ell(f\gamma) = \int_a^b |Df(\dot{\gamma}(t))| dt \geq \mu(\int_a^{t_1} |\dot{\gamma}(t)| dt + \int_{t_2}^b |\dot{\gamma}(t)| dt)$. So

$$\text{diam}(\gamma) \leq \int_a^{t_1} |\dot{\gamma}(t)| dt + \int_{t_2}^b |\dot{\gamma}(t)| dt + r \leq r + \frac{\ell(f\gamma)}{\mu}.$$

For any $x, z \in M$, there exists a shortest geodesic γ from fx to z . Since f is a local diffeomorphism and a covering map (M is compact), there is a smooth lift γ_1 of γ , i.e., a smooth curve γ_1 from x to w satisfying $f\gamma_1 = \gamma$. If $z \in B(fx, \mu'\tau)$, then $d(x, w) \leq \text{diam}(\gamma_1) \leq r + \frac{\mu'\tau}{\mu} < \tau$, i.e., $f(B(x, \tau)) \supset B(fx, \mu'\tau)$. If $z = fy$ and $\frac{\mu}{\mu - \mu'}r < d(x, y) < \delta(f)$, then $d(x, w) \leq r + \frac{d(fx, fy)}{\mu}$. Note that $f(w) = f(y)$ and $d(x, y) < \delta(f)$. Then, by the definition of $\delta(f)$, we see that $d(x, w) \geq d(x, y)$. So $d(fx, fy) > \mu'd(x, y)$. $\square$

Proof of Lemma 4.3. Since $|\det(Df)| > 1$, for any connected open set V of M , there exists a positive integer n_0 such that $\text{diam}(f^{n_0}(V)) \geq \delta(f) = 7\delta$ (otherwise, $\text{Volume}(f^n(V)) \rightarrow \infty$ as $n \rightarrow \infty$). Then we can take a (continuous) curve $\gamma_0 \subset f^{n_0}(V)$ such that $\text{diam}(\gamma_0) = \delta$, $B(\gamma_0, \varepsilon) \subset f^{n_0}(V)$ and $B(\gamma_0, \delta) \cap U = \emptyset$. Note that $B(\gamma_0, \delta) \cap U = \emptyset$ and $m(Df|_{M \setminus U}) = \lambda$. Then, by i) of the above lemma, $f(B(\gamma_0, \varepsilon)) \supset B(f(\gamma_0), \lambda\varepsilon)$. Since $m(Df) > \mu$, by i) of the above lemma again, we see that $f^k(B(\gamma_0, \varepsilon)) \supset f^{k-1}(B(f(\gamma_0), \lambda\varepsilon)) \supset B(f^k(\gamma_0), \lambda\mu^{k-1}\varepsilon) = B(f^k(\gamma_0), \lambda_1\varepsilon)$. Also, note that $\delta(f) = 7\delta > \delta > \frac{\lambda}{\lambda - \sqrt{\lambda}}r$ and $m(Df|_{M \setminus U}) > \lambda$. So, by the latter part of (ii) of the above lemma ($\mu = \lambda > \sqrt{\lambda} = \mu'$), $\text{diam}(f^k(\gamma_0)) > \min\{(\sqrt{\lambda})^k\delta, \delta(f)\} = 7\delta$. Then, the connected set $f^k(\gamma_0)$ contains a curve γ_1 such that $\text{diam}(\gamma_1) = \delta$ and $B(\gamma_1, \delta) \cap U = \emptyset$. Take a positive integer l satisfying $\delta < \lambda_1^l\varepsilon \leq \lambda_1\delta$. Inductively, we can find curves $\{\gamma_n\}_{0 \leq n \leq l}$ such that

$$\text{diam}(\gamma_n) = \delta, B(\gamma_n, \delta) \cap U = \emptyset, \text{ and } f^k(B(\gamma_n, \lambda_1^n\varepsilon)) \supset B(\gamma_{n+1}, \lambda_1^{n+1}\varepsilon), 0 \leq n < l.$$

Then $f^{(n_0+kl)}(V) \supset B(\gamma_l, \delta)$. Since $\delta > \frac{\lambda}{\lambda - \sqrt{\lambda}}r$, $f^{(2j+n_0+lk)}(V)$ contains $B(f^{2j}(\gamma_l), \lambda^j\delta)$ for any positive integer j . Thus, there is a positive integer m such that $f^m(V) = M$. $\square$

The proof of Lemma 4.2 is quite basic. For completion, we give its proof.

Proof of Lemma 4.2. Take a basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ such that the representation of A under the basis is the following: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $|\lambda_1| \geq |\lambda_2| > 1$, or $\begin{pmatrix} \lambda & 0 \\ \alpha & \lambda \end{pmatrix}$, $|\lambda| > |\alpha| + 1$, or $\lambda R_\theta = \lambda \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $|\lambda| > 1$. Take the norm of $\mathbb{R}^2$ such that

$\{e_1, e_2\}$ is an orthonormal basis which induces the flat Riemannian metric on $\mathbb{T}^2$. Here we use this Riemannian metric on $\mathbb{T}^2$.

Fix a small neighborhood U of p . Construct $f \in \text{End}^*(\mathbb{T}^2)$ in the homotopic class of f_A such that $f|_{M \setminus U} = f_A|_{M \setminus U}$, $f|_V(x_1, x_2) = (\lambda_1 x_1, \lambda_2 x_2)$, $V \subset U$ is a neighborhood of p , $|\lambda_1| \geq |\lambda_2| > 1$, and $m(Df) > 1$. Then $|\det(Df)| \geq m(Df)^2 > 1$. The construction is not difficult for the first two cases of A . Suppose $A = \lambda R_\theta$. It is trivial to show that there is a large enough positive integer m , $\alpha = \frac{\theta}{m}$, $\{\varepsilon_i\}_{i=1}^\infty, \varepsilon_i \searrow 0$, and the smooth diffeomorphisms $\{\Phi_i\}_{i=1}^\infty$ on $\mathbb{R}^2$ such that, for any positive integer i ,

$$m(D\Phi_i) > \frac{1}{\lambda}; \quad \Phi_i(x) = R_{-\alpha}(x), |x| < \varepsilon_{i+1}; \quad \Phi_i(x) = x, |x| > \varepsilon_i.$$

Suppose $B(p, \varepsilon_1) \subset U$. Let $f = \lambda \Phi_m \dots \Phi_1 R_\theta$. Note that $f(x) = \lambda \Phi_i R_{(\theta - (i-1)\alpha)}(x)$ for $|x| \in [\varepsilon_{i+1}, \varepsilon_i]$. We can see that $m(Df) > 1$ and $f(x) = \lambda x$ for any $|x| < \varepsilon_{m+1}$. Take any $v, \varepsilon > 0$ satisfying $|v\lambda_1| > 1$ and $v + \varepsilon < 1$. Suppose $V \supset I_1 \times I_2$, I_1 and I_2 are two open intervals whose central points are both 0, and $\frac{3|\lambda_2|\ell(I_2)}{\ell(I_1)} < \varepsilon$. Let α_1 be a smooth bump function such that $\alpha_1(x) = 1$ for any x contained in a small neighborhood of 0, $\alpha_1(x) = 0$ for any $x \notin I_1$, $0 \leq \alpha_1(x) \leq 1$ and $|\alpha_1'(x)| < \frac{3}{\ell(I_1)}$. Take another smooth function α_2 such that $\alpha_2(0) = 0$, $|\alpha_2'(0)| = v + \varepsilon$, $\alpha_2(x) = \lambda_2 x$ for any $x \notin I_2$, and $\lambda_2 \alpha_2'(x) > |\lambda_2|(v + \varepsilon)$. Take g in the homotopic class of f :

$$g(x_1, x_2) = (\lambda_1 x_1, \alpha_2(x_2)\alpha_1(x_1) + \lambda_2 x_2(1 - \alpha_1(x_1))).$$

We can see that g has the hyperbolic saddle p and $\delta(g) > \frac{\delta(f_A)}{2}$. Also, for any $(x_1, x_2) \in U$,

$$Dg_{(x_1, x_2)} = \begin{pmatrix} \lambda_1 & 0 \\ \alpha_1'(x_1)(\alpha_2(x_2) - \lambda_2 x_2) & \alpha_2'(x_2)\alpha_1(x_1) + \lambda_2(1 - \alpha_1(x_1)) \end{pmatrix}.$$

Note that $|\alpha_2'(x_2)\alpha_1(x_1) + \lambda_2(1 - \alpha_1(x_1))| > v + \varepsilon$ and $|\alpha_1'(x_1)(\alpha_2(x_2) - \lambda_2 x_2)| < \frac{3}{\ell(I_1)} \times |\lambda_2|\ell(I_2) < \varepsilon$. Then for any $x \in U$, $|\det(Dg_x)| > |\lambda_1 v| > 1$. Also note that

for any matrix $B = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, $m(B) \geq \min\{|a|, |c|\} - |b|$. So $m(Dg) > v$. $\square$

Proof of Theorem 1.1. Now, Theorem 1.1 follows from Proposition 3.7 and Proposition 4.1. $\square$

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