DUPLICATION FORMULAE INVOLVING JACOBI THETA FUNCTIONS AND GOSPER'S q -TRIGONOMETRIC FUNCTIONS

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ABSTRACT. Using the q -trigonometric definitions of Gosper, we devise a new q -exponential function. Based on this concept, we derive a number of identities involving the Jacobi theta functions. These considerations lead to the answers to Gosper's "mysteries".

1. INTRODUCTION

In 2001, R. W. Gosper [4] introduced a kind of q -analogue of the ordinary sine and cosine functions. Although there are different q -sine and q -cosine functions defined by several authors (see [8]), it seems that Gosper's functions are natural and generalize a number of properties of $\sin$ and $\cos$.

We present here the proofs for the computer-based conjectures of Gosper and introduce a new q -exponential function. Moreover, some theta function identities are deduced, for example,

$$\sqrt{\frac{\pi\sqrt{e^\pi}}{2}} \frac{1}{\Gamma^2\left(\frac{3}{4}\right)} = i \sum_{k=-\infty}^{\infty} e^{\pi(k-2k^2)} \vartheta_1\left(\frac{i\pi}{2}(2k-1), e^{-\pi}\right)$$

and

$$\sqrt{\frac{\pi}{2}} \frac{1}{\Gamma^2\left(\frac{3}{4}\right)} = \sum_{k=-\infty}^{\infty} \frac{\vartheta_4(ik\pi, e^{-\pi})}{e^{2\pi k^2}}.$$

Here ϑ_1 and ϑ_4 are two of the well-known theta functions of Jacobi.

2. PRELIMINARIES

Let $(x; q)_n := (1-x)(1-qx)\cdots(1-q^{n-1}x)$ and $(x; q)_\infty := \lim_{n \rightarrow \infty} (x; q)_n$, if the limit exists.

The q -gamma function [1, 3] is defined as

$$(1) \quad \Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1-q)^{1-z} \quad (0 < q < 1).$$

This really generalizes the ordinary Euler gamma function, since

$$\Gamma_q(z) \rightarrow \Gamma(z) \quad \text{if } q \rightarrow 1-.$$

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The logarithmic derivative of Γ_q is the q -digamma function:

$$(2) \quad \psi_q(z) := \frac{d}{dz} \ln(\Gamma_q(z)) = -\ln(1-q) + \ln(q) \sum_{n \geq 0} \frac{q^{n+z}}{1-q^{n+z}}.$$

The Jacobi theta functions are defined via the following doubly infinite sums [2]:

$$(3) \quad \vartheta_1(z, q) = \sum_{n=-\infty}^{\infty} (-1)^{n-\frac{1}{2}} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)iz},$$

$$(4) \quad \vartheta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)iz},$$

$$(5) \quad \vartheta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz},$$

where $i = \sqrt{-1}$, $z \in \mathbb{C}$ and $0 < q < 1$.

There are representations of the above functions only with simply infinite sums:

$$(6) \quad \vartheta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} \sin[(2n+1)z],$$

$$(7) \quad \vartheta_2(z, q) = 2 \sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} \cos[(2n+1)z],$$

$$(8) \quad \vartheta_4(z, q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz).$$

We use the abbreviation $\vartheta_i(0, q) = \vartheta_i(q)$ ($i = 1, 2, 4$).

Gosper defined his q -trigonometric functions as follows:

$$(9) \quad \sin_q(\pi z) := q^{(z-1/2)^2} \frac{(q^{2z}; q^2)_{\infty} (q^{2-2z}; q^2)_{\infty}}{(q; q^2)_{\infty}^2} \quad (0 < q < 1),$$

$$(10) \quad \cos_q(\pi z) := q^{z^2} \frac{(q^{1-2z}; q^2)_{\infty} (q^{2z+1}; q^2)_{\infty}}{(q; q^2)_{\infty}^2} \quad (0 < q < 1).$$

It can be seen [4] that $\cos_q(z) = \sin_q\left(\frac{\pi}{2} - z\right)$. Moreover,

$$(11) \quad \sin_q(z) = \frac{\vartheta_1(z, p)}{\vartheta_2(p)}, \quad \cos_q(z) = \frac{\vartheta_2(z, p)}{\vartheta_2(p)}.$$

Here p is implicitly defined by the equation

$$(12) \quad \ln p \cdot \ln q = \pi^2.$$

We note a simple restatement of (9):

Theorem 1 ([4]). *For all $z \in \mathbb{C} \setminus \mathbb{Z}$ and $0 < q < 1$,*

$$(13) \quad \sin_q(\pi z) = q^{\frac{1}{4}} \Gamma_{q^2}^2\left(\frac{1}{2}\right) \frac{(q^2)^{\binom{z}{2}}}{\Gamma_{q^2}(z) \Gamma_{q^2}(1-z)}.$$

One may realize that this is just the q -analogue of the classical reflection formula of Euler:

$$\sin(\pi z) = \frac{\pi}{\Gamma(z) \Gamma(1-z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}).$$

Gosper formulated many interesting identities through computer algebra experimentation with MACSYMA (see [5, 6, 7] for such examples). This is so for his q -trigonometric functions; however, the exact proofs are missing. For example, the addition formulae for $\sin_q$ and $\cos_q$ are

$$(14) \quad \sin_q(2z) = \text{const} \cdot \sin_{q^2}(z) \cos_{q^2}(z),$$

$$(15) \quad \cos_q(2z) = \cos_{q^2}^2(z) - \sin_{q^2}^2(z).$$

Here the constant depends only on q and was determined by Gosper heuristically. We shall prove the first formula and a number of consequences. (Such formulae for other types of q -trigonometric functions can be found in [8].)

In addition, we cite Gosper's questions, which he called "mysteries":

- (1) "Are there $\sinh_q$ and $\cosh_q$, perhaps corresponding to ϑ_2 and ϑ_3 ?"
- (2) "Is there a q -exponential and q -deMoivre's theorem that would simplify everything?"

3. A NEW q -EXPONENTIAL FUNCTION

Next, we define a new q -exponential function and introduce the hyperbolic q -trigonometric functions. Thus we answer the "mysteries" above.

Definition 1. For all $z \in \mathbb{C}$ and $0 < q < 1$ let

$$\exp_q(z) := \frac{2}{\vartheta_2(q)} \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} e^{(-1)^n(2n+1)z}.$$

To see why this is an appropriate definition, let us consider the following simple algebraic manipulations:

$$\cosh z + (-1)^n \sinh z = \frac{1}{2} [e^z + e^{-z} + (-1)^n(e^z - e^{-z})] = e^{(-1)^n z};$$

hence

$$e^{(-1)^n(2n+1)z} = \cosh[(2n+1)z] + (-1)^n \sinh[(2n+1)z].$$

Since $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i \sin(z)$, we have

$$\exp_q(iz) = \frac{2}{\vartheta_2(q)} \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} (\cos[(2n+1)z] + (-1)^n i \sin[(2n+1)z]).$$

Definitions (6) and (7) give that

$$(16) \quad \exp_q(iz) = \frac{1}{\vartheta_2(q)} (\vartheta_2(z, q) + i \vartheta_1(z, q)).$$

According to (11), we can write

$$\exp_q(iz) = \cos_p(z) + i \sin_p(z).$$

This motivated our definition. (We remark that if we choose q to be $e^{-\pi}$, then $q = p$ and in this case $\exp_{e^{-\pi}}(iz) = \cos_{e^{-\pi}}(z) + i \sin_{e^{-\pi}}(z)$.) Definitions (3), (4) and (16) yield a doubly infinite sum representation of $\exp_q$:

$$(17) \quad \exp_q(z) = \frac{1}{\vartheta_2(q)} \sum_{n=-\infty}^{\infty} (1 - i(-1)^n) q^{(n+\frac{1}{2})^2} e^{(2n+1)iz}.$$

The function $\exp_q$ offers a definition for q -hyperbolic functions. Since $\cos_q$ is an even function and $\sin_q$ is an odd function, one can easily see that

$$\frac{1}{2i}(\exp_q(iz) - \exp_q(-iz)) = \sin_p(z) \quad \text{and} \quad \frac{1}{2}(\exp_q(iz) + \exp_q(-iz)) = \cos_p(z).$$

Therefore it is reasonable to give the following definitions.

Definition 2. For all $z \in \mathbb{C}$ and $0 < q < 1$ let

$$(18) \quad \sinh_q(z) := -i \sin_q(iz) = \frac{1}{2}(\exp_p(z) - \exp_p(-z)),$$

$$(19) \quad \cosh_q(z) := \cos_q(iz) = \frac{1}{2}(\exp_p(z) + \exp_p(-z)).$$

This definition and (17) immediately imply the following double sum representations:

$$\begin{aligned} \sin_p(z) &= \frac{i}{\vartheta_2(q)} \sum_{n=-\infty}^{\infty} (1 - i(-1)^n) q^{(n+\frac{1}{2})^2} \sinh[(2n+1)z], \\ \cos_p(z) &= \frac{1}{\vartheta_2(q)} \sum_{n=-\infty}^{\infty} (1 - i(-1)^n) q^{(n+\frac{1}{2})^2} \cosh[(2n+1)z], \\ \sinh_p(z) &= \frac{i}{\vartheta_2(q)} \sum_{n=-\infty}^{\infty} (1 - i(-1)^n) q^{(n+\frac{1}{2})^2} \sin[(2n+1)z], \\ \cosh_p(z) &= \frac{1}{\vartheta_2(q)} \sum_{n=-\infty}^{\infty} (1 - i(-1)^n) q^{(n+\frac{1}{2})^2} \cos[(2n+1)z]. \end{aligned}$$

4. SOME INFINITE SUMS INVOLVING JACOBI THETA FUNCTIONS

By using $\cos_q$ we prove some identities according to ϑ_1 and ϑ_4 . Namely:

Theorem 2. For all $0 < q < 1$,

$$(20) \quad \vartheta_4^2(q) = iq^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} q^{2k^2-k} \vartheta_1 \left(\frac{2k-1}{2i} \ln q, q \right),$$

$$(21) \quad \vartheta_4^2(q) = \sum_{k=-\infty}^{\infty} q^{2k^2} \vartheta_4 \left(\frac{k \ln q}{i}, q \right).$$

In particular,

$$(22) \quad \sqrt{\frac{\pi \sqrt{e^\pi}}{2}} \frac{1}{\Gamma^2\left(\frac{3}{4}\right)} = i \sum_{k=-\infty}^{\infty} e^{\pi(k-2k^2)} \vartheta_1 \left(\frac{i\pi}{2}(2k-1), e^{-\pi} \right),$$

$$(23) \quad \sqrt{\frac{\pi}{2}} \frac{1}{\Gamma^2\left(\frac{3}{4}\right)} = \sum_{k=-\infty}^{\infty} \frac{\vartheta_4(ik\pi, e^{-\pi})}{e^{2\pi k^2}}.$$

Proof. Let us take the (10) definition of $\cos_q$ and substitute $\log_q \frac{1}{\sqrt{-x}}$ in place of z ($x < 0$). Then

$$\begin{aligned} q^{z^2} &= \sqrt{-x}^{\log_q \sqrt{-x}}, \\ q^{1-2z} &= -qx, \quad q^{2z+1} = -q/x. \end{aligned}$$

Since $\cos_q$ is an even function, this means that

$$(24) \quad \begin{aligned} \cos_q \left(\pi \log_q \frac{1}{\sqrt{-x}} \right) &= \cos_q \left(\pi \log_q \sqrt{-x} \right) \\ &= \frac{\sqrt{-x}^{\log_q \sqrt{-x}}}{(q; q^2)_\infty^2} \frac{(q^2; q^2)_\infty}{(q^2; q^2)_\infty} (-qx; q^2)_\infty (-q/x; q^2)_\infty. \end{aligned}$$

Recall Jacobi's triple product identity [3, p. 15]:

$$(q^2; q^2)_\infty (-qx; q^2)_\infty (-q/x; q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} x^n \quad (x \in \mathbb{C}).$$

Thus, by (24),

$$\cos_q \left(\pi \log_q \sqrt{-x} \right) = \sqrt{-x}^{\log_q \sqrt{-x}} \frac{\sum_{n=-\infty}^{\infty} q^{n^2} x^n}{\sum_{n=-\infty}^{\infty} q^{n^2} (-1)^n}.$$

After the substitutions $-x := y$, $\sqrt{y} := z$ and $z := q^k$ we get

$$(25) \quad \cos_q(k\pi) = q^{k^2} \frac{\sum_{n=-\infty}^{\infty} q^{n^2} (-1)^n q^{2nk}}{\sum_{n=-\infty}^{\infty} q^{n^2} (-1)^n} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+k)^2} (-1)^n}{\sum_{n=-\infty}^{\infty} q^{n^2} (-1)^n}.$$

This representation is valid for all (not necessarily integer) k and was stated by Gosper [4]. (The denominator is simply $\vartheta_4(q)$.)

Two nice infinite-sum identities can be given by the following sum:

$$\sum_{k=-\infty}^{\infty} \cos_q(\pi k) q^{k^2} = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \vartheta_4(q).$$

On the other hand, by (25),

$$\sum_{k=-\infty}^{\infty} \cos_q(\pi k) q^{k^2} = \frac{1}{\vartheta_4(q)} \sum_{k=-\infty}^{\infty} q^{k^2} \sum_{n=-\infty}^{\infty} q^{(n+k)^2} (-1)^n,$$

so

$$(26) \quad \vartheta_4^2(q) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{(n+k)^2} (-1)^n q^{k^2}.$$

(We remark that these steps do not require the function $\cos_q$ explicitly; the numerator in (25) is equal to the denominator, up to a $\pm$ sign, when $k \in \mathbb{Z}$.)

Next let us remark that $q^{(n+k)^2} = q^{-\frac{1}{4}} q^{k^2} (q^{2k-1})^n q^{(n+\frac{1}{2})^2}$, so the RHS of (26) becomes

$$(27) \quad q^{-\frac{1}{4}} \sum_{k=-\infty}^{\infty} q^{2k^2} \sum_{n=-\infty}^{\infty} (q^{2k-1})^n q^{(n+\frac{1}{2})^2} (-1)^n.$$

Let us modify (3) a bit:

$$ie^{-iz} \vartheta_1(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{2niz}.$$

If we choose $z = \frac{2k-1}{2i} \ln q$, then this sum will be the same as the inner sum in (27). Thus (20) holds. It is known that

$$(28) \quad \vartheta_4(e^{-\pi}) = \sqrt[4]{\frac{\pi}{2}} \frac{1}{\Gamma(\frac{3}{4})}.$$

So the sum (20) has the special value at $q = e^{-\pi}$ as in (22).

Another simple consequence of (26) comes from the (5) definition of $\vartheta_4(z, q)$. Rewrite (26):

$$\vartheta_4^2(q) = \sum_{k=-\infty}^{\infty} q^{2k^2} \sum_{n=-\infty}^{\infty} q^{n^2} (q^{2k})^n (-1)^n.$$

Comparing this and (5), we see that the substitution $z = \frac{1}{i} k \ln q$ yields (21). Moreover, by using (28) again, we get the pair of (22) as in (23). $\square$

5. DUPLICATION FORMULAE

As we mentioned before, Gosper conjectured that the following duplication formulae are valid:

$$(29) \quad \sin_q(2z) = (q^2 + 1) \frac{\pi_q}{\pi_{q^2}} \sin_{q^2}(z) \cos_{q^2}(z),$$

$$(30) \quad \cos_q(2z) = \cos_{q^2}^2(z) - \sin_{q^2}^2(z),$$

where

$$\pi_q = (1 - q^2) \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1})^2}.$$

Since $\pi_q \rightarrow \pi$ (as $q \rightarrow 1-$), the addition formulae above tend to the usual addition formulae for $\sin$ and $\cos$.

In what follows we prove (29) and some of its consequences.

5.1. The duplication formula for $\sin_q$.

Theorem 3. *For all $0 < q < 1$ and $z \in \mathbb{R}$, we have*

$$(31) \quad \sin_q(2z) = C \sin_{q^2}(z) \cos_{q^2}(z),$$

where

$$C \equiv C(q) = q^{-\frac{1}{4}} \frac{(q^2; q^4)_{\infty}^4}{(q; q^2)_{\infty}^2},$$

and thus

$$\sinh_q(2z) = C \sinh_{q^2}(z) \cosh_{q^2}(z).$$

Corollary 1.

$$\frac{\vartheta_1(2z, p)}{\vartheta_2(p)} = C \frac{\vartheta_1(z, \sqrt{p}) \vartheta_2(z, \sqrt{p})}{\vartheta_2^2(\sqrt{p})}.$$

Proof. Take the logarithmic derivative of both sides of (31). Definition (9) gives that

$$(32) \quad \frac{d}{dz} \ln(\sin_q(\pi z)) = \ln(q^2) \left[(z - 1/2) - \sum_{n \geq 0} \frac{(q^2)^{n+z}}{1 - (q^2)^{n+z}} + \sum_{n \geq 0} \frac{(q^2)^{n+1-z}}{1 - (q^2)^{n+1-z}} \right],$$

and the (2) definition of the q -digamma function implies that

$$(33) \quad \frac{d}{dz} \ln(\sin_q(\pi z)) = \ln(q^4)(z - 1/2) - 2\psi_{q^2}(z) + 2\psi_{q^2}(1 - z).$$

We remark that this can be deduced via (13) as well. The logarithmic derivative of the RHS of (29) is

$$\ln(q^4)(z - 1/2) - \psi_{q^4}(z) + \psi_{q^4}(1 - z) + z \ln(q^4) + \psi_{q^4}\left(\frac{1}{2} - z\right) - \psi_{q^4}\left(\frac{1}{2} + z\right).$$

We used that $\cos_q(z) = \sin_q(\frac{\pi}{2} - z)$. Suppose that (29) holds. Then we have

$$2\psi_{q^2}(1 - 2z) - 2\psi_{q^2}(2z) = \psi_{q^4}(1 - z) + \psi_{q^4}\left(\frac{1}{2} - z\right) - \psi_{q^4}(z) - \psi_{q^4}\left(\frac{1}{2} + z\right).$$

We can realize that to prove this latter (a priori) equality, it is enough to show that

$$(34) \quad 2\psi_{q^2}(2z) = \psi_{q^4}(z) + \psi_{q^4}\left(\frac{1}{2} + z\right) + c_q,$$

where c_q is a constant independent from z . Indeed, the substitution $2z \rightsquigarrow 1 - 2z$ leads to

$$2\psi_{q^2}(1 - 2z) = \psi_{q^4}\left(\frac{1}{2} - z\right) + \psi_{q^4}(1 - z) + c_q.$$

When $q \rightarrow 1-$, then (34) tends to the well-known identity

$$2\psi(2z) = \psi(z) + \psi\left(\frac{1}{2} + z\right) + 2\ln(2).$$

(Here $\psi(z) = (\log \Gamma)'$ is the classical digamma function.) This also shows that $c_q \rightarrow 2\ln(2)$ as $q \rightarrow 1-$. Therefore, the only thing to do is prove (34). Since

$$\begin{aligned} 2\psi_{q^2}(2z) &= -2\ln(1 - q^2) + \ln(q^4) \sum_{n \geq 0} \frac{q^{2(n+2z)}}{1 - q^{2(n+2z)}}, \\ \psi_{q^4}(z) &= -\ln(1 - q^4) + \ln(q^4) \sum_{n \geq 0} \frac{q^{4(n+z)}}{1 - q^{4(n+z)}}, \\ \psi_{q^4}\left(z + \frac{1}{2}\right) &= -\ln(1 - q^4) + \ln(q^4) \sum_{n \geq 0} \frac{q^{4(n+z+1/2)}}{1 - q^{4(n+z+1/2)}}, \end{aligned}$$

we can see that $c_q = -2\ln(1 - q^2) + 2\ln(1 - q^4) = 2\ln(1 + q^2)$ (and this really tends to $2\ln(2)$). Moreover, we should prove that

$$(35) \quad \sum_{n \geq 0} \frac{q^{2(n+2z)}}{1 - q^{2(n+2z)}} = \sum_{n \geq 0} \frac{q^{4(n+z)}}{1 - q^{4(n+z)}} + \sum_{n \geq 0} \frac{q^{4(n+z+1/2)}}{1 - q^{4(n+z+1/2)}}.$$

To check the validity of this equality, consider the sum

$$\sum_{n \geq 0} \left[\frac{q^{2(n+2z)}}{1 - q^{2(n+2z)}} - \frac{q^{4(n+z)}}{1 - q^{4(n+z)}} - \frac{q^{4(n+z+1/2)}}{1 - q^{4(n+z+1/2)}} \right]$$

instead. Temporarily, we denote the general term with A_n . Induction shows that

$$\sum_{n=0}^m A_n = - \sum_{n=m+1}^{2m+1} \frac{q^{2(m+2z)}}{1 - q^{2(m+2z)}}.$$

Since the series

$$\sum_{m=0}^{\infty} \frac{q^{2(m+2z)}}{1 - q^{2(m+2z)}}$$

is (absolutely) convergent ($0 < q < 1$), its tails—and, in particular, partial sums from $m + 1$ to $2m + 1$ —tend to zero. This means that

$$\sum_{n=0}^m A_n \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

so (35) holds and the proof of (34) (with $c_q = 2 \ln(1 + q^2)$) is complete.

At this point we have proven only that

$$(36) \quad \frac{\sin_q(2z)}{\sin_{q^2}(z) \cos_{q^2}(z)} = \text{const.}$$

Now we determine the constant. Realize that the logarithmic derivative of

$$\Gamma_{q^2}(2z) = d_q(1 + q^2)^{2z} \Gamma_{q^4}(z) \Gamma_{q^4}\left(z + \frac{1}{2}\right)$$

is exactly (34) (with $c_q = 2 \ln(1 + q^2)$). For the sake of simplicity, we make the substitution $q^2 \rightsquigarrow q$, and, by (1), this equation becomes

$$\begin{aligned} & \frac{(q; q)_{\infty}}{(q^{2z}; q)_{\infty}} (1 - q)^{1-2z} \\ &= d_q(1 + q)^{2z} \frac{(q^2; q^2)_{\infty}}{(q^{2z}; q^2)_{\infty}} (1 - q^2)^{1-z} \frac{(q^2; q^2)_{\infty}}{(q^{2(z+1/2)}; q^2)_{\infty}} (1 - q^2)^{1-(z+1/2)}. \end{aligned}$$

After some rearrangement, we have

$$\frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}^2} \frac{(1 - q)^{-\frac{1}{2}}}{(1 + q)^{\frac{3}{2}}} = d_q \frac{(q^{2z}; q)_{\infty}}{(q^{2z}; q^2)_{\infty} (q^{2z+1}; q^2)_{\infty}}.$$

The factor beside d_q , as one can easily see, is equal to 1. This means that, by using (34), we have proven the (known) formula [3]

$$(37) \quad \Gamma_q(2z) = \frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}^2} \sqrt{\frac{1+q}{1-q}} (1+q)^{2z-2} \Gamma_{q^2}(z) \Gamma_{q^2}\left(z + \frac{1}{2}\right).$$

The constant in (36) will be calculated now. Using Theorem 1, (36) becomes

$$\begin{aligned} & q^{\frac{1}{4}} \Gamma_{q^2}^2\left(\frac{1}{2}\right) \frac{(q^2)^{\binom{2z}{2}}}{\Gamma_{q^2}(2z) \Gamma_{q^2}(1-2z)} \\ &= \text{const} \cdot q^{\frac{1}{2}} \Gamma_{q^4}^2\left(\frac{1}{2}\right) \frac{(q^4)^{\binom{1/2-z}{2}}}{\Gamma_{q^4}(1/2-z) \Gamma_{q^4}(1/2+z)} q^{\frac{1}{2}} \Gamma_{q^4}^2\left(\frac{1}{2}\right) \frac{(q^4)^{\binom{z}{2}}}{\Gamma_{q^4}(z) \Gamma_{q^4}(1-z)}. \end{aligned}$$

(Remember that $\cos_q(z) = \sin_q(\frac{\pi}{2} - z)$.) This reduces to

$$q^{-\frac{1}{4}} \frac{\Gamma_{q^2}^2\left(\frac{1}{2}\right)}{\Gamma_{q^4}^4\left(\frac{1}{2}\right)} = \text{const} \cdot \frac{\Gamma_{q^2}(2z)}{\Gamma_{q^4}(z) \Gamma_{q^4}(1/2+z)} \cdot \frac{\Gamma_{q^2}(1-2z)}{\Gamma_{q^4}(1/2-z) \Gamma_{q^4}(1-z)}.$$

(37) yields that

$$q^{-\frac{1}{4}} \frac{\Gamma_{q^2}^2\left(\frac{1}{2}\right)}{\Gamma_{q^4}^4\left(\frac{1}{2}\right)} = \text{const} \cdot \frac{(q^2; q^2)_{\infty}^2}{(q^4; q^4)_{\infty}^2} \frac{1}{(1 - q^4)(1 + q^2)}.$$

Since

$$\frac{\Gamma_{q^2}^2\left(\frac{1}{2}\right)}{\Gamma_{q^4}^4\left(\frac{1}{2}\right)} = \frac{(q^2; q^2)_\infty^2 (q^2; q^4)_\infty^4}{(q; q^2)_\infty^2 (q^4; q^4)_\infty^4} \frac{1 - q^2}{(1 - q^4)^2},$$

the constant is deduced:

$$\text{const} = q^{-\frac{1}{4}} \frac{(q^2; q^4)_\infty^4}{(q; q^2)_\infty^2}.$$

Since the equality for $\sinh_q$ follows from the definition of this function, Theorem 3 is proved. $\square$

5.2. A new representation for $\vartheta_1(z, q)$ based on the duplication formula of $\sin_q$. The addition theorem for $\sin_q$ implies a special representation for the Jacobi function ϑ_1 . First we introduce the function

$$A_n(z, q) = \sum_{k=0}^n (-1)^k q^{2k(k-n)} \cdot \begin{cases} \sin[(n+1)z] & \text{if } 2 \mid n, \\ \sin[(2k-n)z] & \text{if } 2 \nmid n. \end{cases}$$

With this $A_{n,q}$ we have the following result.

Theorem 4. *If $\ln p \ln q = \pi^2$, then*

$$\vartheta_1(z, q^2) = 2 \left(\frac{q^2}{p} \right)^{\frac{1}{8}} \frac{(p; p^2)_\infty^4}{(\sqrt{p}; p)_\infty^2} \frac{\vartheta_2(q^2)}{\vartheta_2^2(q)} \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} A_n(2z, q).$$

Proof. First realize that

(38)

$$\Im \exp_q^2(iz) = 2 \sin_p z \cos_p z = 2p^{\frac{1}{8}} \frac{(\sqrt{p}, p)_\infty^2}{(p, p^2)_\infty^4} \sin_{\sqrt{p}}(2z) = 2p^{\frac{1}{8}} \frac{(\sqrt{p}, p)_\infty^2}{(p, p^2)_\infty^4} \frac{\vartheta_1(2z, q^2)}{\vartheta_2(q^2)}.$$

On the other hand,

$$\begin{aligned} \exp_q^2(iz) &= \frac{4}{\vartheta_2^2(q)} \sum_{n=0}^{\infty} \sum_{k=0}^n q^{(k+\frac{1}{2})^2} q^{(n-k+\frac{1}{2})^2} \exp[(-1)^k(2k+1)iz] \\ &\quad \cdot \exp[(-1)^{n-k}(2(n-k)+1)iz]. \end{aligned}$$

The double sum can be reduced to the following:

$$q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \sum_{k=0}^n q^{2k(k-n)} \cdot \begin{cases} \exp[(-1)^k 2iz(n+1)] & \text{if } 2 \mid n, \\ \exp[(-1)^k 2iz(2k-n)] & \text{if } 2 \nmid n. \end{cases}$$

If we take the imaginary part, we can see that the inner sum is $A_n(2z, q)$. Comparing these with (38), we have

$$2p^{\frac{1}{8}} \frac{(\sqrt{p}, p)_\infty^2}{(p, p^2)_\infty^4} \frac{\vartheta_1(2z, q^2)}{\vartheta_2(q^2)} = \frac{4q^{\frac{1}{4}}}{\vartheta_2^2(q)} \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} A_n(2z, q).$$

Then a rearrangement yields our formula in Theorem 4. $\square$

5.3. A note on Gosper's “ q -de Moivre mystery”. If we use the addition formula (15) for $\cos_q$, which we were unable to prove, we can easily deduce the generalization of the elementary identity

$$\exp(2z) = \exp^2(z).$$

The q -version of the left-hand side can be given easily (if (15) is true!):

$$\exp_q(2z) = \sinh_p(2z) + \cosh_p(2z).$$

Using Theorem 3 and (15), we easily get that

$$\exp_q(2z) = \frac{1}{4} \left[(C + 2) \exp_{\sqrt{p}}^2(x) + (2 - C) \exp_{\sqrt{p}}^2(-2) \right].$$

If $q \rightarrow 1-$, then $C \rightarrow 2$ and our formula goes to $\exp(2z) = \exp^2(z)$, but the presence of C obstructs the existence of a “simple” q -de Moivre identity asked by Gosper.

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