EXISTENCE OF POLYNOMIALS ON SUBSPACES WITHOUT EXTENSION

MAITE FERNÁNDEZ-UNZUETA

(Communicated by Marius Junge)

ABSTRACT. We prove the existence of a polynomial of degree d defined on a closed subspace that cannot be extended to the Banach space E (in particular, the existence of a *nonextendible polynomial*) in the following cases: (1) $d \geq 2$ and E does not have type p for some $1 < p < 2$; (2) the space ℓ_k , $k \in \mathbb{N}$, $2 < k \leq d$, is finitely representable in E . In each of these cases we prove, equivalently, the existence of a closed subspace $F \subset E$ such that the subspace $\hat{\otimes}_{s,\pi}^d F$ is not closed in $\hat{\otimes}_{s,\pi}^d E$.

1. INTRODUCTION

In the context of infinite-dimensional Banach spaces, a given polynomial defined on a closed subspace is not necessarily the restriction of a polynomial defined on the whole space. For each degree $d \geq 2$, the polynomial $P(\sum_{i=1}^{\infty} a_i e_i) := \sum_{i=1}^{\infty} a_i^d$ defined on ℓ_d is an example of this fact: P is not the restriction of a polynomial defined on a Banach space such that it contains ℓ_d and has the Dunford-Pettis property (as is the case of ℓ_{∞}) since, by means of [19, Proposition 5], given such a Q , the sequence $(Q(e_i))_{i=1}^{\infty}$ would necessarily tend to zero.

The study of this phenomenon, which has no parallels in the case of linear forms nor in the case of finite-dimensional linear spaces, has been carried out from very diverse points of view. For example, the problem of extending analytic vector-valued mappings defined on an open subset of a subspace is discussed in [2]. The extension to the bidual space introduced there is used in [6] to prove that there exists a norm-preserving extension of a polynomial defined on a Banach space to the bidual space. The problem of the uniqueness of the norm-preserving extensions of polynomials is treated in [13], [14] and [16]. In [4] and [15] the authors studied the so-called extendible polynomials, that is, the polynomials defined on a Banach space that extend to any larger space. One may find a broad list of references on the topic in [23].

In this article we provide geometric conditions on a Banach space that imply the existence of a polynomial defined on a closed subspace which does not extend or, equivalently, the existence of a closed subspace such that its symmetric tensor product is not closed in the symmetric tensor product of the space. In this way,

Received by the editors February 1, 2011 and, in revised form, October 20, 2011.

2010 *Mathematics Subject Classification.* Primary 47H60, 46B07.

The author has been partially supported by P48363 CONACyT, México. She would also like to thank the Fields Institute, since part of the research was done while the author was a visiting professor there.

the extension of polynomials is studied as in [11], where the attention is focused on the geometry of the space. It is worth mentioning that, thus far, besides the spaces having type 2 (respectively, the spaces isomorphic to a Hilbert space), it is not known whether there exists a Banach space such that every quadratic form (resp. every homogeneous polynomial of degree $d > 2$) defined on a subspace of it extends to the whole space (see Problems 1 and 2 in [11]).

The degree 2 case is studied in Section 2. We use the local nature of the property (see [11, Theorem 1.3]) to prove that if a Banach space E has no type p for some $1 < p < 2$, then there exist a subspace and a quadratic form defined on it which does not extend (see Theorem 2.2). The case of polynomials of degree $d > 2$ is studied in Section 3. There, we prove that if ℓ_k is a closed subspace of E (or even more, if ℓ_k is finitely represented in it) for $2 < k \leq d$, $k \in \mathbb{N}$, then there exists a polynomial of degree d defined on a subspace which does not extend (see Theorem 3.3). Moreover, in the cases $d = 2$ and $k = d > 2$, the polynomial $P(\sum_i a_i e_i) := \sum_i a_i^d$ of the previous example, defined on a suitable copy of ℓ_d^m inside the space, is proved to be a polynomial with this bad extension behavior. (Note that a polynomial defined on a subspace which does not extend to the space is a *nonextendible polynomial* in the sense of P. Kirwan and R. Ryan in [15]; that is, it does not extend to every space containing the subspace.)

These results follow from the link between symmetric linear operators and polynomials defined on suitable ℓ_p^m spaces, established in Proposition 3.2. This proposition makes it possible to apply techniques of the *local (linear) theory of Banach spaces* to the context of polynomial maps. It is worth mentioning that the symmetry of the multilinear maps underlying the polynomial context has a significant impact in the use of these techniques, with respect to the extension of not necessarily symmetric multilinear maps, as was studied in [12]. In the last section we discuss these differences and put in context the 2-Extension Property with the following properties: the type 2 property and the Maurey Extension Property.

Given an integer $d \geq 2$ and the Banach spaces E and F , we denote the space of multilinear operators and symmetric multilinear operators defined on $E \times \dots \times E$ with values in F by $\mathcal{L}(E \times \dots \times E, F)$ and $\mathcal{L}_s(E \times \dots \times E, F)$ respectively. The projective tensor product of order d and the projective symmetric tensor product of E of order d will be denoted by $\hat{\otimes}_\pi^d E$ and $\hat{\otimes}_{s,\pi}^d E$, respectively. By definition, a map between two Banach spaces $P : E \rightarrow F$ is a d -homogeneous polynomial if there exists a continuous multilinear operator $T \in \mathcal{L}(E \times \dots \times E, F)$ such that $P(x) = T(x, \dots, x)$ for every $x \in E$. The space of d -homogeneous scalar-valued polynomials will be denoted by $\mathcal{P}^d(E)$. The notation and basic definitions that are not explicitly defined here can be found in [10].

In this paper we are centered on the extendibility of nonlinear scalar-valued polynomials. It is worth noting, however, that whenever there exists a scalar-valued polynomial on a subspace $F \subset E$ with no extension to E , then, for every Banach space Y , there exists a polynomial on F with values in Y , with no extension to E .

The proof of the following lemma, as well as other general properties of tensor products, operators between them and polynomials, can be consulted in [7], [10], [18] or [21].

Lemma 1.1. *Let E and F be Banach spaces. Given $x, x_i \in E$, $i = 1, \dots, d$, for each $1 \leq k < d$, the relations $(\eta_k(T)(x_1, \dots, x_k))(x_{k+1}, \dots, x_d) := T(x_1, \dots, x_d)$, $\nu_d(T)(x_1 \otimes \dots \otimes x_d) := T(x_1, \dots, x_d)$, $\nu_s(T)(x \otimes \dots \otimes x) := T(x, \dots, x)$ and $\rho(S)(x) := S(x \otimes \dots \otimes x)$ induce the following canonical isomorphisms:*

- (1) $\mathcal{L}(E \times \dots \times E, F) \xrightarrow{\eta_k} \mathcal{L}(E \times \dots \times E, \mathcal{L}(E \times \dots \times E, F))$ which is onto and isometric.
- (2) $\mathcal{L}(E \times \dots \times E, F) \xrightarrow{\nu_d} \mathcal{L}(\hat{\otimes}_{s,\pi}^d E, F)$ which is onto and isometric.
- (3) $\mathcal{L}_s(E \times \dots \times E, F) \xrightarrow{\nu_s} \mathcal{L}(\hat{\otimes}_{s,\pi}^d E, F) \xrightarrow{\rho} P(dE, F)$, where ν_s is onto and ρ is onto and isometric.

In the following lemma we prove some other basic facts concerning the extension of polynomials from subspaces that will be used throughout this paper. First, let us fix some notation. Given $k \leq d$, let us denote by $i_k : \hat{\otimes}_{s,\pi}^k E \rightarrow \hat{\otimes}_{\pi}^k E$ the closed inclusion map and let $i_k^* : (\hat{\otimes}_{\pi}^k E)^* \rightarrow (\hat{\otimes}_{s,\pi}^k E)^*$ be its adjoint operator. Since $\hat{\otimes}_{s,\pi}^k E$ is complemented in $\hat{\otimes}_{\pi}^k E$, i_k^* is a projection operator. For Φ^s and Φ_k as in:

$$\begin{aligned} \mathcal{L}(E \times \dots \times E) &\xrightarrow{\Phi_k} \mathcal{L}(\hat{\otimes}_{\pi}^k E, (\hat{\otimes}_{\pi}^{d-k} E)^*) \xrightarrow{\Phi^s} \mathcal{L}(\hat{\otimes}_{s,\pi}^k E, (\hat{\otimes}_{s,\pi}^{d-k} E)^*) \\ T &\mapsto \nu_{d-k}((\nu_k(\eta_k(T))) (\cdot)) \mapsto i_{d-k}^* \circ (\nu_{d-k}((\nu_k(\eta_k(T))) (\cdot))) \circ i_k, \end{aligned}$$

the composition map $\Phi^s \circ \Phi_k$ is an onto bounded operator. The closed subspace $(\Phi^s \circ \Phi_k)(\mathcal{L}_s(E \times \dots \times E))$ will be denoted by $\mathcal{L}_s(\hat{\otimes}_{s,\pi}^k E, (\hat{\otimes}_{s,\pi}^{d-k} E)^*)$. Then,

Lemma 1.2. *Let E be a Banach space and let $k, d \in \mathbb{N}$, $1 \leq k \leq d$ be fixed.*

- (1) *The map $\Psi_k : \mathcal{P}(dE) \rightarrow \mathcal{L}_s(\hat{\otimes}_{s,\pi}^k E, (\hat{\otimes}_{s,\pi}^{d-k} E)^*)$ determined by the relation $(\Psi_k(P)(x \otimes \dots \otimes x))(x \otimes \dots \otimes x) := P(x)$, for every $x \in E$, is an onto isomorphism with $\|\Psi_k\| \leq \frac{d^d}{d!}$ and $\|(\Psi_k)^{-1}\| \leq 1$.*
- (2) *Let $F \xrightarrow{i} E$ be the inclusion map defined on the closed subspace F . A d -homogeneous polynomial $\hat{P} \in \mathcal{P}(dE)$ is an extension of the d -homogeneous polynomial $P \in \mathcal{P}(dF)$ if and only if for some (resp. for all) $1 \leq k < d$ the diagram*

$$\begin{array}{ccc} \hat{\otimes}_{s,\pi}^k E & \xrightarrow{\Psi_k(\hat{P})} & (\hat{\otimes}_{s,\pi}^{d-k} E)^* \\ i_k^s := i \otimes \dots \otimes i \uparrow & & \downarrow (i_{d-k}^s)^* \\ \hat{\otimes}_{s,\pi}^k F & \xrightarrow{\Psi_k(P)} & (\hat{\otimes}_{s,\pi}^{d-k} F)^* \end{array}$$

commutes, where Ψ_k is as in (1).

- (3) *The space $\hat{\otimes}_{s,\pi}^d F$ is a closed subspace of $\hat{\otimes}_{s,\pi}^d E$ if and only if every $P \in \mathcal{P}(dF)$ has an extension $\hat{P} \in \mathcal{P}(dE)$.*

Proof. The proof of (1) follows directly from the canonical isomorphisms in Lemma 1.1 and the fact that an arbitrary symmetric map is completely determined by its values on the elements of the form $x \otimes \dots \otimes x$, when $x \in E$, via the polarization formula (see [10, Corollary 1.3]). The proof of (2) follows from the fact that the map $(i_{d-k}^s)^*$ restricts every element in the dual of $\hat{\otimes}_{s,\pi}^{d-k} E$ to the not necessarily closed subspace $\hat{\otimes}_{s,\pi}^{d-k} F$. To prove (3) observe that every $P \in \mathcal{P}(dF)$ has an

extension $\hat{P} \in \mathcal{P}(^dE)$ if and only if the operator $(i_d^s)^* : (\hat{\otimes}_{s,\pi}^d E)^* \rightarrow (\hat{\otimes}_{s,\pi}^d F)^*$ is surjective. The surjectivity of this map is, in turn, equivalent to $i_d^s : \hat{\otimes}_{s,\pi}^d F \rightarrow \hat{\otimes}_{s,\pi}^d E$ being an isomorphism onto its image. This holds, precisely, when $\hat{\otimes}_{s,\pi}^d F$ is closed in $\hat{\otimes}_{s,\pi}^d E$ (see for instance [9, Lemma 4.2]). $\square$

Following [11, Definition 1.1], we say that

Definition 1.3. A Banach space E has the d -Extension Property, with $d \in \mathbb{N}$, if for every polynomial $P \in \mathcal{P}(^dF)$ defined on a closed subspace $F \subset E$ there exists a polynomial $\hat{P} \in \mathcal{P}(^dE)$ such that $\hat{P}|_F = P$.

This, by means of (3) in Lemma 1.2, is equivalent to saying that for every closed subspace $F \subset E$, the subspace $\hat{\otimes}_{s,\pi}^d F$ is closed in $\hat{\otimes}_{s,\pi}^d E$.

We will frequently use the following results from [11]. If a Banach space E has the d -Extension Property, then it has the k -Extension Property for $2 \leq k \leq d$, as well (see [11, Proposition 1.2]). Moreover, it is essential for our purposes that whenever a Banach space E has the d -Extension Property, there exists a constant $M > 0$ such that for each polynomial $P \in \mathcal{P}(^dF)$ defined on a closed subspace of E , there exists a polynomial $\hat{P}$ on E extending P such that $\|\hat{P}\| \leq M\|P\|$ [11, Theorem 1.3].

2. EXTENDIBILITY OF QUADRATIC FORMS IMPLIES TYPE p FOR ALL $1 \leq p < 2$

In the particular case of a quadratic form $P \in \mathcal{P}(^2F)$, the extension diagram stated in Lemma 1.1 is as follows:

$$(2.1) \quad \begin{array}{ccccc} E & \xrightarrow{\Psi_1(\hat{P})} & E^* \\ i \uparrow & & \downarrow i^* \\ F & \xrightarrow{\Psi_1(P)} & F^*, \end{array}$$

where $\Psi_1(P)$ and $\Psi_1(\hat{P})$ are symmetric and satisfy $P(x) = (\Psi_1(P)(x))(x)$ if $x \in F$, and $\hat{P}(x) = (\Psi_1(\hat{P})(x))(x)$ if $x \in E$.

This immediately refers us to the well-known Maurey's Theorem [17, Corollaire 3] in the following form: If the Banach space E has type 2, then every map $\Psi_1(P)$ has an extension in such a way that diagram (2.1) holds (for a proof of this fact, see [8, Corollary 12.23]). As stated in [11, Problem 1], Type 2 spaces are the only known examples of spaces having the 2-Extension Property. In this direction, we prove in Theorem 2.2 that having type p for all $1 \leq p < 2$ is, indeed, a necessary condition for the 2-Extension Property to hold.

Let $d(E, F)$ denote the Banach-Mazur distance between two isomorphic Banach spaces E and F : $d(E, F) = \inf\{\|T\| \cdot \|T^{-1}\| : T \in \mathcal{L}(E, F), T \text{ an onto isomorphism}\}$. We have:

Proposition 2.1. *Let E be a Banach space with the 2-Extension Property. Then, there exists some constant $M > 0$ such that for every n and every n -dimensional subspace $F \subset E$ there exists a projection $\Pi \in \mathcal{L}(E, E)$ onto F of norm $\|\Pi\| \leq 2Md(F, \ell_2^n)^2$.*

Proof. Let $M > 0$ be the extension constant provided by [11, Theorem 1.3]. For an arbitrary n -dimensional subspace F of E , consider an isomorphism $T \in \mathcal{L}(F, \ell_2^n)$ such that $\|T\| \cdot \|T^{-1}\| = d(F, \ell_2^n)$. The symmetric isomorphism $\phi = T^* \circ T \in \mathcal{L}_s(F, F^*)$ determines a 2-homogeneous polynomial $P \in \mathcal{P}^2(F)$ by the relation $(\Psi_1)^{-1}$ stated in Lemma 1.2. Note that $\|\phi\| \cdot \|\phi^{-1}\| \leq d(F, \ell_2^n)^2$. Using that E has the 2-Extension Property with constant M , and writing it in terms of diagram (2.1), we find a symmetric linear map $\tilde{\phi} \in \mathcal{L}_s(E, E^*)$ for which $\|\tilde{\phi}\| \leq \frac{2^2}{2!} M \|\phi\|$ and such that diagram (2.1) commutes.

Now we can directly check that the map $\Pi : i \circ \phi^{-1} \circ i^* \circ \tilde{\phi} \in \mathcal{L}(E, E)$ is a projection onto $i(F)$ with $\|\Pi\| \leq 2Md(F, \ell_2^n)^2$. $\square$

This result proves, in particular, that a Banach space with the 2-Extension Property is K -convex (see [8, Theorem 19.3]). Consequently, it also has nontrivial type (see [8, Theorem 13.3]); that is, for some $1 < p \leq 2$ there exists $\theta > 0$ such that for any finitely many vectors $x_1, \dots, x_n$ in the space, the estimation

$$(2.2) \quad \left(\int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\|^2 \right)^{\frac{1}{2}} \leq \theta \left(\sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}}$$

holds, where r_k denotes de Rademacher functions $r_k(t) = \text{sign}(\sin 2^k \pi t)$, for $t \in [0, 1]$ and $k \in \mathbb{N}$. This fact will be used in the proof of Theorem 2.2 below, where we show that, indeed, the space has type p for all $p < 2$. We will also use the following result, implicitly stated in the proof of [3, Theorem 3.1], due to Bennett, Dor et al. There is proved the existence in ℓ_p of nonuniformly complemented copies of subspaces which are uniformly isomorphic to ℓ_2^n , $n \in \mathbb{N}$, when $1 < p < 2$:

Proposition (BDGJN). *Let $1 < p < 2$. There exists $\lambda > 0$ satisfying that for every $K > 0$ there exist a pair of natural numbers $m < n$, a subspace $E_{m,n}^{(2)} \subset l_p^n$ and an isomorphism $\varphi \in \mathcal{L}(E_{m,n}^{(2)}, l_2^m)$, with $\|\varphi\| = 1$, $\|(\varphi)^{-1}\| \leq \lambda$, such that if $\Pi_{m,n} : l_p^n \rightarrow E_{m,n}^{(2)}$ is an onto projection operator, then $\|\Pi_{m,n}\| \geq K$.*

Recall that a Banach space X is called μ -representable in E , for $\mu \geq 1$, if given a finite-dimensional space $Y \subset X$, there exists a finite-dimensional subspace $F \subset E$ and an isomorphism $u : F \rightarrow Y$ such that $\|u\| \cdot \|u^{-1}\| \leq \mu$ (see [8, Chapter 8]).

Theorem 2.2. *A Banach space which has the 2-Extension Property has type p for all $1 \leq p < 2$.*

Proof. Let E be a Banach space with the 2-Extension Property. Using [11, Theorem 1.3] again, we get a constant $M > 0$ such that for every subspace $F \subset E$ and every $P \in \mathcal{P}^2(F)$, there exists an extension $\tilde{P} \in \mathcal{P}^2(E)$ such that $\|\tilde{P}\| \leq M\|P\|$. Assume further that E does not have type p for some $1 < p < 2$. Consider p_E defined as $p_E := \sup_{1 \leq p \leq 2} \{E \text{ has type } p\}$. By Maurey-Pisier's Theorem (see [8, Theorem 14.1]), the space l_{p_E} is finitely representable in E . By the assumption on the type of E and the fact that 'type q ' always implies 'type r ' when $1 \leq r < q \leq 2$, we have that $p_E \leq p < 2$. On the other side, as we said immediately after Proposition 2.1, the 2-Extension Property implies that $1 < p_E$. For fixed $\epsilon > 0$, for every n , there exists an n -dimensional subspace $G_n \subset E$ such that $d(G_n, l_{p_E}^n) < 1 + \epsilon$. Let $T_n \in \mathcal{L}(G_n, l_{p_E}^n)$ be an isomorphism such that $\|T_n\| \cdot \|T_n^{-1}\| \leq 1 + \epsilon$ and let $\lambda > 0$ be the constant provided by Proposition (BDGJN) applied to the case $p = p_E$.

For a fixed $K > 2M(1 + \epsilon)^3\lambda^2$, let us consider $m < n$ and $E_{m,n}^{(2)} \subset \ell_{p_E}^n$ as in Proposition (BDGJN). Thus, we know that any projection map $\Pi_{m,n}$ onto $E_{m,n}^{(2)}$ has $\|\Pi_{m,n}\| > K$.

The subspace $T_n^{-1}(E_{m,n}^{(2)}) \subset G_n \subset E$ satisfies $d(T_n^{-1}(E_{m,n}^{(2)}), \ell_2^m) < (1 + \epsilon)\lambda$. Then, by Proposition 2.1, the subspace $T_n^{-1}(E_{m,n}^{(2)})$ is complemented in E (thus in G_n) with some projection Π of norm $\|\Pi\| \leq 2M(1 + \epsilon)^2\lambda^2$. Finally, we can check that $T_{m,n} : T_n \circ \Pi \circ T_n^{-1} \in \mathcal{L}(\ell_{p_E}^n, \ell_{p_E}^m)$ is a projection onto $E_{n,m}^{(2)}$ of norm $\leq (1 + \epsilon)\|\Pi\| \leq 2M(1 + \epsilon)^3\lambda^2 < K$. This fact contradicts the assertion in Proposition (BDGJN). This proves that, necessarily, $p_E = 2$. $\square$

3. THE CASE OF POLYNOMIALS OF DEGREE $d > 2$

It was proved in [11, Proposition 1.2] that when every homogeneous polynomial of degree d defined on a fixed closed subspace has an extension defined on the space, the same holds for the homogeneous polynomials defined on the space, which have degree k , for $1 \leq k \leq d$. Using this, we get from Theorem 2.2:

Corollary 3.1. *If a Banach space E does not have type p for some $1 < p < 2$, then there exists a closed subspace $F \subset E$ such that for every degree $d \geq 2$, there exists a scalar homogeneous polynomial of degree d defined on F which is not the restriction of any polynomial defined on E . Equivalently, for each $d \geq 2$, the subspace $\hat{\otimes}_{s,\pi}^d F$ is not closed in $\hat{\otimes}_{s,\pi}^d E$.*

There are Banach spaces which do not verify the hypotheses in the corollary (even spaces which have type 2) and do not have the d -Extension Property. We will prove that this occurs, indeed, whenever ℓ_l is finitely representable in the space, with $l \in \mathbb{N}$ and $2 < l \leq d$. To that end we will use that a specific space of symmetric linear operators defined on some ℓ_p spaces can be viewed as a subspace of the space of polynomials. The precise statement we need is the next proposition.

For $k, m \in \mathbb{N}$, $1 < r < \infty$, let $\Pi_{D_r^k}^m \in \mathcal{L}(\hat{\otimes}_{s,\pi}^k \ell_r^m, \hat{\otimes}_{s,\pi}^k \ell_r^m)$ denote the projection onto the subspace $\text{span}\{e_i \otimes \dots \otimes e_i, i = 1, \dots, m\}$. It is proved in [1] that $\|\Pi_{D_r^k}^m\| = 1$ and that the image is a subspace isometrically isomorphic to ℓ_p^m , where $\frac{1}{p} := \min\{1, \frac{k}{r}\}$.

For $m \in \mathbb{N}$, let the canonical basis $\beta_m := \{e_1, \dots, e_m\}$ of $\mathbb{R}^m$ be fixed. Given $1 \leq p, q \leq \infty$, we say that an operator $T \in \mathcal{L}(\ell_p^m, \ell_q^m)$ is *symmetric* if it is symmetric with respect to the canonical basis β_m both in ℓ_p^m and ℓ_q^m , that is, if $(T(e_i))_j = (T(e_j))_i$, for $i, j \in \{1, \dots, m\}$.

Proposition 3.2. *Given $l, d \in \mathbb{N}$, $2 \leq d, l < d$, $1 \leq r < \infty$, consider $1 \leq p, q < \infty$ such that $\frac{1}{p} := \min\{1, \frac{l}{r}\}$ and $q = \frac{q^*}{q^* - 1}$, where $\frac{1}{q^*} := \min\{1, \frac{d-l}{r}\}$. Then, for $m \in \mathbb{N}$, the Banach space of symmetric operators $\mathcal{L}_s(\ell_p^m, \ell_q^m)$ can be identified with a complemented subspace of $\mathcal{P}(\ell_r^m)$ through the following isomorphism Ψ :*

$$\begin{aligned} \Psi : \mathcal{L}_s(\ell_p^m, \ell_q^m) &\longrightarrow \mathcal{L}_s(\hat{\otimes}_{s,\pi}^l \ell_r^m, (\hat{\otimes}_{s,\pi}^{d-l} \ell_r^m)^*) & \xrightarrow{\Psi_l^{-1}} & \mathcal{P}(\ell_r^m) \\ S &\longrightarrow T_S := (\Pi_{D_r^{d-l}}^m)^* \circ S \circ \Pi_{D_r^l}^m & \mapsto & \Psi_l^{-1}(T_S), \end{aligned}$$

where Ψ_l is the isomorphism in Lemma 1.2.

Observe that in this case, the images of $\Pi_{D_l}^m$ and $\Pi_{D_r^{d-l}}^m$ are, respectively, isometric to ℓ_p^m and ℓ_q^m . We omit the proof since it can be easily done following the proof of [12, Proposition 3.6]. It is enough to check that each step in that proof respects the symmetry that is required in this new context.

Theorem 3.3. *Let E be a Banach space and let $d, k \in \mathbb{N}$, $2 < k \leq d$. If ℓ_k is μ -representable in E , then E does not have the d -Extension Property. Equivalently, there exists a closed subspace $F \subset E$ such that $\hat{\otimes}_{s,\pi}^d F$ is not closed in $\hat{\otimes}_{s,\pi}^d E$.*

Proof. Consider the case when $k = d$. Let us assume that, contrary to the assertion in the theorem, there exists E with the d -Extension Property such that ℓ_d is μ -representable in it.

Let $M > 0$ be the uniform constant given by [11, Theorem 1.3] such that for every closed subspace $F \subset E$, each polynomial $P \in \mathcal{P}(^d F)$ has an extension $\hat{P} \in \mathcal{P}(^d E)$ with norm $\|\hat{P}\| \leq M\|P\|$.

The space of symmetric operators $\mathcal{L}_s(\ell_d^m, \ell_d^m)$ can be identified with a complemented subspace of $\mathcal{P}(^d \ell_d^m)$, for each $m \in \mathbb{N}$: This follows from Proposition 3.2, choosing the parameters $r := d$ and $l := 1$: In this case, p and q become $p = d$ and $q = \frac{q^*}{q^*-1} = d$.

We will show that these two facts and the assumption on the d -Extension Property are in contradiction with the following statement:

Claim (Rosenthal's Theorem). There exists $\lambda_d > 1$ such that for every fixed constant $K > 0$, there exist integer numbers $m < N$ and a linear subspace $B_d^m \subset \ell_d^N$ with $d(B_d^m, \ell_d^m) \leq \lambda_d$, such that the norm of any linear projection $\Pi \in \mathcal{L}(\ell_d^N, \ell_d^N)$ onto B_d^m satisfies $\|\Pi\| \geq K$ (see [22, Theorem 6]).

Consider λ_d as in the claim. For each $K > 0$, consider $m < N$ and $B_d^m \subset \ell_d^N$ as in the claim, too, and let $\phi \in \mathcal{L}(B_d^m, \ell_d^m)$ be an isomorphism with $\|\phi\| \cdot \|\phi^{-1}\| \leq \lambda_d$. Since ℓ_d is μ -representable in E , there exist a subspace $F \subset E$ and a linear isomorphism $T \in \mathcal{L}(\ell_d^N, F)$, such that $\|T\| = 1$, $\|T^{-1}\| \leq \mu$. If $G := T(B_p^m) \subset F$, we have that $\psi := (\phi \circ T|_G^{-1}) \otimes \dots \otimes (\phi \circ T|_G^{-1}) \in \mathcal{L}(\hat{\otimes}_{s,\pi}^{d-1} G, \hat{\otimes}_{s,\pi}^{d-1} \ell_d^m)$ is an isomorphism. It satisfies that $\|\psi\| \cdot \|\psi^{-1}\| = \|\psi^*\| \cdot \|(\psi^*)^{-1}\| \leq (\|\phi\| \cdot \|\phi^{-1}\| \cdot \|T\| \cdot \|T^{-1}\|)^{d-1} \leq \lambda_d^{d-1} \mu^{d-1}$.

Let $P \in \mathcal{P}(^d G)$ be the polynomial uniquely determined by the operator $\psi^* \circ (\Pi_{D_d^m})^* \circ \phi \circ T|_G^{-1} \in \mathcal{L}(G, (\hat{\otimes}_{s,\pi}^{d-1} G)^*)$. Here $\Pi_{D_d^m} \in \mathcal{L}(\hat{\otimes}_{s,\pi}^{d-1} \ell_d^m, \hat{\otimes}_{s,\pi}^{d-1} \ell_d^m)$ is the projection map onto the main diagonal described immediately before Proposition 3.2. In this case, this diagonal is isometric to $\ell_{\frac{d}{d-1}}^m$. We denote this isometry $i_{D_d^m} \in \mathcal{L}(\ell_{\frac{d}{d-1}}^m, \hat{\otimes}_{s,\pi}^{d-1} \ell_d^m)$ and identify the projection $\Pi_{D_d^m}$ with its corresponding element in $\mathcal{L}(\hat{\otimes}_{s,\pi}^{d-1} \ell_d^m, \ell_{\frac{d}{d-1}}^m)$.

Now, by the assumption of the d -Extension Property on the space E , there exists a polynomial $\hat{P} \in \mathcal{P}(^d E)$ which extends P with a norm satisfying $\|\hat{P}\| \leq M\|P\|$. Thus, by the estimates written in Lemma 1.2, the linear operator $\Psi_1(\hat{P}|_F) \in \mathcal{L}_s(F, (\hat{\otimes}_{s,\pi}^{d-1} F)^*)$ has a norm bounded by $M \frac{d!}{d!} \|\psi^* \circ (\Pi_{D_d^m})^* \circ \phi \circ T|_G^{-1}\|$.

Now we construct a projection map defined on ℓ_d^N onto B_d^m . To facilitate the reading, we collect in a diagram the action of the operators involved in the projection:

$$\begin{array}{ccccccc}
 \ell_d^N & \xrightarrow{T} & F & \xrightarrow{\Psi_1(\hat{P}|_F)} & (\hat{\otimes}_{s,\pi}^{d-1} F)^* \\
 \uparrow & & \uparrow & & \downarrow \\
 B_d^m & \xrightarrow{T|_{B_d^m}} & G & \xrightarrow{T|_G^{-1}} & B_d^m & \xrightarrow{\phi} & \ell_d^m & \xrightarrow{(\Pi_{D_d^m})^*} & (\hat{\otimes}_{s,\pi}^{d-1} \ell_d^m)^* & \xrightarrow{\psi^*} & (\hat{\otimes}_{s,\pi}^{d-1} G)^*.
 \end{array}$$

Define $R := \phi^{-1} \circ (i_{D_d^m})^* \circ (\psi^*)^{-1} \circ (i \otimes \hat{\otimes}_{s,\pi}^{d-1} i)^* \circ \Psi_1(\hat{P}|_F) \circ T$. Using that $(i \otimes \hat{\otimes}_{s,\pi}^{d-1} i)^* \circ \Psi_1(\hat{P}|_F)$ comes from an extension of $\psi^* \circ (\Pi_{D_d^m})^* \circ \phi \circ T|_G^{-1}$ in G , it is direct to check that $R|_{B_d^m} = Id|_{B_d^m}$ and thus that R is a projection onto B_d^m .

We have the following estimation of the norm of this projection operator: $\|R\| \leq \|\phi^{-1}\| \cdot \|(i_{D_d^m})^*\| \cdot \|(\psi^*)^{-1}\| \cdot \|(i \otimes \hat{\otimes}_{s,\pi}^{d-1} i)^*\| \cdot \|\Psi_1(\hat{P}|_F)\| \cdot \|T\| \leq \|\phi^{-1}\| \cdot \|(\psi^*)^{-1}\| \cdot M \frac{d^d}{d!} \|\psi^* \circ (\Pi_{D_d^m})^* \circ \phi \circ T|_G^{-1}\| \cdot \|T\| \leq M \frac{d^d}{d!} \|\phi^{-1}\| \cdot \|\phi\| \cdot \|(\psi^*)^{-1}\| \cdot \|\psi^*\| \cdot \|T\| \cdot \|T^{-1}\| \leq M \frac{d^d}{d!} \lambda_d^d \mu^d$.

Whenever $K > M \frac{d^d}{d!} \lambda_d^d \mu^d$, this estimate gives rise to a contradiction with the claim: R being a projection onto B_d^m , it must necessarily satisfy that $\|R\| > K$.

The case $k < d$ of the theorem follows from the fact that the d -Extension Property implies the k -Extension Property if $d > k$ (see [11, Proposition 1.2]).

Finally, the equivalence in terms of symmetric tensor products is a consequence of (3) in Lemma 1.2. $\square$

The proofs of Theorem 2.2 and Theorem 3.3 provide the following constructive procedure of a polynomial which does not extend:

Proposition 3.4. *Let $d \in \mathbb{N}$, $d > 2$ (resp. $d = 2$) and let E be a Banach space such that ℓ_d (resp. ℓ_p , $1 < p < 2$) is μ -representable in E , for some $\mu > 0$. There exist a constant λ_d such that for every $K > 0$ there exist a pair of natural numbers $m, N \in \mathbb{N}$, $m < N$; a pair of subspaces $G \subset F \subset E$; and isomorphisms $T \in \mathcal{L}(\ell_d^N, F)$, $\|T\| \cdot \|T^{-1}\| \leq \lambda_d$, $S \in \mathcal{L}(\ell_d^m, G)$, $\|S\| \cdot \|S^{-1}\| \leq \lambda_d$ such that if $\hat{P} \in \mathcal{P}(^d F)$ is an extension of the polynomial $P(\sum_{i=1}^m a_i S(e_i)) := \sum_{i=1}^m a_i^d$, then $\|\hat{P}\| \geq K \|P\|$.*

In the particular case when the Banach space E has a subspace F isomorphic to ℓ_d , then, for every copy G of ℓ_d uncomplemented in F , the polynomial $P(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^\infty a_i^d$ cannot be extended to F , neither to E , where $(e_i)_i$ stands for the basis of G equivalent to the canonical basis of ℓ_d . Equivalently, for $l \in \mathbb{N} \cup \{0\}$, $\hat{\otimes}_{s,\pi}^{d+l} G$ is not closed in $\hat{\otimes}_{s,\pi}^{d+l} F$, both spaces being isomorphic to $\hat{\otimes}_{s,\pi}^{d+l} \ell_d$.

Proof. To avoid notation, we assume that the isomorphic copies of ℓ_d^m inside F are ℓ_d^m . It is proved, both in the case $d = 2$ (Theorem 2.2) and $d > 2$ (Theorem 3.3), that the polynomial which does not extend is $\Psi(Id_{\ell_d^m \rightarrow \ell_d^m})$, where Ψ is the isomorphism in Proposition 3.2 and the parameters are $k = p = q = r$ and $l = 1$. This polynomial is, precisely, $P(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^\infty a_i^d$. The argument for the case of infinite-dimensional subspaces is analogous, since so is Proposition 3.2. $\square$

4. IN CONTEXT WITH TWO OTHER EXTENSION PROPERTIES: OPEN QUESTIONS AND CLOSING REMARKS

In this final section we put the d -Extension Property in context with other extension properties by means of the canonical isomorphisms between polynomials, multilinear maps and linear operators on tensor product spaces, summarized in Lemma 1.1.

From these canonical isomorphisms one can prove the following equivalent form of the d -Extension Property, as was introduced in Definition 1.3:

Proposition 4.1. *Given $d \in \mathbb{N}$, a Banach space E has the d -Extension Property if and only if for every symmetric multilinear form $T \in \mathcal{L}_s(F \times \dots \times F)$ there exists a multilinear form $\hat{T} \in \mathcal{L}(E \times \dots \times E)$ such that $\hat{T}|_{F \times \dots \times F} = T$.*

In this case, it is possible to choose $\hat{T}$ symmetric, since the symmetrization of $\hat{T}$ is a symmetric extension of T .

This form of the property immediately reveals a relation with the so-called d -Multilinear Extension Property studied in [12]. For $d \in \mathbb{N}$, $d > 1$, recall that a Banach space E is said to have the d -Multilinear Extension Property if every continuous multilinear form defined on the product $F_1 \times \dots \times F_n$ of closed subspaces of E has an extension to a continuous multilinear form defined on $E \times \dots \times E$ (see [12, Definition 2.1]). The following relation is easily derived:

Proposition 4.2. *Given $d \in \mathbb{N}$, if E has the d -Multilinear Extension Property, then E has the d -Extension Property.*

Proof. Let E be a Banach space which has the d -Multilinear Extension Property. If we use the equivalent form in Proposition 4.1 the proof is immediate: Given a symmetric multilinear form T defined on $F \times \dots \times F$, there exists, by hypothesis, a multilinear operator $\hat{T}$ defined on $E \times \dots \times E$ which is an extension of T . $\square$

Thus far, we do not know if these two properties are actually different. Moreover, it is not known if any of them is different from having type 2, in the case where $d = 2$, or different from the space to be isomorphic to a Hilbert space when $d > 2$ (see [12, Question 3.4], [11, Problem 1], [12, Question 3.13] and [11, Problem 2], respectively). We can see, however, that symmetry plays a significant role in the study of the extendibility of multilinear maps. This is the situation in the following case:

Consider a Banach space E such that the space ℓ_p is finitely representable in E , for $2 < p < \infty$. Then:

- (1) If $\frac{p}{2} + 1 \leq d$, then E does not have the d -Multilinear Extension Property ([12, Theorem 3.8]).
- (2) If $p \leq d$ and $p \in \mathbb{N}$, then E does not have the d -Extension Property (Theorem 3.3).

We do not know if statement (2) remains true when condition “ $p \in \mathbb{N}$ ” is removed. That is, if it is true that whenever $2 < p < \infty$, $p \in \mathbb{R}$ and ℓ_p is finitely representable in a Banach space E , the space E does not have the d -Extension Property for every $p \leq d$. We can prove that this is true if the following question has an affirmative answer:

Question. Let $2 < p < q < \infty$. Do there exist a closed subspace $X_p \subset \ell_p$ isomorphic to ℓ_p and a symmetric (with respect to the corresponding canonical basis) bounded operator $S \in \mathcal{L}_s(X_p, \ell_q)$ which does not have an extension to an operator $\tilde{S} \in \mathcal{L}_s(\ell_p, \ell_q)$?

This question asks for a symmetric version of the statement “ ℓ_q does not have the M_p property, when $2 < p < q < \infty$ ”, found in [5]. It is worth mentioning, however, that for our purposes, it is necessary to ask the subspace X_p to be isomorphic to ℓ_p .

Note that the symmetry involved in the polynomial case has another effect: When $d > 2$, statement (1) holds for $\frac{p}{2} + 1 \leq d$, while the inequality in statement (2) is $p \leq d$. Here the difference arises from the loss of freedom in the choice of the subspaces: in the polynomial case, the closed subspaces $F_i \subset E$ $i = 1, \dots, d$, must all coincide.

The loss of freedom in the choice of the subspaces is also reflected in the specially relevant case of degree 2 forms: Every Banach space which has the Bilinear Extension Property has weak type 2 (see Theorem 3.1 and Remark 3.2 in [12] for this result and [20] for an extensive study of weak type 2 spaces). In the polynomial context, the question remains open:

Question. Let E be a Banach space with the 2-Extension Property. Does E have weak type 2?

Another local property comes into the picture, with a slightly different origin: the so-called *Maurey Extension Property*. Recall that a Banach space E is said to have the *Maurey Extension Property* if every bounded operator defined on an arbitrary closed subspace of E with values on an arbitrary Banach space of cotype 2 extends to a bounded operator on E . As in the 2-Extension Property and the Bilinear Extension Property cases, it is not known if the Maurey Extension Property is different from that of the space having type 2. P.G. Casazza and N. J. Nielsen have an extensive study of the Maurey Extension Property in [5], where sufficient conditions for both properties (having type 2 and the Maurey Extension Property) to be equivalent are given. The unified treatment of these local properties makes it reasonable to ask if the conditions provided in [5, Theorem 2.4] (as is, for example, the Gordon-Lewis property) are also sufficient conditions for a Banach space with the 2-Extension Property or the Bilinear Extension Property to be a type 2 space.

REFERENCES

1. A. Arias and J. D. Farmer, *On the structure of tensor products of l_p -spaces*. Pacific J. Math. 175 (1996), no. 1, 13–37. MR1419470 (98d:46015)
2. R. M. Aron and P. D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France 106 (1978), no. 1, 3–24. MR508947 (80e:46029)
3. G. Bennett, L. E. Dor, V. Goodman, W. B. Johnson and C. M. Newman, *On uncomplemented subspaces of L_p , $1 < p < 2$* . Israel J. Math. 26 (1977), no. 2, 178–187. MR0435822 (55:8778)
4. D. Carando, *Extendibility of polynomials and analytic functions on l_p* . Studia Math. 145 (2001), no. 1, 63–73. MR1828993 (2002d:46042)
5. P. G. Casazza and N. J. Nielsen, *The Maurey extension property for Banach spaces with the Gordon-Lewis property and related structures*, Studia Math. 155 (1) (2003), 1–21. MR1961157 (2003k:46016)
6. A. M. Davie and T. W. Gamelin, *A theorem on polynomial-star approximation*. Proc. Amer. Math. Soc. 106 (1989), no. 2, 351–356. MR947313 (89k:46023)
7. A. Defant and K. Floret, *Tensor norms and operator ideals*. North-Holland Mathematics Studies, 176. North-Holland Publishing Co., Amsterdam, 1993. MR1209438 (94e:46130)

8. J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995. MR1342297 (96i:46001)
9. S. Dineen, *Quasinormable spaces of holomorphic functions*. Note Mat. 13 (1993), no. 1, 155-195. MR1283528 (95i:46050)
10. S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Mathematics, 1999. MR1705327 (2001a:46043)
11. M. Fernández-Unzueta and A. Prieto, *Extension of polynomials defined on subspaces*, Math. Proc. Cambridge Philos. Soc. 148 (2010), no. 3, 505-518. MR2609305 (2011g:46081)
12. M. Fernández-Unzueta, *Extension of multilinear maps defined on subspaces*. Israel J. Math. 188 (2012), 301-322. MR2897734
13. A. Kamińska and H. J. Lee, *On uniqueness of extension of homogeneous polynomials*. Houston J. Math. 32 (2006), no. 1, 227-252. MR2202363 (2006j:46054)
14. A. Kamińska and P. K. Lin, *On the extensions of homogeneous polynomials*. Proc. Amer. Math. Soc. 135 (2007), no. 8, 2471-2482. MR2302568 (2008e:46055)
15. P. Kirwan and R. A. Ryan, *Extendibility of homogeneous polynomials on Banach spaces*. Proc. Amer. Math. Soc. 126 (1998), no. 4, 1023-1029. MR1415346 (98f:46042)
16. P. K. Lin, *Extensions of 2-homogeneous polynomials on c_0* . Acta Math. Sin. (Engl. Ser.) 24 (2008), no. 5, 877-880. MR2403121 (2009d:46080)
17. B. Maurey, *Un théorème de prolongement*, C. R. Acad. Sci. Paris, A 279 (1974), 329-332. MR0355539 (50:8013)
18. J. Mujica, *Complex analysis in Banach spaces. Holomorphic functions and domains of holomorphy in finite and infinite dimensions*. North-Holland Mathematics Studies, 120. North-Holland Publishing Co., Amsterdam, 1986. MR842435 (88d:46084)
19. A. Pelczyński, *A theorem of Dunford-Pettis type for polynomial operators*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 11 (1963), 379-386. MR0161161 (28:4370)
20. G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics, 94. Cambridge University Press, 1989. MR1036275 (91d:52005)
21. R. A. Ryan, *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002. MR1888309 (2003f:46030)
22. H. P. Rosenthal, *On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273-303. MR0271721 (42:6602)
23. I. Zalduendo, *Extending polynomials on Banach spaces—a survey*. Rev. Un. Mat. Argentina 46 (2005), no. 2, 45-72. MR2281672 (2008k:46133)

CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS (CIMAT), A.P. 402, 36000 GUANAJUATO, GTO., MÉXICO

E-mail address: maite@cimat.mx