EQUIVALENT NORMS ON FOCK SPACES WITH SOME APPLICATION TO EXTENDED CESARO OPERATORS

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ABSTRACT. Let F_γ^p be the Fock space of all holomorphic functions f in $\mathbf{C}^n$ with the Fock norm

$$\|f\|_{p,\gamma} = \left\{ \int_{\mathbf{C}^n} \left| f(z) e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}} < \infty,$$

where p, γ are positive numbers. We prove that, given any positive integer m , the Fock norm $\|f\|_{p,\gamma}$ is equivalent to

$$\sum_{|\alpha| \leq m-1} |\partial^\alpha f(0)| + \left\{ \sum_{|\alpha|=m} \int_{\mathbf{C}^n} \left| \partial^\alpha f(z) (1+|z|)^{-m} e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}}.$$

As some application we characterize these holomorphic functions g in $\mathbf{C}^n$ for which the induced extended Cesaro operator T_g is bounded (or compact) from one Fock space F_γ^p to another F_γ^q .

1. INTRODUCTION

Let $\mathbf{C}^n$ be the n -dimensional complex Euclidean space. For any $z = (z_1, \dots, z_n)$, we write $|z| = \sqrt{\sum_{j=1}^n |z_j|^2}$. Let $H(\mathbf{C}^n)$ be the family of all holomorphic functions in $\mathbf{C}^n$. For $f \in H(\mathbf{C}^n)$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ a multi-index with $|\alpha| = \sum_{j=1}^n \alpha_j$, write $\partial^\alpha f(z) = \frac{\partial^{|\alpha|} f}{\partial \alpha z}(z)$. Set $dA(z)$ to be the ordinary Lebesgue volume measure on $\mathbf{C}^n$. For $0 < p < \infty$ and $\gamma > 0$, the Lebesgue space L_γ^p consists of all measurable functions f for which

$$\|f\|_{p,\gamma} = \left\{ \int_{\mathbf{C}^n} \left| f(z) e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}} < \infty.$$

We define the Fock space $F_\gamma^p = L_\gamma^p \cap H(\mathbf{C}^n)$. It is clear that, for $p \geq 1$, L_γ^p is a Banach space under the norm $\|\cdot\|_{p,\gamma}$, F_γ^p is a closed subspace of L_γ^p , and F_γ^2 is a Hilbert space. For $0 < p < 1$, F_γ^p is an F-space under $d(f, g) = \|f - g\|_{p,\gamma}^p$. The Fock space has been studied in [4, 5, 6] and by some other authors.

In the setting of a Bergman space on the unit ball B , it is well known that f is in some p -th Bergman space if and only if $\sum_{|\alpha|=m} |\partial^\alpha f| (1 - |z|)^m \in L^p(B, dA)$ for some (or any) positive integer m . This kind of equivalence makes the Bergman space

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be connected with Bloch space and Besov space. Also, it can be applied to some other topics such as Gleason's problem on the Bergman space. See [13, 14, 16, 17] for details. Now it is natural to ask if we can characterize Fock space functions via their partial derivatives. We will prove in the next section, for $f \in H(\mathbf{C}^n)$, that $f \in F_\gamma^p$ if and only if $\sum_{|\alpha|=m} |\partial^\alpha f| (1 + |z|)^{-m} \in L_\gamma^p$ for some (or any) positive integral m .

In [8], the extended Cesaro operator T_g was introduced on the unit ball of $\mathbf{C}^n$. The behavior of T_g on the Bergman space and the Schatten ideal property was studied respectively in [8] and [10]. It is trivial that T_g can also be well defined on $\mathbf{C}^n$. Given a function $g \in H(\mathbf{C}^n)$, set $\Re g = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}$ to be the radial derivative of g ; see [15]. For $m = 1, 2, \dots$, set $\Re^m f = \Re(\Re^{m-1} f)$. We give the following.

Definition. Let $g \in H(\mathbf{C}^n)$. The extended Cesoro operator T_g on $H(\mathbf{C}^n)$ is defined as

$$T_g(f)(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad z \in \mathbf{C}^n.$$

In section 3, with the help of our equivalent norms on F_γ^p , we will characterize those $g \in H(\mathbf{C}^n)$ for which the induced operator T_g is bounded (or compact) from F_γ^p to F_γ^q with $0 < p, q < \infty$.

In what follows, C and C_1, C_2 will stand for positive constants whose value may change from line to line. Two quantities A and B are called equivalent, denoted by $A \simeq B$, if there exist C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$.

2. EQUIVALENT NORMS ON FOCK SPACES

In this section we will give the equivalent norms on F_γ^p . To do this, we need more notation. Given $z \in \mathbf{C}^n$ and $\delta > 0$, set the Euclidean ball $D(z, \delta) = \{w \in \mathbf{C}^n : |w - z| < \delta\}$ and its volume $|D(z, \delta)| \simeq \delta^{2n}$. Set $S = \{z \in \mathbf{C}^n : |z| = 1\}$, and set $d\sigma$ to be the rotation-invariant positive Borel measure on S with $\sigma(S) = 1$; see [15]. For $f \in H(\mathbf{C}^n)$, the p -th integral mean on rS , $0 < p < \infty$, is defined as $M_p(f, r) = \left\{ \int_S |f(r\xi)|^p d\sigma(\xi) \right\}^{\frac{1}{p}}$. It is well known that $M_p(f, r)$ is increasing with $r \in [0, \infty)$. We write $\omega(z) = \frac{1}{1+|z|}$ for $z \in \mathbf{C}^n$.

Lemma 2.1. Let $\{k_m\}$ be a sequence of positive numbers satisfying

$$(1) \quad \frac{k_{m+1}}{k_m} \leq c$$

for some $c \in (0, 1)$, and let $\beta > 0$. Then there exist some constants C_1 and C_2 depending only on c and β such that, for all sequences $\{A_m\}$ of complex numbers,

$$(2) \quad C_1 \sum_{m=1}^{\infty} k_m |A_m|^\beta \leq \sum_{m=2}^{\infty} k_m |A_m - A_{m-1}|^\beta + k_1 |A_1|^\beta \leq C_2 \sum_{m=1}^{\infty} k_m |A_m|^\beta.$$

Proof. It is easy to see that, for $a, b \in (0, \infty)$,

$$(3) \quad (a + b)^\beta \leq \max(1, 2^{\beta-1})(a^\beta + b^\beta).$$

Let $\{A_m\}$ be a sequence of complex numbers. By (3) and (1),

$$\sum_{m=2}^J k_m |A_m - A_{m-1}|^\beta \leq C \sum_{m=2}^J k_m (|A_m|^\beta + |A_{m-1}|^\beta) \leq C \sum_{m=1}^{\infty} k_m |A_m|^\beta.$$

Let $J \rightarrow \infty$ to obtain

$$(4) \quad \sum_{m=2}^{\infty} k_m |A_m - A_{m-1}|^{\beta} + k_1 |A_1|^{\beta} \leq C \sum_{m=1}^{\infty} k_m |A_m|^{\beta}.$$

Now we prove the reverse inequality. If $0 < \beta < 1$, then by (3) and (1) again,

$$\sum_{m=1}^J k_m |A_m|^{\beta} \leq \sum_{k=2}^J k_m |A_m - A_{m-1}|^{\beta} + c \sum_{m=2}^J k_{m-1} |A_{m-1}|^{\beta} + k_1 |A_1|^{\beta}.$$

This implies

$$(5) \quad \sum_{m=1}^J k_m |A_m|^{\beta} \leq C \left(\sum_{m=2}^{\infty} k_m |A_m - A_{m-1}|^{\beta} + k_1 |A_1|^{\beta} \right).$$

If $\beta \in [1, \infty)$, by the Minkowsky inequality

$$\begin{aligned} \left\{ \sum_{m=2}^J k_m |A_m|^{\beta} \right\}^{\frac{1}{\beta}} &= \left\{ \sum_{m=2}^J k_m |(A_m - A_{m-1}) - A_{m-1}|^{\beta} \right\}^{\frac{1}{\beta}} \\ &\leq \left\{ \sum_{m=2}^J k_m |A_m - A_{m-1}|^{\beta} \right\}^{\frac{1}{\beta}} + c^{\frac{1}{\beta}} \left\{ \sum_{m=2}^J k_{m-1} |A_{m-1}|^{\beta} \right\}^{\frac{1}{\beta}}. \end{aligned}$$

With this we have (5) again. Then we have

$$(6) \quad \sum_{m=1}^{\infty} k_m |A_m|^{\beta} \leq C \left\{ \sum_{m=2}^{\infty} k_m |A_m - A_{m-1}|^{\beta} + k_1 |A_1|^{\beta} \right\}.$$

The estimates (2) come from (4) and (6). The proof is completed. $\square$

Lemma 2.2. Suppose $1 < p < \infty$, $0 < s < \infty$ and $k \in \mathbf{R}$. There is some constant C such that, for all locally integrable function f on $[0, \infty)$,

$$(7) \quad \int_0^{\infty} \left| \int_0^r f(t) dt \right|^p (1+r)^k e^{-sr^2} dr \leq C \int_0^{\infty} |f(r)|^p (1+r)^{k-p} e^{-sr^2} dr.$$

Proof. We set $\varphi(r) = (1+r)^k e^{-sr^2}$ and $\psi(r) = (1+r)^{k-p} e^{-sr^2}$, and denote by p' the conjugate of p . Then by L'Hospital's rule we know that

$$\lim_{r \rightarrow \infty} \frac{\int_r^{\infty} \varphi(t) dt}{r^{k-1} e^{-sr^2}} = \frac{1}{2s} > 0$$

and

$$\lim_{r \rightarrow \infty} \frac{\int_0^r \psi(t)^{1-p'} dt}{r^{-\frac{k-p}{p-1}-1} e^{\frac{s}{p-1}r^2}} = \frac{p-1}{2s} > 0.$$

Therefore

$$\begin{aligned} &\lim_{r \rightarrow \infty} \left(\int_r^{\infty} \varphi(r) dr \right)^{\frac{1}{p}} \left(\int_0^r \psi(r)^{1-p'} dr \right)^{\frac{1}{p'}} \\ &= \left(\frac{1}{2s} \right)^{\frac{1}{p}} \left(\frac{p-1}{2s} \right)^{\frac{1}{p'}} \lim_{r \rightarrow \infty} \left(r^{k-1} e^{-sr^2} \right)^{\frac{1}{p}} \left(r^{-\frac{k-p}{p-1}-1} e^{\frac{s}{p-1}r^2} \right)^{\frac{1}{p'}} \\ &= \left(\frac{1}{2s} \right)^{\frac{1}{p}} \left(\frac{p-1}{2s} \right)^{\frac{1}{p'}} \in (0, \infty). \end{aligned}$$

Notice that $\int_0^\infty \varphi(t)dt < \infty$, and by $\psi(r)^{1-p'} \in C[0, R]$ for all $R > 0$, we have

$$\sup_{r \in [0, \infty)} \left(\int_r^\infty \varphi(r)dr \right)^{\frac{1}{p}} \left(\int_0^r \psi(r)^{1-p'} dr \right)^{\frac{1}{p'}} < \infty.$$

Apply the Riemann-Liouville Integral Theorem (see [12]) to obtain (7). The lemma is proved. $\square$

Here is the main result of the paper.

Theorem 2.1. *Let $0 < p, \gamma < \infty$, and let m be some (or any) positive integer. Then the following three quantities are equivalent for all $f \in H(\mathbf{C}^n)$:*

(A) $\|f\|_{p, \gamma}$;

(B) $\sum_{|\alpha| \leq m-1} |\partial^\alpha f(0)| + \left\{ \sum_{|\alpha|=m} \int_{\mathbf{C}^n} \left| \partial^\alpha f(z) \omega^m(z) e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}}$;

(C) $|f(0)| + \left\{ \int_{\mathbf{C}^n} \left| \Re^m f(z) \omega^{2m}(z) e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}}$.

In other words, we have

$$\begin{aligned} \|f\|_{p, \gamma} &\simeq \sum_{|\alpha| \leq m-1} |\partial^\alpha f(0)| + \left\{ \sum_{|\alpha|=m} \int_{\mathbf{C}^n} \left| \partial^\alpha f(z) \omega^m(z) e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}} \\ &\simeq |f(0)| + \left\{ \int_{\mathbf{C}^n} \left| \Re^m f(z) \omega^{2m}(z) e^{-\frac{\gamma|z|^2}{2}} \right|^p dA(z) \right\}^{\frac{1}{p}}. \end{aligned}$$

Proof. To simplify our notation, set $t = \frac{\gamma}{2}$. First, we prove that (B) can be dominated by (A). We claim that

$$(8) \quad \frac{1}{|D(u, 2\omega(u))|} \int_{D(u, 2\omega(u))} e^{-t|z|^2} dA(z) \leq C e^{-t|u|^2} \quad \text{for } u \in \mathbf{C}^n.$$

To see this, we can only deal with $|u| \geq 2$. For $z \in D(u, 2\omega(u))$, then $|z| > |u| - 2\omega(u)$; hence

$$\frac{1}{|D(u, 2\omega(u))|} \int_{D(u, 2\omega(u))} e^{-t|z|^2} dA(z) \leq \sup_{|z| \geq |u| - 2\omega(u)} e^{-t|z|^2} \leq C e^{-t|u|^2}.$$

Meanwhile, by the subharmonicity of $|f(z)|^p$, with the Cauchy estimate as in [11] we have some constant depending only on p and m such that for all $z \in \mathbf{C}^n$ and $|\alpha| = m$,

$$(9) \quad |\partial^\alpha f(z)|^p \leq C \frac{1}{\omega(z)^{2n+mp}} \int_{D(z, \omega(z))} |f(w)|^p dA(w).$$

This gives

$$(10) \quad |\partial^\alpha f(0)| \leq C \left(\int_{D(0,1)} |f(u)^p dA(u) \right)^{\frac{1}{p}} \leq C \|f\|_{p,\gamma}.$$

Also, it is easy to verify $|u - z| < 2\omega(u)$ if $|u - z| < \omega(z)$. Therefore, (8) and (9) yield

$$\begin{aligned} & \int_{\mathbf{C}^n} \left| \partial^\alpha f(z) \omega^m(z) e^{-t|z|^2} \right|^p dA(z) \\ & \leq C \int_{\mathbf{C}^n} \frac{e^{-tp|z|^2}}{\omega^{2n}(z)} dA(z) \int_{\mathbf{C}^n} |f(u)|^p \chi_{D(z, \omega(z))}(u) dA(u) \\ & \leq C \int_{\mathbf{C}^n} \left| f(w) e^{-t|w|^2} \right|^p dA(w). \end{aligned}$$

This and (10) yield (B) $\leq C(A)$.

For (A) $\leq C(C)$, we suppose $n = 1$ temporarily and prove

$$(11) \quad \int_{\mathbf{C}} \left| f(z) \omega^k(z) e^{-t|z|^2} \right|^p dA(z) \leq C \int_{\mathbf{C}} \left| f'(z) \omega^{k+1}(z) e^{-t|z|^2} \right|^p dA(z)$$

for fixed $p > 0$, $k \geq 0$, and for all $f \in H(\mathbf{C})$ with $f(0) = 0$. If $0 < p \leq 1$, Hardy's inequality implies, for $0 < \rho \leq r < \infty$,

$$(12) \quad M_p^p(f, r) - M_p^p(f, \rho) \leq C(r - \rho)^p M_p^p(f', r);$$

see Lemma 3.3 in [13]. Notice that $M_p(f, r)$ is increasing with r ; hence

$$\begin{aligned} & \int_{\mathbf{C}} \left| f(z) \omega^k(z) e^{-t|z|^2} \right|^p dA(z) \\ & = \sum_{s=1}^{\infty} \int_{\sqrt{s-1}}^{\sqrt{s}} M_p^p(f, r) \omega^{pk}(r) e^{-ptr^2} r dr \\ & \leq C \sum_{s=1}^{\infty} M_p^p(f, \sqrt{s}) \omega^{pk}(\sqrt{s}) e^{-pts} \sqrt{s} [\sqrt{s} - \sqrt{s-1}] \\ & \leq C \sum_{s=1}^{\infty} M_p^p(f, \sqrt{s}) \omega^{pk}(\sqrt{s}) e^{-pts}. \end{aligned}$$

Set $k_s = \omega^{pk}(\sqrt{s})e^{-pts}$ and $A_s = M_p^p(f, \sqrt{s})$ for $s = 1, 2, \dots$, and $A_0 = M_p^p(f, 0) = f(0) = 0$. Apply Lemma 2.1 and (12) to obtain

$$\begin{aligned}
 & \int_{\mathbf{C}} \left| f(z) \omega^k(z) e^{-t|z|^2} \right|^p dA(z) \\
 & \leq C \sum_{s=1}^{\infty} \omega^{pk}(\sqrt{s}) e^{-pts} \left| M_p^p(f, \sqrt{s}) - M_p^p(f, \sqrt{s-1}) \right| \\
 & \leq C \sum_{s=1}^{\infty} \omega^{pk}(\sqrt{s}) e^{-pts} [\sqrt{s} - \sqrt{s-1}]^p M_p^p(f', \sqrt{s}) \\
 & \leq C \left\{ \sum_{s=1}^{\infty} \omega^{pk}(\sqrt{s}) e^{-pts} \left(\frac{1}{s} \right)^{\frac{p}{2}} M_p^p(f', \sqrt{s}) \right\} \\
 & \leq C \left\{ \sum_{s=1}^{\infty} \int_{\sqrt{s}}^{\sqrt{s+1}} M_p^p(f', r) \omega^{p(k+1)}(r) e^{-ptr^2} r dr \right\} \\
 & \leq C \int_1^{\infty} M_p^p(f', r) \omega^{p(k+1)}(r) e^{-ptr^2} r dr \\
 (13) \quad & = C \int_{\mathbf{C}} |f'(z) \omega^{k+1}(z)|^p e^{-t|z|^2} dA(z).
 \end{aligned}$$

If $p > 1$, by Lemma 2.2 we know

$$\begin{aligned}
 \int_0^{\infty} \left| f(r\zeta) \omega^k(z) e^{-tr^2} \right|^p dr &= \int_0^{\infty} \left| \left(\int_0^r f'(re^{i\theta}) dr \right) \omega^k(r) e^{-tr^2} \right|^p dr \\
 &\leq C \int_0^{\infty} \left| f'(re^{i\theta}) \omega^{k+1}(r) e^{-tr^2} \right|^p dr.
 \end{aligned}$$

Hence, by the monotonicity of $M_p^p(f', r)$ we have

$$\begin{aligned}
 & \int_{\mathbf{C}} \left| f(z) \omega^k(z) e^{-t|z|^2} \right|^p dA(z) \\
 & \leq C \int_0^{2\pi} \left(\int_0^{\infty} \left| f(re^{i\theta}) \omega^k(r) e^{-tr^2} \right|^p (1+r) dr \right) d\theta \\
 & \leq C \int_0^{2\pi} \left(\int_0^{\infty} \left| f'(re^{i\theta}) \omega^{k+1}(r) e^{-tr^2} \right|^p (1+r) dr \right) d\theta \\
 & = C \int_0^{\infty} \left(\int_0^{2\pi} \left| f'(re^{i\theta}) \omega^{k+1}(r) e^{-tr^2} \right|^p (1+r) d\theta \right) dr \\
 & = C \int_0^{\infty} M_p^p(f', r) \left(\omega^{k+1}(r) e^{-tr^2} \right)^p (1+r) dr \\
 & = C \left\{ \int_0^1 + \int_1^{\infty} \right\} M_p^p(f', r) \left(\omega^{k+1}(r) e^{-tr^2} \right)^p (1+r) dr \\
 & \leq C \left\{ M_p^p(f', 1) + \int_1^{\infty} M_p^p(f', r) \left(\omega^{k+1}(r) e^{-tr^2} \right)^p r dr \right\} \\
 & \leq C \left\{ \int_1^2 + \int_1^{\infty} \right\} M_p^p(f', r) \left(\omega^{k+1}(r) e^{-tr^2} \right)^p r dr \\
 (14) \quad & \leq C \int_{\mathbf{C}} |f'(z) \omega^{k+1}(z) e^{-t|z|^2}|^p dA(z).
 \end{aligned}$$

From (13) and (14), (11) follows. Now we go back to a higher dimension. Let $f \in H(\mathbf{C}^n)$ with $f(0) = 0$. For each $\zeta \in S$ set the slice function $f_\zeta(\cdot) = f(\cdot, \zeta) \in H(\mathbf{C})$. By 1.4.3 and 1.4.7 in [15] we know from (11) that, for $p > 0$ and $k \geq 0$,

$$\begin{aligned} & \int_{\mathbf{C}^n} |f(z) \omega^k(z) e^{-t|z|^2}|^p dA(z) \\ &= \frac{1}{2\pi} \int_0^\infty r^{2n-1} dr \int_S d\sigma(\zeta) \frac{1}{2\pi} \int_{-\pi}^\pi |f_\zeta(re^{i\theta}) \omega^k(r) e^{-tr^2}|^p d\theta \\ &= \int_S d\sigma(\zeta) \int_{\mathbf{C}} |f_\zeta(u) \omega^k(u) e^{-t|u|^2}|^p |u|^{2(n-1)} dA(u) \\ &\leq C \int_S d\sigma(\zeta) \int_{\mathbf{C}} |(f_\zeta)'(u) \omega^{k+1}(u) e^{-t|u|^2}|^p |u|^{2(n-1)} dA(u). \end{aligned}$$

Since $\frac{\Re f(u\zeta)}{u}$ is holomorphic in $u \in \mathbf{C}$, we get

$$\begin{aligned} & \int_{\{z \in \mathbf{C}: |z| < 1\}} \left| \frac{\Re f(u\zeta)}{u} \omega^{k+1}(u) e^{-t|u|^2} \right|^p |u|^{2(n-1)} dA(u) \\ &\leq C \frac{1}{2\pi} \int_{-\pi}^\pi \left| \frac{\Re f(e^{i\theta}\zeta)}{e^{i\theta}} \right|^p d\theta \\ &\leq C \int_{\{z \in \mathbf{C}: 1 < |z| < 2\}} |\Re f(u\zeta) \omega^{k+2}(u) e^{-t|u|^2}|^p |u|^{2(n-1)} dA(u). \end{aligned}$$

Then

$$\begin{aligned} & \int_{\mathbf{C}} \left| \frac{\Re f(u\zeta)}{u} \omega^{k+1}(u) e^{-t|u|^2} \right|^p |u|^{2(n-1)} dA(u) \\ &\leq C \int_{\mathbf{C}} |\Re f(u\zeta) \omega^{k+2}(u) e^{-t|u|^2}|^p |u|^{2(n-1)} dA(u). \end{aligned}$$

Apply Proposition 6.4.4 in [15] to obtain

$$\begin{aligned} & \int_{\mathbf{C}^n} |f(z) \omega^k(z) e^{-t|z|^2}|^p dA(z) \\ &\leq C \int_S d\sigma(\zeta) \int_{\mathbf{C}} \left| \frac{\Re f(u\zeta)}{u} \omega^{k+1}(u) e^{-t|u|^2} \right|^p |u|^{2(n-1)} dA(u) \\ &\leq C \int_S d\sigma(\zeta) \int_{\mathbf{C}} |\Re f(u\zeta) \omega^{k+2}(u) e^{-t|u|^2}|^p |u|^{2(n-1)} dA(u) \\ (15) \quad &\leq C \int_{\mathbf{C}^n} |\Re f(z) \omega^{k+2}(z)|^p e^{-t|z|^2} dA(z), \end{aligned}$$

provided $f(0) = 0$. Take $k = 0, 2, 4, \dots, 2(m-1)$ inductively in the above estimate (15) and notice that $\Re^k f(0) = 0$ for $k \geq 1$:

$$\begin{aligned} & \int_{\mathbf{C}^n} \left| f(z) e^{-t|z|^2} \right|^p dA(z) \\ & \leq C \left\{ \int_{\mathbf{C}^n} \left| [f(z) - f(0)] e^{-t|z|^2} \right|^p dA(z) + |f(0)|^p \right\} \\ & \leq C \left\{ \int_{\mathbf{C}^n} \left| \Re f(z) \omega^2(z) e^{-t|z|^2} \right|^p dA(z) + |f(0)|^p \right\} \\ & \dots\dots \\ & \leq C \left\{ \int_{\mathbf{C}^n} \left| \Re^m f(z) \omega^{2m}(z) e^{-t|z|^2} \right|^p dA(z) + |f(0)|^p \right\}. \end{aligned}$$

This gives the desired estimate (A) $\leq C$ (C).

To prove (A) $\leq C$ (B), we notice that $|\Re f(z)| \leq |z| |\nabla f(z)| \leq C|z| \sum_{j=1}^n \left| \frac{\partial f}{\partial z_j} \right|$. By (14) we have

$$\int_{\mathbf{C}^n} \left| f(z) \omega^k(z) e^{-t|z|^2} \right|^p dA(z) \leq C \left\{ |f(0)|^p + \int_{\mathbf{C}^n} \sum_{j=1}^n \left| \frac{\partial f}{\partial z_j} \omega^{k+1}(z) e^{-t|z|^2} \right|^p dA(z) \right\}.$$

Inductively,

$$\begin{aligned} & \int_{\mathbf{C}^n} \left| f(z) \omega^k(z) e^{-t|z|^2} \right|^p dA(z) \\ & \leq C \left\{ \sum_{|\alpha|=m} \int_{\mathbf{C}^n} \left| \partial^\alpha f \omega^{k+m}(z) e^{-t|z|^2} \right|^p dA(z) + \sum_{|\alpha| \leq m-1} |\partial^\alpha f(0)|^p \right\}. \end{aligned}$$

Taking $k = 0$ in the above inequality we see that (A) $\leq C$ (B).

We now prove that (C) can be dominated by (B). An elementary calculation shows that there are some constants a_α such that

$$\Re^m f(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha \partial^\alpha f(z) \quad \text{for } f \in H(\mathbf{C}^n).$$

Then

$$|\Re^m f(z) \omega^{2m}(z)|^p \leq C \sum_{|\alpha| \leq m} |\partial^\alpha f(z) \omega^m(z)|^p \leq C \sum_{|\alpha| \leq m} \left| \partial^\alpha f(z) \omega^{|\alpha|}(z) \right|^p.$$

It yields

$$\begin{aligned} & \int_{\mathbf{C}^n} \left| \Re^m f(z) \omega^{2m}(z) e^{-t|z|^2} \right|^p dA(z) \\ & \leq C \sum_{|\alpha| \leq m} \int_{\mathbf{C}^n} \left| \partial^\alpha f(z) \omega^{|\alpha|}(z) e^{-t|z|^2} \right|^p dA(z) \\ & \leq C \sum_{|\alpha|=m} \left\{ \sum_{|\beta| \leq m-|\alpha|-1} |\partial^{\beta+\alpha} f(0)|^p + \int_{\mathbf{C}} \left| \partial^\alpha f(z) \omega^m(z) e^{-t|z|^2} \right|^p \right\} \\ & = C \left\{ \sum_{|\alpha| \leq m-1} |\partial^\alpha f(0)|^p + \sum_{|\alpha|=m} \int_{\mathbf{C}} \left| \partial^\alpha f(z) \omega^m(z) e^{-t|z|^2} \right|^p \right\}. \end{aligned}$$

Therefore, (C) can be dominated by (B). The proof of the theorem is completed. $\square$

Remark. In [7], it is claimed that “in general, the Fock space cannot be characterized in terms of membership of partial derivatives in L^p spaces” (see pages 470 and 483 therein). Our Theorem 2.1 shows that the claim in [7] is false.

3. THE EXTENDED CESARO OPERATOR

In this section we are going to give some application of the estimate in the previous section. For this purpose, we need Fock Carleson measures which appear in [9].

Definition. Let $0 < p, q < \infty$ and let μ be a positive Borel measure on $\mathbf{C}^n$ (denoted by $\mu \geq 0$). We call μ a (p, q) -Fock Carleson measure if the embedding operator $i : F_\alpha^p \rightarrow L_\alpha^q(d\mu)$ is bounded; i.e. there exists some constant C such that for all $f \in F_\alpha^p$,

$$\|f\|_{L_\alpha^q(d\mu)} = \left(\int_{\mathbf{C}^n} \left| f(z) e^{-\frac{\alpha}{2}|z|^2} \right|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{p,\alpha}.$$

We will use $\|\mu\|$ to stand for the operator norm of i from F_α^p to $L_\alpha^q(d\mu)$. Also, we call μ a vanishing (p, q) -Fock Carleson measure if

$$\lim_{j \rightarrow \infty} \int_{\mathbf{C}^n} \left| f_j(z) e^{-\frac{\alpha}{2}|z|^2} \right|^q d\mu(z) = 0$$

whenever $\{f_j\}$ is a bounded sequence in F_α^p that converges to 0 uniformly on compact subsets of $\mathbf{C}^n$ as $j \rightarrow \infty$.

The following three lemmas (Lemma 3.1, 3.2 and 3.3) are from [9] as well.

Lemma 3.1. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent:*

- (1) μ is a (p, q) -Fock Carleson measure;
- (2) $\mu(B(\cdot, \delta))$ is bounded on $\mathbf{C}^n$ for some (or any) $\delta > 0$.

Furthermore,

$$\|\mu\| \simeq \sup_{z \in \mathbf{C}^n} \mu(B(z, \delta))^{\frac{1}{q}}.$$

Lemma 3.2. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent:*

- (1) μ is a vanishing (p, q) -Fock Carleson measure;
- (2) $\mu(B(z, \delta)) \rightarrow 0$ as $z \rightarrow \infty$ for some (or any) $\delta > 0$.

Lemma 3.3. *Let $0 < q < p < \infty$, and let $\mu \geq 0$. Set $s = \frac{p}{q}$. Then the following statements are equivalent:*

- (1) μ is a (p, q) -Fock Carleson measure;
- (2) μ is a vanishing (p, q) -Fock Carleson measure;
- (3) $\mu(B(\cdot, \delta)) \in L^{s'}(dA)$ for some (or any) $\delta > 0$ with s' the conjugate exponent of s .

Furthermore,

$$\|\mu\| \simeq \|\mu(B(\cdot, \delta))\|_{L^{s'}(dA)}^{\frac{1}{q}}.$$

The next lemma is a generalization of the well-known Liouville theorem in higher dimension.

Lemma 3.4. *Let $0 < p \leq \infty$ and $m > 0$. Then for $f \in H(\mathbf{C}^n)$, the limit $\lim_{r \rightarrow \infty} \frac{M_p(f, r)}{r^m}$ exists (or $\frac{M_p(f, r)}{r^m}$ remains bounded on $[0, \infty)$) if and only if f is a polynomial of degree not more than m .*

Proof. If $f(z) = \sum_{|\alpha| \leq J} a_\alpha z^\alpha$ with $J \leq m$, it is easy to verify that

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{M_p(f, r)}{r^m} &= \lim_{r \rightarrow \infty} \left\{ \int_S \left| \sum_{|\alpha| \leq J} a_\alpha \frac{1}{r^{J-|\alpha|}} \zeta^\alpha \right|^p d\sigma(\zeta) \right\}^{\frac{1}{p}} \frac{1}{r^{m-J}} \\ &= \left\{ \int_S \left| \sum_{|\alpha|=J} a_\alpha \zeta^\alpha \right|^p d\sigma(\zeta) \right\}^{\frac{1}{p}} \lim_{r \rightarrow \infty} \frac{1}{r^{m-J}} \\ &= \left\{ \int_S \left| \sum_{|\alpha|=J} a_\alpha \zeta^\alpha \right|^p d\sigma(\zeta) \right\}^{\frac{1}{p}} \delta_{m,J}, \end{aligned}$$

where $\delta_{m,J} = 1$ if $m = J$ and $\delta_{m,J} = 0$ if $m \neq J$. Conversely, if $\lim_{r \rightarrow \infty} \frac{M_p(f, r)}{r^m}$ exists, then we have some constant C such that

$$\frac{M_p(f, r)}{r^m} \leq C$$

for all $r > 1$. Write $f(z) = \sum_{|\alpha| \geq 0} b_\alpha z^\alpha$. Similar to the proof of the Liouville theorem in standard complex analysis, for any multi-index α with $|\alpha| > m$, with the Cauchy estimate and the subharmonicity of $|f|^p$,

$$|b_\alpha| \leq C \frac{\max_{|\zeta|=r} |f(\zeta)|}{r^{|\alpha|}} \leq C \frac{M_p(f, 2r)}{r^m} \frac{1}{r^{|\alpha|-m}} \rightarrow 0$$

as $r \rightarrow \infty$. Therefore, f is a polynomial of degree not more than m . The proof is completed. $\square$

Theorem 3.1. *Let $0 < p \leq q < \infty$, and let $g \in H(\mathbf{C}^n)$. Then T_g is bounded from F_γ^p to F_γ^q if and only if g is a polynomial of degree not more than two. Also, T_g is compact from F_γ^p to F_γ^q if and only if g is a polynomial of degree not more than one. Furthermore,*

$$(16) \quad \|T_g\| \simeq \sup_{z \in \mathbf{C}^n} |\Re g(z) \omega^2(z)|.$$

Proof. Suppose $g \in H(\mathbf{C}^n)$; as in [8] we have $\Re T_g f(z) = f(z) \Re g(z)$. Then by Theorem 2.1,

$$(17) \quad \|T_g f\|_{q,\gamma} \simeq \left\{ \int_{\mathbf{C}^n} \left| f(z) e^{-\alpha|z|^2} \right|^q |\Re g(z) \omega^2(z)|^q dA(z) \right\}^{\frac{1}{q}}.$$

Therefore, the operator T_g is bounded if and only if

$$(18) \quad \left\{ \int_{\mathbf{C}^n} \left| f(z) e^{-\alpha|z|^2} \right|^q |\Re g(z) \omega^2(z)|^q dA(z) \right\}^{\frac{1}{q}} \leq C \|f\|_{p,\gamma},$$

equivalently, if and only if

$$d\mu = |\Re g(z) \omega^2(z)|^q dA(z) \quad \text{is a } (p, q)\text{-Fock Carleson measure.}$$

By the subharmonicity of $|\Re g(z)|^q$, for $z \in \mathbf{C}^n$ there holds

$$\begin{aligned} |\Re g(z)\omega^2(z)|^q &\leq \omega^{2q}(z) \frac{1}{|D(z,1)|} \int_{D(z,1)} |\Re g(z)|^q dA(z) \\ &\leq C \int_{D(z,1)} |\Re g(z)\omega^2(z)|^q dA(z). \end{aligned}$$

Lemma 3.1 tells us that $d\mu = |\Re g(z)\omega^2(z)|^q dA(z)$ is a (p, q) -Fock Carleson measure if and only if $\Re g(z)\omega^2(z)$ is bounded in $\mathbf{C}^n$. This and Lemma 3.4 imply that T_g is bounded from F_γ^p to F_γ^q if and only if $\Re g(z)$ is a polynomial of degree not more than 2. In other words, g is a polynomial of degree not more than 2.

For the compactness of T_g , estimate (17) tells us that T_g is compact from F_γ^p to F_γ^q if and only if $d\mu = |\Re g(z)\omega^2(z)|^q dA(z)$ is a vanishing (p, q) -Fock Carleson measure which is equivalent to the fact that $\lim_{z \rightarrow \infty} \Re g(z)\omega^2(z) = 0$, as Lemma 3.2 implies. Lemma 3.4 tells us that $\Re g$ (and equivalently g) is a polynomial of degree not more than 1.

The operator norm estimate (16) comes from (18) and Lemma 3.1. The proof is completed. $\square$

Theorem 3.2. *Let $0 < q < p < \infty$, and let $g \in H(\mathbf{C}^n)$. Then the following statements are equivalent:*

- (A) T_g is bounded from F_γ^p to F_γ^q ;
- (B) T_g is compact from F_γ^p to F_γ^q ;
- (C) g is a polynomial of degree not more than 1 if $\frac{pq}{p-q} > 2n$ and g is constant if $\frac{pq}{p-q} \leq 2n$.

Furthermore,

$$(19) \quad \|T_g\| \simeq \left\{ \int_{\mathbf{C}^n} |\Re g(z)\omega^2(z)|^{\frac{pq}{p-q}} dA(z) \right\}^{\frac{1}{q}}.$$

Proof. Lemma 3.3 and (17) imply that (A) and (B) are equivalent, and each of them is equivalent to

$$(20) \quad \int_{\mathbf{C}^n} |\Re g(z)\omega^2(z)|^{\frac{pq}{p-q}} dA(z) = \int_0^\infty \left\{ \frac{M_{\frac{pq}{p-q}}(\Re g, r)}{(1+r)^2} \right\}^{\frac{pq}{p-q}} r^{2n-1} dr < \infty.$$

With the help of Lemma 3.4 we see that (20) is equivalent to the fact that g is a polynomial of degree not more than 1 if $\frac{pq}{p-q} > 2n$ or that g is constant if $\frac{pq}{p-q} \leq 2n$. Estimate (19) comes from Lemma 3.3 and (18). The proof is completed. $\square$

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