A CHANGE OF SCALE FORMULA FOR A FUNCTION SPACE INTEGRAL ON $C_{a,b}[0, T]$

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ABSTRACT. Cameron and Storvick discovered change of scale formulas for Wiener integrals of functionals in a Banach algebra $\mathcal{S}$ on classical Wiener space. Yoo and Skoug extended these results for functionals in the Fresnel class $\mathcal{F}(B)$ and in a generalized Fresnel class $\mathcal{F}_{A_1, A_2}$ on abstract Wiener space. We establish a relationship between a function space integral and a generalized analytic Feynman integral on $C_{a,b}[0, T]$ for functionals in a Banach algebra $\mathcal{S}(L^2_{a,b}[0, T])$. Moreover, we obtain a change of scale formula for a function space integral on $C_{a,b}[0, T]$ of these functionals.

1. INTRODUCTION

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [2] and under translations [1]. Cameron and Storvick [5] expressed the analytic Feynman integral on classical Wiener space as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals on classical Wiener space $(C_0[0, 1], m_w)$ [4]. In [12, 13], Yoo and Skoug extended these results to an abstract Wiener space (H, B, ν) . Moreover, Yoo, Song, Kim and Chang [14, 15] established a change of scale formula for Wiener integrals of some unbounded functionals on (a product) abstract Wiener space.

On the other hand, Chang, Choi, Lee and Skoug [6, 7, 8, 9] studied a generalized analytic Feynman integral on a very general function space $C_{a,b}[0, T]$ rather than the Wiener space $C_0[0, T]$. The Wiener process W on $C_0[0, T] \times [0, T]$ is free of drift and is stationary in time, while the stochastic process Y on $C_{a,b}[0, T] \times [0, T]$ is subject to a drift $a(t)$ and is nonstationary in time. For more details, see [6, 9].

In this paper, we establish a relationship between a function space integral and a generalized analytic Feynman integral on $C_{a,b}[0, T]$ for functionals in a Banach algebra $\mathcal{S}(L^2_{a,b}[0, T])$. That is, we express a generalized analytic Feynman integral on $C_{a,b}[0, T]$ as a limit of function space integrals on $C_{a,b}[0, T]$. Moreover, we obtain a change of scale formula for a function space integral on $C_{a,b}[0, T]$ of these functionals.

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By choosing $a(t) = 0$ and $b(t) = t$ on $[0, T]$, the function space $C_{a,b}[0, T]$ reduces to the Wiener space $C_0[0, T]$, and so the results in [4, 5] are immediate corollaries of the results in this paper.

2. DEFINITIONS AND PRELIMINARIES

Let $a(t)$ be an absolutely continuous real valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ be a strictly increasing continuous differentiable real valued function with $b(0) = 0$. The generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$. For more detail, see Section 2 of [6]. By Theorem 14.2 in [11], the probability measure μ induced by Y , taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions x on $[0, T]$ with $x(0) = 0$ under the sup norm). Let $W(C_{a,b}[0, T])$ denote the class of all μ measurable subsets of $C_{a,b}[0, T]$.

A subset E of $C_{a,b}[0, T]$ is said to be scale-invariant measurable provided ρE is in $W(C_{a,b}[0, T])$ for every $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (*s-a.e.*) [10].

Let $L^2_{a,b}[0, T]$ be the Hilbert space of continuous functions on $[0, T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0, T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0, T] = \{v : \int_0^T v^2(t) db(t) < \infty \text{ and } \int_0^T v^2(t) d|a|(t) < \infty\},$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0, t]$.

For $u, v \in L^2_{a,b}[0, T]$, let

$$(u, v)_{a,b} = \int_0^T u(t)v(t) d[b(t) + |a|(t)].$$

Then $(\cdot, \cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0, T]$ and $\|u\|_{a,b} = \sqrt{(u, u)_{a,b}}$ is a norm on $L^2_{a,b}[0, T]$. Furthermore, $(L^2_{a,b}[0, T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}$ be a complete orthonormal set of real valued functions of bounded variation on $[0, T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases}$$

and for each $v \in L^2_{a,b}[0, T]$, let

$$v_n(t) = \sum_{j=1}^n (v, \phi_j)_{a,b} \phi_j(t)$$

for $n = 1, 2, \dots$. Then for each $v \in L^2_{a,b}[0, T]$, the Paley-Wiener-Zygmund stochastic integral $\langle v, x \rangle$ is defined by the formula

$$\langle v, x \rangle = \lim_{n \rightarrow \infty} \int_0^T v_n(t) dx(t)$$

for all $x \in C_{a,b}[0, T]$ for which the limit exists.

We denote the function space integral of a $W(C_{a,b}[0, T])$ -measurable functional F by

$$\int_{C_{a,b}[0, T]} F(x) d\mu(x)$$

whenever the integral exists.

Let $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the set of complex numbers with positive real part and complex numbers with nonnegative real part, respectively.

Definition 2.1. Let F be a complex valued measurable functional on $C_{a,b}[0, T]$ such that the function space integral

$$(2.1) \quad J_F(\lambda) = \int_{C_{a,b}[0, T]} F(\lambda^{-1/2}x) d\mu(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0, T]$ with parameter λ , and for $\lambda \in \mathbb{C}_+$ we write

$$(2.2) \quad \int_{C_{a,b}[0, T]}^{\text{an}_\lambda} F(x) d\mu(x) = J_F^*(\lambda).$$

If the following limit exists for nonzero real q , then we call it the generalized analytic Feynman integral of F over $C_{a,b}[0, T]$ with parameter q and we write

$$(2.3) \quad \int_{C_{a,b}[0, T]}^{\text{anf}_q} F(x) d\mu(x) = \lim_{\lambda \rightarrow -iq} \int_{C_{a,b}[0, T]}^{\text{an}_\lambda} F(x) d\mu(x),$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Definition 2.2. Let $M(L_{a,b}^2[0, T])$ denote the space of complex valued countably additive Borel measures on $L_{a,b}^2[0, T]$. The Banach algebra $\mathcal{S}(L_{a,b}^2[0, T])$ is the space of all s -equivalence classes of functionals F on $C_{a,b}[0, T]$ which have the form

$$(2.4) \quad F(x) = \int_{L_{a,b}^2[0, T]} \exp\{i\langle v, x \rangle\} df(v),$$

where the associated measure f is an element of $M(L_{a,b}^2[0, T])$.

Remark 2.3. Let $a(t) = 0$ and $b(t) = t$ on $[0, T]$. Then:

- (i) The analytic function space integral $\int_{C_{a,b}[0, T]}^{\text{an}_\lambda} F(x) d\mu(x)$ reduces to the analytic Wiener integral $\int_{C_0[0, T]}^{\text{anw}_\lambda} F(x) dm_w(x)$.
- (ii) The generalized analytic Feynman integral $\int_{C_{a,b}[0, T]}^{\text{anf}_q} F(x) d\mu(x)$ reduces to the analytic Feynman integral $\int_{C_0[0, T]}^{\text{anf}_q} F(x) dm_w(x)$.
- (iii) The Banach algebra $\mathcal{S}(L_{a,b}^2[0, T])$ reduces to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [3].

The following notation is used throughout this paper:

$$(u, a') = \int_0^T u(t)a'(t) dt = \int_0^T u(t) da(t)$$

and

$$(u^2, b') = \int_0^T u^2(t)b'(t) dt = \int_0^T u^2(t) db(t)$$

for $u \in L^2_{a,b}[0, T]$. Furthermore, for all $\lambda \in \mathbb{C}_+^\sim$, $\sqrt{\lambda}$ is always chosen to have positive real part.

Now we state a very fundamental integration formula on the function space $C_{a,b}[0, T]$. It is a generalization of the Wiener integration formula on Wiener space.

Let $\{\phi_1, \dots, \phi_n\}$ be an orthonormal set in $L^2_{a,b}[0, T]$ and, for $k = 1, \dots, n$, let

$$(2.5) \quad A_k \equiv (\phi_k, a') = \int_0^T \phi_k(t) da(t)$$

and

$$(2.6) \quad B_k \equiv (\phi_k^2, b') = \int_0^T \phi_k^2(t) db(t).$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lebesgue measurable function. Then

$$(2.7) \quad \begin{aligned} & \int_{C_{a,b}[0,T]} f(\langle \phi_1, x \rangle, \dots, \langle \phi_n, x \rangle) d\mu(x) \\ &= \left(\prod_{k=1}^n 2\pi B_k \right)^{-1/2} \int_{\mathbb{R}^n} f(u_1, \dots, u_n) \exp \left\{ - \sum_{k=1}^n \frac{(u_k - A_k)^2}{2B_k} \right\} du_1 \cdots du_n \end{aligned}$$

in the sense that if either side exists, both sides exist and equality holds.

We finish this section by introducing the existence theorem (Remark 4.2 of [9]) of the analytic function space integral and the generalized analytic Feynman integral on $C_{a,b}[0, T]$ for functionals in $\mathcal{S}(L^2_{a,b}[0, T])$.

Theorem 2.4. *Let $\lambda \in \mathbb{C}_+$. Let $F \in \mathcal{S}(L^2_{a,b}[0, T])$ be given by (2.4) and suppose that the associated measure f satisfies the condition*

$$(2.8) \quad \int_{L^2_{a,b}[0,T]} \exp \left\{ -\beta \int_0^T |v(t)| d|a|(t) \right\} |df(v)| < \infty,$$

where $\beta = \text{Im}(\lambda^{-1/2})$. Then the analytic function space integral of F exists and is given by

$$(2.9) \quad \int_{C_{a,b}[0,T]}^{\text{an}_\lambda} F(x) d\mu(x) = \int_{L^2_{a,b}[0,T]} \exp \left\{ -\frac{1}{2\lambda}(v^2, b') + i\lambda^{-1/2}(v, a') \right\} df(v).$$

Furthermore, let q be a nonzero real number and suppose that the associated measure f satisfies the condition

$$(2.10) \quad \int_{L^2_{a,b}[0,T]} \exp \left\{ |2q|^{-1/2} \int_0^T |v(t)| d|a|(t) \right\} |df(v)| < \infty.$$

Then the generalized analytic Feynman integral of F exists and is given by

$$(2.11) \quad \int_{C_{a,b}[0,T]}^{\text{anf}_q} F(x) d\mu(x) = \int_{L^2_{a,b}[0,T]} \exp \left\{ -\frac{i}{2q}(v^2, b') + i\left(\frac{i}{q}\right)^{1/2}(v, a') \right\} df(v).$$

3. A CHANGE OF SCALE FORMULA FOR A FUNCTION SPACE INTEGRAL ON $C_{a,b}[0, T]$

We begin this section by giving a function space integration formula that is useful in the proof of Theorem 3.2.

Lemma 3.1. *Let $\lambda \in \mathbb{C}_+$, let $\{\phi_1, \dots, \phi_n\}$ be an orthonormal set in $L^2_{a,b}[0, T]$ and let $v \in L^2_{a,b}[0, T]$. Then if*

$$(3.1) \quad K \equiv \int_{C_{a,b}[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n \frac{\langle \phi_k, x \rangle^2}{B_k} + (\lambda^{1/2} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x \rangle}{B_k} + i \langle v, x \rangle\right\} d\mu(x),$$

it follows that

$$(3.2) \quad \begin{aligned} K = & \lambda^{-n/2} \exp\left\{-\frac{1}{2}(\psi_n^2, b') + i(\psi_n, a')\right. \\ & \left. + \sum_{k=1}^n \left[-\frac{1}{2\lambda} B_k (\phi_k, v)_{a,b}^2 + i\lambda^{-1/2} A_k (\phi_k, v)_{a,b}\right]\right\}, \end{aligned}$$

where

$$(3.3) \quad \psi_n = v - \sum_{k=1}^n (\phi_k, v)_{a,b} \phi_k.$$

Proof. By the Gram-Schmidt process, we obtain $\phi_{n+1} \in L^2_{a,b}[0, T]$ such that $\{\phi_1, \dots, \phi_n, \phi_{n+1}\}$ is an orthonormal set in $L^2_{a,b}[0, T]$ and

$$v = \sum_{k=1}^{n+1} c_k \phi_k,$$

where

$$c_k = \begin{cases} (\phi_k, v)_{a,b}, & k = 1, \dots, n, \\ [\|v\|_{a,b}^2 - \sum_{j=1}^n (\phi_j, v)_{a,b}^2]^{1/2}, & k = n+1. \end{cases}$$

In fact $\psi_n = \|\psi_n\|_{a,b} \phi_{n+1} = c_{n+1} \phi_{n+1}$, where ψ_n is given by (3.3). Hence

$$\begin{aligned} K = & \int_{C_{a,b}[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n \frac{\langle \phi_k, x \rangle^2}{B_k} + (\lambda^{1/2} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x \rangle}{B_k}\right. \\ & \left. + i \sum_{k=1}^{n+1} c_k \langle \phi_k, x \rangle\right\} d\mu(x). \end{aligned}$$

By the integration formula (2.7), we have

$$\begin{aligned} K = & \left(\prod_{k=1}^{n+1} 2\pi B_k\right)^{-1/2} \int_{\mathbb{R}^{n+1}} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n \frac{u_k^2}{B_k} + (\lambda^{1/2} - 1) \sum_{k=1}^n \frac{A_k u_k}{B_k}\right. \\ & \left. + i \sum_{k=1}^{n+1} c_k u_k - \sum_{k=1}^{n+1} \frac{(u_k - A_k)^2}{2B_k}\right\} du_1 \cdots du_{n+1}. \end{aligned}$$

Now we evaluate the above integral using the integration formula

$$\int_{\mathbb{R}} \exp\{-\alpha u^2 + i\beta u\} du = \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left\{-\frac{\beta^2}{4\alpha}\right\}, \quad \alpha \in \mathbb{C}_+,$$

to obtain

$$K = \lambda^{-n/2} \exp \left\{ -\frac{B_{n+1}}{2} c_{n+1}^2 + i A_{n+1} c_{n+1} + \sum_{k=1}^n \left[-\frac{1}{2\lambda} B_k c_k^2 + i \lambda^{-1/2} A_k c_k \right] \right\}.$$

Finally, since

$$B_{n+1} c_{n+1}^2 = (\phi_{n+1}^2, b') c_{n+1}^2 = (\phi_{n+1}^2 c_{n+1}^2, b') = (\psi_n^2, b')$$

and

$$A_{n+1} c_{n+1} = (\phi_{n+1}, a') c_{n+1} = (\psi_n, a'),$$

we have (3.2). $\square$

Now we give a relationship between the function space integral and the generalized analytic Feynman integral on $C_{a,b}[0, T]$ for functionals in $\mathcal{S}(L_{a,b}^2[0, T])$. But we need a further assumption on $a(t)$ to obtain the relationship.

Theorem 3.2. *Let $|a(t)| = cb(t)$ on $[0, T]$ for some constant $c \geq 0$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_{a,b}^2[0, T]$. Let q be a nonzero real number and let $\{\lambda_n\}$ be a sequence of complex numbers in $\mathbb{C}_+$ such that $\lambda_n \rightarrow -iq$. Let $F \in \mathcal{S}(L_{a,b}^2[0, T])$ be given by (2.4) and suppose that the associated measure f satisfies condition (2.10). Then we have*

$$(3.4) \quad \int_{C_{a,b}[0, T]}^{\text{anf}_q} F(x) d\mu(x) = \lim_{n \rightarrow \infty} \lambda_n^{n/2} \int_{C_{a,b}[0, T]} \exp \left\{ \frac{1 - \lambda_n}{2} \sum_{k=1}^n \frac{\langle \phi_k, x \rangle^2}{B_k} \right. \\ \left. + (\lambda_n^{1/2} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x \rangle}{B_k} \right\} F(x) d\mu(x).$$

Proof. Let $\Gamma(\lambda_n)$ be the function space integral on the right hand side of (3.4). Then by (2.4) and the Fubini theorem,

$$\Gamma(\lambda_n) = \int_{L_{a,b}^2[0, T]} K(v, \lambda_n) df(v),$$

where

$$K(v, \lambda_n) = \int_{C_{a,b}[0, T]} \exp \left\{ \frac{1 - \lambda_n}{2} \sum_{k=1}^n \frac{\langle \phi_k, x \rangle^2}{B_k} \right. \\ \left. + (\lambda_n^{1/2} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x \rangle}{B_k} + i \langle v, x \rangle \right\} d\mu(x).$$

By Lemma 3.1,

$$(3.5) \quad K(v, \lambda_n) = \lambda_n^{-n/2} \exp \left\{ -\frac{1}{2} (\psi_n^2, b') + i (\psi_n, a') \right. \\ \left. + \sum_{k=1}^n \left[-\frac{1}{2\lambda_n} B_k (\phi_k, v)_{a,b}^2 + i \lambda_n^{-1/2} A_k (\phi_k, v)_{a,b} \right] \right\},$$

where ψ_n is given as in (3.3). Now we examine each term in the exponential on the right hand side of the above expression. Since $\psi_n \rightarrow 0$ in $L_{a,b}^2[0, T]$ as $n \rightarrow \infty$,

$$(3.6) \quad |(\psi_n^2, b')| = \left| \int_0^T (\psi_n(t))^2 db(t) \right| \leq \|\psi_n\|_{a,b}^2 \rightarrow 0$$

as $n \rightarrow \infty$. Furthermore, by the Hölder inequality,

$$(3.7) \quad |(\psi_n, a')| = \left| \int_0^T \psi_n(t) da(t) \right| \leq \|\psi_n\|_{a,b} \|a\|^{1/2} \rightarrow 0$$

as $n \rightarrow \infty$, where $\|a\|$ denotes the total variation of a on $[0, T]$. Hence we know that the first two terms in the exponential of (3.5) go to 0 as $n \rightarrow \infty$. Moreover,

$$\left| \sum_{k=1}^n A_k(\phi_k, v)_{a,b} - (v, a') \right| = \left| \left(\sum_{k=1}^n (\phi_k, v)_{a,b} \phi_k - v, a' \right) \right| = |(\psi_n, a')| \rightarrow 0$$

as above, and so

$$(3.8) \quad \sum_{k=1}^n A_k(\phi_k, v)_{a,b} \rightarrow (v, a')$$

as $n \rightarrow \infty$. Now it remains to consider the third term in the exponential of (3.5). Since

$$0 = (\phi_j, \phi_k)_{a,b} = (1+c) \int_0^T \phi_j(t) \phi_k(t) db(t)$$

for all $j \neq k$, we have

$$\int_0^T \phi_j(t) \phi_k(t) db(t) = 0$$

for all $j \neq k$, and so

$$(3.9) \quad \sum_{k=1}^n B_k(\phi_k, v)_{a,b}^2 = (\psi_n^2, b') + 2 \sum_{k=1}^n (\phi_k, v)_{a,b} (\phi_k v, b') - (v^2, b').$$

But

$$|(\psi_n^2, b')| \leq \|\psi_n\|_{a,b}^2 \rightarrow 0,$$

and by the Hölder inequality

$$\begin{aligned} & \left| \sum_{k=1}^n (\phi_k, v)_{a,b} (\phi_k v, b') - (v^2, b') \right| = |(\psi_n v, b')| \\ & \leq \left(\int_0^T (\psi_n(t))^2 db(t) \right)^{1/2} \left(\int_0^T (v(t))^2 db(t) \right)^{1/2} \leq \|\psi_n\|_{a,b} \|v\|_{a,b} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence from (3.9) we have

$$(3.10) \quad \sum_{k=1}^n B_k(\phi_k, v)_{a,b}^2 \rightarrow (v^2, b')$$

as $n \rightarrow \infty$.

From the four convergences in (3.6), (3.7), (3.8) and (3.10), we see that

$$\lim_{n \rightarrow \infty} \lambda_n^{n/2} K(v, \lambda_n) = \exp \left\{ -\frac{i}{2q} (v^2, b') + i \left(\frac{i}{q} \right)^{1/2} (v, a') \right\}.$$

Since f satisfies condition (2.10) we apply the dominated convergence theorem to obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_n^{n/2} \Gamma(\lambda_n) &= \lim_{n \rightarrow \infty} \lambda_n^{n/2} \int_{L_{a,b}^2[0,T]} K(v, \lambda_n) df(v) \\ &= \int_{L_{a,b}^2[0,T]} \exp \left\{ -\frac{i}{2q} (v^2, b') + i \left(\frac{i}{q} \right)^{1/2} (v, a') \right\} df(v). \end{aligned}$$

By (2.11) in Theorem 2.4, the proof is completed. $\square$

Let $a(t) = 0$ and $b(t) = t$ on $[0, T]$. Then $a(t)$ satisfies the assumption in Theorem 3.2, condition (2.10) is satisfied for any measure f in $M(L^2_{a,b}[0, T])$ and

$$A_k = (\phi_k, 0) = 0, \quad B_k = (\phi_k^2, 1) = (\phi_k^2, 1)_{a,b} = 1.$$

Moreover, as we see in Remark 2.3 above, the generalized analytic Feynman integral $\int_{C_{a,b}[0,T]}^{\text{anf}_q} F(x) d\mu(x)$ on the function space $C_{a,b}[0, T]$ reduces to the analytic Feynman integral $\int_{C_0[0,T]}^{\text{anf}_q} F(x) dm_w(x)$ on Wiener space $C_0[0, T]$. Hence we have the following relationship between the Wiener integral and the analytic Feynman integral on $C_0[0, T]$ for functionals in a Banach algebra $\mathcal{S}$.

Corollary 3.3 (Theorem 2 of [5]). *Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2[0, T]$. Let q be a nonzero real number and let $\{\lambda_n\}$ be a sequence of complex numbers in $\mathbb{C}_+$ such that $\lambda_n \rightarrow -iq$. Let F be a functional in a Banach algebra $\mathcal{S}$; that is,*

$$(3.11) \quad F(x) = \int_{L^2[0,T]} \exp\{i\langle v, x \rangle\} df(v),$$

where f is a complex Borel measure on $L^2[0, T]$. Then

$$(3.12) \quad \int_{C_0[0,T]}^{\text{anf}_q} F(x) dm_w(x) = \lim_{n \rightarrow \infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\left\{\frac{1-\lambda_n}{2} \sum_{k=1}^n \langle \phi_k, x \rangle^2\right\} F(x) dm_w(x).$$

In our next example we give $a(t), b(t)$ and a functional F in $\mathcal{S}(L^2_{a,b}[0, T])$ to show that the relationship (3.4) is not necessarily valid if $a(t)$ does not satisfy the assumption in Theorem 3.2. For notational and computational convenience let $T = 1$.

Example 3.4. Let $a(t) = t^2/2$ and $b(t) = t$ for $0 \leq t \leq 1$ and let q be a nonzero real number. Note that $a(t)$ does not satisfy the assumption in Theorem 3.2. Let $\phi_1(t) = 1/\sqrt{1+t}$ for $0 \leq t \leq 1$. Then

$$\|\phi_1\|_{a,b} = \int_0^1 \frac{1}{1+t} d[b(t) + |a|(t)] = \int_0^1 \frac{1}{1+t} (1+t) dt = 1.$$

Of course we can choose $\phi_2, \phi_3, \dots$ in $L^2_{a,b}[0, 1]$ such that $\{\phi_n\}$ is a complete orthonormal set in $L^2_{a,b}[0, 1]$. Let f be the Dirac measure concentrated at $v \equiv 1$ and let F be given by (2.4). Obviously f satisfies the condition (2.10),

$$(1^2, b') = \int_0^1 1 dt = 1, \quad (1, a') = \int_0^1 t dt = \frac{1}{2}.$$

Hence by Theorem 2.4, we have

$$(3.13) \quad \int_{C_{a,b}[0,1]}^{\text{anf}_q} F(x) d\mu(x) = \exp\left\{-\frac{i}{2q} + i\left(\frac{i}{q}\right)^{1/2} \frac{1}{2}\right\}.$$

Now we consider the right hand side of (3.4). Let $\Gamma(\lambda_n)$ be the function space integral on the right hand side of (3.4). By the same method as in the proof of

Theorem 3.2, we have

$$\begin{aligned}\Gamma(\lambda_n) = K(1, \lambda_n) = & \lambda_n^{-n/2} \exp\left\{-\frac{1}{2}(\psi_n^2, 1) + i(\psi_n, t)\right. \\ & \left. + \sum_{k=1}^n \left[-\frac{1}{2\lambda_n} B_k(\phi_k, 1)_{a,b}^2 + i\lambda_n^{-1/2} A_k(\phi_k, 1)_{a,b}\right]\right\},\end{aligned}$$

where

$$\psi_n = 1 - \sum_{k=1}^n (\phi_k, 1)_{a,b} \phi_k.$$

A careful look at the proof of Theorem 3.2 tells us that the convergences in (3.6), (3.7) and (3.8) are valid even if $a(t)$ does not satisfy the assumption in Theorem 3.2; that is,

$$(\psi_n^2, 1) \rightarrow 0, \quad (\psi_n, t) \rightarrow 0, \quad \sum_{k=1}^n A_k(\phi_k, 1)_{a,b} \rightarrow (1, t) = \frac{1}{2}$$

as $n \rightarrow \infty$. On the other hand,

$$\begin{aligned}B_1 &= (\phi_1^2, 1) = \int_0^1 \frac{1}{1+t} dt = \ln 2, \\ (\phi_1, 1)_{a,b} &= \int_0^1 \frac{1}{\sqrt{1+t}} (1+t) dt = \frac{2}{3}(2\sqrt{2} - 1),\end{aligned}$$

and so

$$B_1(\phi_1, 1)_{a,b}^2 = \frac{4}{9}(9 - 4\sqrt{2}) \ln 2 \approx 1.0299 > 1.$$

Since $B_k \geq 0$ for all $k = 1, 2, \dots$, we have

$$\sum_{k=1}^n B_k(\phi_k, 1)_{a,b}^2 \geq B_1(\phi_1, 1)_{a,b}^2 > 1,$$

and so the left hand side of the above inequality does not converge to 1 as $n \rightarrow \infty$. Hence

$$(3.14) \quad \lim_{n \rightarrow \infty} \lambda_n^{n/2} \Gamma(\lambda_n) \neq \exp\left\{-\frac{i}{2q} + i\left(\frac{i}{q}\right)^{1/2} \frac{1}{2}\right\}.$$

Comparing (3.13) and (3.14), we see that the relationship (3.4) is not valid in this case.

Theorem 3.5. *Let $|a(t)| = cb(t)$ on $[0, T]$ for some constant $c \geq 0$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_{a,b}^2[0, T]$. Let $\lambda \in \mathbb{C}_+$. Let $F \in \mathcal{S}(L_{a,b}^2[0, T])$ be given by (2.4) and suppose that the associated measure f satisfies condition (2.8). Then*

$$\begin{aligned}(3.15) \quad & \int_{C_{a,b}[0, T]}^{\text{an}_\lambda} F(x) d\mu(x) \\ &= \lim_{n \rightarrow \infty} \lambda^{n/2} \int_{C_{a,b}[0, T]} \exp\left\{\frac{1-\lambda}{2} \sum_{k=1}^n \frac{\langle \phi_k, x \rangle^2}{B_k} + (\lambda^{1/2} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x \rangle}{B_k}\right\} F(x) d\mu(x).\end{aligned}$$

Proof. To prove this corollary, we modify the first part of the proof of Theorem 3.2 by replacing λ_n by λ whenever it occurs. Then we have

$$\lim_{n \rightarrow \infty} \lambda^{n/2} K(v, \lambda) = \exp \left\{ -\frac{1}{2\lambda} (v^2, b') + i\lambda^{-1/2} (v, a') \right\}.$$

Since f satisfies condition (2.8) we apply the dominated convergence theorem to obtain

$$\lim_{n \rightarrow \infty} \lambda^{n/2} \Gamma(\lambda) = \int_{L_{a,b}^2[0,T]} \exp \left\{ -\frac{1}{2\lambda} (v^2, b') + i\lambda^{-1/2} (v, a') \right\} df(v).$$

By (2.9) in Theorem 2.4, the proof is completed. $\square$

Our main result, namely a change of scale formula for a function space integral on $C_{a,b}[0, T]$, now follows from Theorem 3.5.

Theorem 3.6. *Let $|a(t)| = cb(t)$ on $[0, T]$ for some constant $c \geq 0$. Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L_{a,b}^2[0, T]$. Let $\rho > 0$ and let $F \in \mathcal{S}(L_{a,b}^2[0, T])$ be given by (2.4). Then*

$$(3.16) \quad \begin{aligned} & \int_{C_{a,b}[0,T]} F(\rho x) d\mu(x) \\ &= \lim_{n \rightarrow \infty} \rho^{-n} \int_{C_{a,b}[0,T]} \exp \left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^n \frac{\langle \phi_k, x \rangle^2}{B_k} + (\rho^{-1} - 1) \sum_{k=1}^n \frac{A_k \langle \phi_k, x \rangle}{B_k} \right\} F(x) d\mu(x). \end{aligned}$$

Proof. By letting $\lambda = \rho^{-2}$ in (3.15), we have equation (3.16). Note that since $\text{Im}(\lambda^{-1/2}) = \text{Im}\rho = 0$, condition (2.8) is satisfied for every f in $M(L_{a,b}^2[0, T])$. $\square$

Taking $a(t) = 0$ and $b(t) = t$ in Theorem 3.6, we have the following change of scale formula for Wiener integrals on classical Wiener space.

Corollary 3.7 (Theorem 2 of [4]). *Let $\{\phi_n\}$ be a complete orthonormal set of functionals in $L^2[0, T]$. Let $\rho > 0$ and let $F \in \mathcal{S}$ be given by (3.11). Then*

$$(3.17) \quad \int_{C_0[0,T]} F(\rho x) dm_w(x) = \lim_{n \rightarrow \infty} \rho^{-n} \int_{C_0[0,T]} \exp \left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^n \langle \phi_k, x \rangle^2 \right\} F(x) dm_w(x).$$

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