INVERSION FORMULAE FOR THE cosh-WEIGHTED HILBERT TRANSFORM

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ABSTRACT. In this paper we develop formulae for inverting the so-called cosh-weighted Hilbert transform H_μ , which arises in Single Photon Emission Computed Tomography (SPECT). The formulae are theoretically exact, require a minimal amount of data, and are similar to the classical inversion formulae for the finite Hilbert transform (FHT) H_0 . We also find the null-space and the range of H_μ in L^p with $p > 1$. Similarly to the FHT, the null-space turns out to be one-dimensional in L^p for any $p \in (1, 2)$ and trivial – for $p \geq 2$. We prove that H_μ is a Fredholm operator of index -1 when it acts between the L^p spaces, $p \in (1, \infty)$, $p \neq 2$. Finally, in the case where $p = 2$ we find the range condition for H_μ , which is similar to that for the FHT H_0 . Our work is based on the method of the Riemann-Hilbert problem.

1. INTRODUCTION

A relationship between the cone-beam transform of a function f and the Hilbert transform of f along lines was found by Gelfand and Graev in [GG91]. In combination with a formula for the finite Hilbert transform (FHT) inversion, the two results led to a development of new, accurate, and flexible algorithms for image reconstruction in transmission tomography (CT); for some of the references see [ZP04, NCP04, PNC05, ZYZW05]. The key property of FHT inversion is that it can be performed using an efficient convolution-type formula and requires only a minimal amount of data. More precisely, for each line L one needs to compute the Hilbert transform of f from the CT data only for points in $I := L \cap \text{supp } f$ (assuming this intersection is an interval). Since the amount of CT data required is minimal (i.e., any algorithm that uses data on a subinterval of I is severely unstable), reconstruction algorithms based on FHT inversion have the potential to reduce the x-ray dose to the patients.

It was shown recently by Rullgard that in the case of Single Photon Emission Computed Tomography (SPECT) with constant attenuation there is a relation between the attenuated projections of f and a modified Hilbert transform of f along lines [Rul04]. This relation is analogous to the one in transmission CT. Let μ be the constant attenuation coefficient of the medium. It is known that by using simple weighting of the attenuated projections of f , one can compute the

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exponential Radon transform of f :

$$(1.1) \quad R_\mu f(\alpha, p) := \int_{-\infty}^{\infty} f(\alpha p + \alpha^\perp t) e^{\mu t} dt.$$

Here α is a unit vector, and $\alpha^\perp$ is obtained by rotating α counterclockwise by 90° . The identity, which was found in [Rul04], gives us

$$(1.2) \quad \int_0^\pi \frac{\partial R_\mu f(\alpha, p = \alpha \cdot x)}{\partial p} e^{-\mu x \cdot \alpha^\perp} d\alpha = 2 \int_{-\infty}^{\infty} f(x_1 + t, x_2) \frac{\cosh \mu t}{t} dt.$$

Thus we can compute an integral transform of f along lines L knowing the attenuated projections of f . Inverting the integral transform for f along a family of lines which cover the support of f solves the image reconstruction problem. From a theoretical perspective, the lines L are not related to each other. Hence we can consider only one such line L , assume that it coincides with the x -axis, and assume that f is a function of x only. Thus, using (1.2) and rescaling $\text{supp } f$ so that it coincides with the interval $[-1, 1]$, we arrive at the the following problem.

Problem 1.1. Let

$$(1.3) \quad H_\mu(f) := \frac{1}{\pi} \int_{-1}^1 \frac{\cosh \mu(x-y)}{x-y} f(x) dx = g(y), \quad \mu \geq 0.$$

Given $g(y)$ on $[-1, 1]$, reconstruct the function $f(x)$, $|x| \leq 1$.

Here and henceforth, singular integrals are understood in the Cauchy principal value sense. These integrals are well defined for Hölder-continuous (on $[-1, 1]$) functions f , but can be extended to $f \in L^p := L^p([-1, 1])$ with $p \in (1, \infty)$. To avoid certain technical issues, we first derive the inversion formulae for the case when f and, thus, $g = H_\mu f$, are Hölder-continuous, and then extend the results to L^p , $p \in (1, \infty)$, functions. Therefore we assume that, unless specified otherwise, the functions f and g in (1.3) are Hölder-continuous on $[-1, 1]$.

It was shown in [Rul04] that the transform H_μ can be inverted with the help of a distribution, which can be computed numerically. However, this approach requires the data far outside the support of f . In [NDPC07] the problem of inverting H_μ from the minimal amount of data (which, for each line L , is the interval $I = L \cap \text{supp } f$) was investigated. In particular, a numerical method for inverting H_μ based on solving an integral equation was proposed. Additional results on numerical implementation of H_μ inversion are in [HYZG09].

In this paper we develop several inversion formulae for H_μ . They are theoretically exact, require a minimal amount of data, and are similar to the classical formulae for the FHT inversion. In particular, the inversion formulae are convolution-based; i.e. they admit efficient numerical implementation based on FFT (fast Fourier transform). We also find the null-space of H_μ . Similarly to the FHT, the null-space turns out to be one-dimensional in L^p for any $p \in (1, 2)$, and trivial for $p \geq 2$. Finally, we prove that H_μ is a Fredholm operator when it acts between the L^p spaces, $p \in (1, \infty)$, $p \neq 2$. In the case where $p = 2$, we find the range condition for H_μ , which is similar to that for the FHT H_0 . Our work is based on the reduction of equation (1.3) to a vector Riemann-Hilbert problem (RHP).

The paper is organized as follows. In Section 2 we reduce Problem 1.1 to a non-homogeneous vector RHP. We also formulate a homogeneous matrix RHP and show how to obtain a solution to the nonhomogeneous problem from a solution

to the homogeneous one. In Section 3 we study the null-space of $H_\mu : L^p \rightarrow L^p$ for all $p > 1$. Two inversion formulae for H_μ are obtained in Section 4 under the assumption that f is Hölder-continuous. The action of H_μ on L^p spaces is studied in Section 5. We also obtain two more inversion formulae. One of them applies in the case $p \in (1, 2)$, and the other in the case $p \in (2, \infty)$. In Section 6 we show that $H_\mu : L^2 \rightarrow L^2$ is not Fredholm and describe its range. Some technical lemmas are contained in Appendix A. A different approach leading to the derivation of a two-parameter family of inversion formulae for H_μ directly from the solution of a general RHP is presented in Appendix B.

2. REDUCTION TO A VECTOR RHP

Define $Q(z)$ with the help of the complex integral transform

$$(2.1) \quad Q(z) := \frac{1}{2\pi i} \int_{-1}^1 \frac{\cosh \mu(\zeta - z)}{\zeta - z} f(\zeta) d\zeta.$$

It is clear that $Q(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and attains some limiting values $Q_\pm(z)$, $z \in [-1, 1]$, on the upper and lower sides of the cut $[-1, 1]$. Here and in what follows, the subscripts $\pm$ denote the boundary values on the upper/lower sides of the branchcut $[-1, 1]$, respectively. It is understood that these values can be infinite at the ends of the cut. By the Sokhotski-Plemelj theorem,

$$(2.2) \quad -ig(z) = Q_+(z) + Q_-(z), \quad z \in [-1, 1],$$

where $g(z)$ is the same as in (1.3). Formula (2.1) can be written as

$$(2.3) \quad Q(z) = \frac{\cosh \mu z}{2\pi i} \int_{-1}^1 \frac{\cosh \mu \zeta}{\zeta - z} f(\zeta) d\zeta - \frac{\sinh \mu z}{2\pi i} \int_{-1}^1 \frac{\sinh \mu \zeta}{\zeta - z} f(\zeta) d\zeta.$$

In a similar fashion, (1.3) can be written in the form

$$(2.4) \quad \cosh \mu z [(F_c)_+ + (F_c)_-] - \sinh \mu z [(F_s)_+ + (F_s)_-] = -ig(z),$$

where

$$(2.5) \quad F_c = C(\cosh \mu z \cdot f(z)), \quad F_s = C(\sinh \mu z \cdot f(z))$$

and C denotes the Cauchy operator

$$(2.6) \quad (C\phi)(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\phi(\zeta)}{\zeta - z} d\zeta.$$

In the rest of the paper, we will routinely use the standard identities $C_+ + C_- = -iH$ and $C_+ - C_- = \text{Id}$ on $[-1, 1]$. Combining (2.4) with the obvious consequence of the last identity,

$$(2.7) \quad \sinh \mu z [(F_c)_+ - (F_c)_-] - \cosh \mu z [(F_s)_+ - (F_s)_-] = 0,$$

we obtain after some algebra the following nonhomogeneous RHP for the vector $F = (F_c, F_s)$.

Problem 2.1. Find a vector-function $F(z)$ with the following properties:

- (a) $F(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$;

(b) $F(z)$ satisfies the jump condition

$$(2.8) \quad F_+ = F_- M + G \text{ on } (-1, 1),$$

where

$$(2.9) \quad M = \begin{bmatrix} -\cosh(2\mu z) & -\sinh(2\mu z) \\ \sinh(2\mu z) & \cosh(2\mu z) \end{bmatrix}, \quad G = -ig(\cosh \mu z, \sinh \mu z);$$

(c) $F(z) = \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$; and

(d) $F(z) = \mathcal{O}((z \mp 1)^{-1+\varepsilon})$ as $z \rightarrow \pm 1$, where ε is a small positive number.

Remark 2.2. If f is Hölder-continuous on $[-1, 1]$, then $F(z)$ has no more than $\mathcal{O}(\ln(z \mp 1))$ behavior at the endpoints $z = \pm 1$. The more general limiting condition (d), used in the RHP 2.1, does not affect our analysis.

To solve the RHP 2.1, we first consider the following homogeneous RHP.

Problem 2.3. Find a matrix-function $\Gamma(z)$ with the following properties:

(a) Γ is analytic and invertible in $\mathbb{C} \setminus [-1, 1]$;

(b) Γ satisfies the jump condition

$$(2.10) \quad \Gamma_+ = \Gamma_- M \text{ on } (-1, 1);$$

(c)

$$(2.11) \quad \Gamma(z) = \mathbf{1} + \mathcal{O}(z^{-1}), \quad z \rightarrow \infty,$$

where $\mathbf{1}$ is the identity matrix; and

(d)

$$(2.12) \quad \Gamma(z) = \mathcal{O}((z \mp 1)^{-1+\varepsilon}) \text{ as } z \rightarrow \pm 1,$$

where ε is a small positive number.

Proposition 2.4. If $\Gamma(z)$ is a solution to the RHP 2.3 and $\Gamma^{-1}(z)$ satisfies endpoint conditions (2.12), then

$$(2.13) \quad F(z) = C(G\Gamma_+^{-1})\Gamma$$

satisfies conditions (a)-(c) of the nonhomogeneous RHP 2.1.

Proof. Since $G(z)$ is Hölder-continuous on $[-1, 1]$, the Cauchy operator in (2.13) is well defined. Then

$$(2.14) \quad F_+ = C_+(G\Gamma_+^{-1})\Gamma_+ = [C_-(G\Gamma_+^{-1}) + G\Gamma_+^{-1}]\Gamma_+ = C_-(G\Gamma_+^{-1})\Gamma_- M + G = F_- M + G.$$

□

Condition (d) of the RHP 2.1 will be addressed in Remark 2.6 below. For convenience of matrix calculations below, we will use the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proposition 2.5. The matrix

$$(2.15) \quad \begin{aligned} \Gamma(z) &= \frac{\sigma_1}{2} \text{diag}(1, i\beta)(\sigma_3 + \sigma_1) e^{\mu(z - \sqrt{z^2 - 1})\sigma_3} (\mathbf{1} + i\sigma_2) \\ &= \begin{bmatrix} i\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh \mu \mathbf{r} & \sinh \mu \mathbf{r} \\ \sinh \mu \mathbf{r} & \cosh \mu \mathbf{r} \end{bmatrix} = \begin{bmatrix} i\beta & 0 \\ 0 & 1 \end{bmatrix} (\cosh \mu \mathbf{r} \mathbf{1} + \sinh \mu \mathbf{r} \sigma_1) \end{aligned}$$

is a solution of the RHP 2.3. Here $\mathbf{1}$ is the identity matrix, $\mathbf{r} := z - \sqrt{z^2 - 1}$ and $\beta = \beta(z) := \sqrt{\frac{1-z}{1+z}}$, with the determination of β such that $\beta_+ > 0$ for $z \in (-1, 1)$.

Proof. It is a straightforward verification by observing $\beta_+ = -\beta_-$ and then elementary matrix algebra. $\square$

Remark 2.6. A simple calculation yields the matrix

$$(2.16) \quad \Gamma^{-1}(z) = (\cosh \mu \mathbf{r} \mathbf{1} - \sinh \mu \mathbf{r} \sigma_1) \begin{bmatrix} (i\beta)^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

which satisfies condition (2.12) in Proposition 2.4. Moreover, if $\Gamma(z)$ is given by (2.15), then $F(z) = C(G\Gamma_+^{-1})\Gamma$ satisfies all the conditions of the RHP 2.1.

3. NULL-SPACE OF THE TRANSFORM H_μ

In the case $\mu = 0$, it is a well-known fact that if $Hv = 0$ on $(-1, 1)$ and $v \in L^p$ for some $p \in (1, 2)$, then v is a constant multiple of the function $v_0 = (1 - z^2)^{-\frac{1}{2}}$ (see [Gak66]). In this section we prove that, similarly to $H = H_0$, the transform $H_\mu : L^p \rightarrow L^p$ with $\mu > 0$ has a nontrivial one-dimensional null space $\mathcal{N}_\mu$ for any $p \in (1, 2)$ and construct a function $v_\mu \in \mathcal{N}_\mu$. This function is not in L^2 , and the null-space of H_μ in the case $p \in [2, \infty)$ is trivial. We start with a result describing analytic properties of $\phi \in \mathcal{N}_\mu$.

Proposition 3.1. *If $\phi \in \mathcal{N}_\mu$ and $\phi \in L^p$ for some $p \in (1, 2)$, then $\phi(z) = \frac{\Phi(z)}{\sqrt{1-z^2}}$, where $\Phi(z)$ is an entire function. If $\phi \in \mathcal{N}_\mu$ and $\phi \in L^p$ for some $p \in [2, \infty)$, then $\phi(z) = \Phi(z)\sqrt{1-z^2}$, where $\Phi(z)$ is an entire function.*

Proof. Pick any $\phi \in \mathcal{N}_\mu$ and suppose $\phi \in L^p$ for some $p \in (1, 2)$. Then

$$(3.1) \quad \int_{-1}^1 \frac{\cosh \mu(x-y)}{x-y} \phi(x) dx \equiv 0, \quad |y| < 1.$$

Denote $K(t) := (\cosh(\mu t) - 1)/t$. Then $K(z)$, $z \in \mathbb{C}$, is entire. Rewrite (3.1) as follows:

$$(3.2) \quad (H\phi)(y) = -b(y), \quad |y| < 1, \quad b(y) := \int_{-1}^1 K(x-y)\phi(x) dx.$$

Clearly $b(z)$, $z \in \mathbb{C}$, is entire as well. In particular, $b \in L^p$ for any $p \in (1, 2)$. Thus, by Corollary 2.5 of [OE91],

$$(3.3) \quad \phi(x) = \frac{C}{\sqrt{1-x^2}} + \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \int_{-1}^1 b(y) \frac{\sqrt{1-y^2}}{y-x} dy, \quad |x| < 1,$$

for some constant C . Equation (3.3) can be rewritten as

$$(3.4) \quad \phi(x) = \frac{1}{\sqrt{1-x^2}} \left(C + \frac{1}{\pi} \int_{-1}^1 \frac{b(y) - b(x)}{y-x} \sqrt{1-y^2} dy + \frac{b(x)}{\pi} \int_{-1}^1 \frac{\sqrt{1-y^2}}{y-x} dy \right),$$

where $|x| < 1$. A simple residue calculation (see also formula 2.2.5.10 in [PBM86]) shows that the last integral in parentheses in (3.4) equals $-\pi x$. Thus, the desired assertion is proved.

Next, suppose $p \in [2, \infty)$. Similarly to (3.3), by Proposition 2.6 of [OE91] for $p \in (2, \infty)$ and by Theorem 4.2 of [OE91] for $p = 2$, we get from (3.2):

$$(3.5) \quad \phi(x) = \frac{1}{\pi} \sqrt{1-x^2} \int_{-1}^1 \frac{b(y)}{\sqrt{1-y^2}(y-x)} dy, \quad |x| < 1.$$

Since $\int_{-1}^1 (\sqrt{1-y^2}(y-x))^{-1} dy = 0$ for $|x| < 1$, we can write (3.5) in the form

$$(3.6) \quad \phi(x) = \frac{1}{\pi} \sqrt{1-x^2} \int_{-1}^1 \frac{b(y) - b(x)}{\sqrt{1-y^2}(y-x)} dy, \quad |x| < 1,$$

and the desired assertion follows immediately. $\square$

We continue with the observation that if $f \in \mathcal{N}_\mu$, then the corresponding vector $F = (F_c, F_s)$ (cf. (2.5)) has the jump $F_+ = F_- M$ on $(-1, 1)$ and satisfies the normalization condition $F(z) = \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$. Because of this normalization, rows of the solution $\Gamma(z)$ from (2.15) do not produce v_μ . However, $F(z)$, and consequently v_μ , can be obtained as a difference of two solutions to the RHP 2.3.

Proposition 3.2. *Let $\tilde{\Gamma}(z)$ be a solution to the RHP 2.3. If $U(z) = \tilde{\Gamma}(z)\Gamma^{-1}(z)$, where $\Gamma(z)$ is given by (2.15), then*

$$(3.7) \quad U(z) = \mathbf{1} + \frac{A}{z-1} + \frac{B}{z+1},$$

where A, B are constant matrices.

Proof. Matrix U is analytic in $\mathbb{C} \setminus [-1, 1]$ and $U_+ = \tilde{\Gamma}_+ \Gamma_+^{-1} = \tilde{\Gamma}_- M (M^{-1} \Gamma_-^{-1}) = U_-$ on $[-1, 1]$; that is, U does not have a jump on $(-1, 1)$. Thus, U is analytic in $\mathbb{C} \setminus \{-1, 1\}$ and bounded at infinity. According to (2.16) and the endpoint behavior (2.12) of $\tilde{\Gamma}(z)$, it has at most simple poles at the endpoints ± 1 . Thus, U has the form (3.7). $\square$

Let $\phi \in \mathcal{N}_\mu$. By Proposition 3.1, the row $F = C((\cosh \mu z, \sinh \mu z)\phi(z))$ has the following properties: (a) F is analytic in $\mathbb{C} \setminus [-1, 1]$ with, at worst, $\mathcal{O}((z \mp 1)^{-\frac{1}{2}})$ behaviour at the endpoints ± 1 ; and (b) $F(z) = \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.

Let $\tilde{F}$ be a matrix with the rows F and 0 respectively. Then $\tilde{\Gamma} = \Gamma + \tilde{F}$ satisfies RHP 2.3 with $\varepsilon = \frac{1}{2}$. Thus, according to Proposition 3.2, $\tilde{\Gamma} = U\Gamma$, where U has the form (3.7), so that $\tilde{F} = \left(\frac{A}{z-1} + \frac{B}{z+1}\right)\Gamma$. Since Γ is invertible, it is clear that the second rows of A, B are zeros. Taking into account (2.15), we obtain

$$(3.8) \quad F = \left(\frac{(ik\beta, l)}{z-1} + \frac{(im\beta, n)}{z+1} \right) \begin{bmatrix} \cosh \mu(z - \sqrt{z^2 - 1}) & \sinh \mu(z - \sqrt{z^2 - 1}) \\ \sinh \mu(z - \sqrt{z^2 - 1}) & \cosh \mu(z - \sqrt{z^2 - 1}) \end{bmatrix},$$

where (k, l) and (m, n) denote the first rows of the constant matrices A, B respectively. It is now clear that the requirement $F = \mathcal{O}(z \mp 1)^{-\frac{1}{2}}$ as $z \rightarrow \pm 1$ respectively implies $l = m = n = 0$. Thus, up to a constant factor,

$$(3.9) \quad F(z) = (1 - z^2)^{-\frac{1}{2}} (\cosh \mu(z - \sqrt{z^2 - 1}), \sinh \mu(z - \sqrt{z^2 - 1})).$$

Now using (2.5), we obtain

$$(3.10) \quad \begin{aligned} \phi(z) &= \cosh \mu z [(F_c)_+ - (F_c)_-] - \sinh \mu z [(F_s)_+ - (F_s)_-] \\ &= \left(F \begin{bmatrix} \cosh \mu z \\ \sinh \mu z \end{bmatrix} \right)_+ - \left(F \begin{bmatrix} \cosh \mu z \\ \sinh \mu z \end{bmatrix} \right)_- = \frac{2 \cos(\mu \sqrt{1 - z^2})}{\sqrt{1 - z^2}}. \end{aligned}$$

If $p \in [2, \infty)$, Proposition 3.1 implies that $\phi(z) \in \mathcal{N}_\mu$ is bounded near $z = \pm 1$. Hence the matrices A and B in (3.7) are zero, and $\phi(z) \equiv 0$. Thus, we have obtained the following result.

Theorem 3.3. *For any $p \in (1, 2)$, the null-space of $H_\mu : L^p \rightarrow L^p$ is one-dimensional and is spanned by*

$$(3.11) \quad v_\mu = \frac{\cos(\mu\sqrt{1-z^2})}{\sqrt{1-z^2}}.$$

If $p \in [2, \infty)$, the null-space of H_μ is trivial.

4. INVERSION FORMULA FOR THE TRANSFORM H_μ

In the case $\mu = 0$ the integral transform H_μ coincides with the FHT $H = H_0$ on $[-1, 1]$. The inverse transforms H_L^{-1}, H_R^{-1} for H with the square root singularities at the left/right endpoint of the contour, respectively, are given by

$$(4.1) \quad H_L^{-1}g = -\frac{1}{\pi}\sqrt{\frac{1-z}{1+z}}\int_{-1}^1\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{g(\zeta)d\zeta}{\zeta-z}, \quad H_R^{-1}g = -\frac{1}{\pi}\sqrt{\frac{1+z}{1-z}}\int_{-1}^1\sqrt{\frac{1-\zeta}{1+\zeta}}\frac{g(\zeta)d\zeta}{\zeta-z}.$$

In this section we derive analogous formulae for inverting H_μ , $\mu > 0$, using the solution $\Gamma(z)$ to the RHP 2.3 given by (2.15). Here we still assume that g is Hölder continuous.

Let $H_\mu f = g$ on $[-1, 1]$. Then, according to (2.8), (2.9) and (2.13),

$$(4.2) \quad \begin{aligned} f(z) = & -i[C_+(g(\cosh \mu\zeta, \sinh \mu\zeta)\Gamma_+^{-1})(z)\Gamma_+(z) \\ & - C_-(g(\cosh \mu\zeta, \sinh \mu\zeta)\Gamma_+^{-1})(z)\Gamma_-(z)](\cosh \mu z, -\sinh \mu z)^T. \end{aligned}$$

We start with the calculation of

$$(4.3) \quad \Gamma(z)(\cosh \mu z, -\sinh \mu z)^T = (i\beta \cosh(\mu\sqrt{z^2-1}), -\sinh(\mu\sqrt{z^2-1}))^T =: P^T(z).$$

Equation (4.3) implies that $P_+(z) = -P_-(z)$ on the cut $[-1, 1]$. Then (4.2) becomes

$$(4.4) \quad \begin{aligned} f &= -i[C_+(g(\cosh \mu\zeta, \sinh \mu\zeta)\Gamma_+^{-1})(z) + C_-(g(\cosh \mu\zeta, \sinh \mu\zeta)\Gamma_+^{-1})(z)]P_+^T(z) \\ &= H(g(\cosh \mu\zeta, \sinh \mu\zeta)\Gamma_+^{-1})(z)P_+^T(z). \end{aligned}$$

Next we use (2.16) to calculate

$$(4.5) \quad (\cosh \mu z, \sinh \mu z)\Gamma^{-1} = (-i\beta^{-1} \cosh(\mu\sqrt{z^2-1}), \sinh(\mu\sqrt{z^2-1})) =: Q(z).$$

Combining (4.3), (4.4) and (4.5), we obtain

$$(4.6) \quad \begin{aligned} f(z) &= -H(Q_+(\zeta)g(\zeta))(z)P_-^T(z) \\ &= \cos(\mu\sqrt{1-z^2})H_L^{-1}(\cos(\mu\sqrt{1-\zeta^2})g) - \sin(\mu\sqrt{1-z^2})H(\sin(\mu\sqrt{1-\zeta^2})g), \end{aligned}$$

where the inverse Hilbert transforms H_L^{-1}, H_R^{-1} were defined in (4.1). Let $\tilde{G}$ be obtained from (2.15) via replacing $\beta = \sqrt{\frac{1-z}{1+z}}$ by $1/\beta$. Then $\tilde{G}$ is a solution of the RHP 2.3. Using this solution instead of Γ in (4.2), we can obtain another inversion formula by replacing H_L^{-1} by H_R^{-1} in (4.6).

Thus, we obtained the following expressions for the transforms $(H_\mu)_L^{-1}$ and $(H_\mu)_R^{-1}$ with the square root singularities at ∓ 1 respectively that are inverse to the cosh transform H_μ :

$$(4.7) \quad \begin{aligned} (H_\mu)_{L,R}^{-1}g &= \cos(\mu\sqrt{1-z^2})H_{L,R}^{-1}(\cos(\mu\sqrt{1-\zeta^2})g) \\ &\quad - \sin(\mu\sqrt{1-z^2})H(\sin(\mu\sqrt{1-\zeta^2})g). \end{aligned}$$

In the case $\mu = 0$ they coincide with $H_{L,R}^{-1}g$. The precise meaning of the inversion formulae will be discussed in Section 5.

Remark 4.1. Direct calculations show that

$$(4.8) \quad (H_\mu)_R^{-1}g - (H_\mu)_L^{-1}g = \frac{2}{\pi}(v_\mu, g)v_\mu(z),$$

where v_μ is given by (3.11), and

$$(4.9) \quad (v_\mu, g) = \int_{-1}^1 v_\mu(\zeta)g(\zeta)d\zeta.$$

Thus the following three conditions are equivalent: (a) both $(H_\mu)_L^{-1}g$ and $(H_\mu)_R^{-1}g$ are bounded on $[-1, 1]$; (b) $(H_\mu)_L^{-1}g = (H_\mu)_R^{-1}g$ on $[-1, 1]$; and (c) $(v_\mu, g) = 0$.

Remark 4.2. Inversion formulae (4.7) appear to reconstruct a Hölder continuous function f up to a multiple of the null-space function v_μ , since $(H_\mu)_{L,R}^{-1}g$ appear to have square root singularities at $z = \mp 1$. As shown in Remark 4.1, these singularities disappear if g satisfies $(v_\mu, g) = 0$. It will be shown below that this condition is, in fact, the range condition for H_μ in L^p with $p > 2$.

5. ANALYSIS OF H_μ IN L^p SPACES

In this section we study the action of H_μ in L^p -spaces and obtain two more inversion formulae. We start the section by reminding the reader of a few facts about Fredholm operators. Let X be a Banach space. The space of linear continuous operators $X \rightarrow X$ is denoted $L(X)$. An operator $A \in L(X)$ is said to be Fredholm if (a) $\mathcal{N}(A)$ and $\mathcal{N}(A^*)$, the null-spaces of A and its adjoint A^* , are finite-dimensional; and (b) the range of A is closed. If $A \in L(X)$ is Fredholm, denote

$$(5.1) \quad \alpha(A) := \dim \mathcal{N}(A), \quad \beta(A) := \dim \mathcal{N}(A^*).$$

The number $\chi(A) := \alpha(A) - \beta(A)$ is called the index of A . It is proven in [Jor82], §13, that the FHT $H : L^p \rightarrow L^p$ is Fredholm for $p \in (1, \infty)$, $p \neq 2$. Moreover,

$$(5.2) \quad \alpha(H) = 1, \quad \beta(H) = 0, \quad p \in (1, 2), \quad \alpha(H) = 0, \quad \beta(H) = 1, \quad p \in (2, \infty).$$

From Theorem 3.3,

$$(5.3) \quad \alpha(H_\mu) = 1, \quad p \in (1, 2); \quad \alpha(H_\mu) = 0, \quad p \in (2, \infty).$$

Using the kernel K , which was introduced in the proof of Proposition 3.1, we see that $H_\mu - H$ is a compact operator. Hence, by Theorem 5.12 of [Jor82]: (a) $H_\mu : L^p \rightarrow L^p$ is also Fredholm for $p \in (1, \infty)$, $p \neq 2$, and (b) H and H_μ have the same index. From (5.2) and (5.3),

$$(5.4) \quad \beta(H_\mu) = 0, \quad p \in (1, 2); \quad \beta(H_\mu) = 1, \quad p \in (2, \infty).$$

Since the adjoint of H_μ is $-H_\mu$ and $v_\mu \in L^p$ for any $p \in (1, 2)$, we have

$$(5.5) \quad (H_\mu v_\mu, f) = -(v_\mu, H_\mu f) = 0$$

for any $f \in L^q$, $q \in (2, \infty)$. From (5.4) we conclude that (5.5) is the only range condition for H_μ on L^p , $p \in (2, \infty)$. This is a generalization of Remark 4.1 from Hölder continuous to L^p , $p > 2$, functions.

Next we obtain two new inversion formulae for H_μ that are more convenient for L^p classes. Assume first that $p \in (1, 2)$. Since H_μ has a one-dimensional null-space spanned by v_μ , we can subtract any multiple of v_μ from the right side of (4.7). Considering the first integral in $(H_\mu)_L^{-1}$,

$$(5.6) \quad \begin{aligned} & \sqrt{\frac{1-z}{1+z}} \cos(\mu\sqrt{1-z^2}) \int_{-1}^1 \sqrt{\frac{1+\zeta}{1-\zeta}} \cos(\mu\sqrt{1-\zeta^2}) \frac{g(\zeta)}{\zeta-z} d\zeta \\ & - \frac{\cos(\mu\sqrt{1-z^2})}{\sqrt{1-z^2}} \int_{-1}^1 \sqrt{\frac{1+\zeta}{1-\zeta}} \cos(\mu\sqrt{1-\zeta^2}) g(\zeta) d\zeta \\ & = \frac{\cos(\mu\sqrt{1-z^2})}{\sqrt{1-z^2}} \int_{-1}^1 \sqrt{\frac{1+\zeta}{1-\zeta}} \cos(\mu\sqrt{1-\zeta^2}) \left(\frac{1-z}{\zeta-z} - 1 \right) g(\zeta) d\zeta \\ & = \frac{\cos(\mu\sqrt{1-z^2})}{\sqrt{1-z^2}} \int_{-1}^1 \cos(\mu\sqrt{1-\zeta^2}) \sqrt{1-\zeta^2} \frac{g(\zeta)}{\zeta-z} d\zeta. \end{aligned}$$

Suppose now that $p \in (2, \infty)$. If $g \in H_\mu(L^2)$, then $(g, v_\mu) = 0$. Hence we can subtract from the kernel of $(H_\mu)_L^{-1}$ any multiple of v_μ . Arguing similarly to (5.6), we find

$$(5.7) \quad \begin{aligned} & \sqrt{\frac{1-z}{1+z}} \cos(\mu\sqrt{1-z^2}) \int_{-1}^1 \frac{\cos(\mu\sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}} \left(\frac{1+\zeta}{\zeta-z} - 1 \right) g(\zeta) d\zeta \\ & = \cos(\mu\sqrt{1-z^2}) \sqrt{1-z^2} \int_{-1}^1 \frac{\cos(\mu\sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}} \frac{g(\zeta)}{\zeta-z} d\zeta. \end{aligned}$$

Apart from the cosine factors in (5.6) and (5.7), these formulae are completely analogous to those for inverting the FHT H . In particular, they have the same continuity properties as their counterparts for H (see [Jor82], §13, and [OE91], Propositions 2.4 and 2.6). Thus, the operator in (5.6) is $L^p \rightarrow L^p$ continuous for $p \in (1, 2)$, and the operator in (5.7) is $L^p \rightarrow L^p$ continuous for $p \in (2, \infty)$. Since $H : L^p \rightarrow L^p$ is continuous for all $p \in (1, \infty)$ (which describes the second part of the inversion formulae in (4.7)), our argument proves the following theorem.

Theorem 5.1. *The map $H_\mu : L^p \rightarrow L^p$ is Fredholm for all $p \in (1, \infty)$, $p \neq 2$. For $p \in (1, 2)$: (a) the null-space of H_μ is one-dimensional and spanned by v_μ ; (b) the range of H_μ is all of L_p ; (c) the pseudo-inverse of H_μ modulo the null-space is given by*

$$(5.8) \quad \begin{aligned} H_\mu^{-1} g = & -\frac{1}{\pi} \frac{\cos(\mu\sqrt{1-z^2})}{\sqrt{1-z^2}} \int_{-1}^1 \cos(\mu\sqrt{1-\zeta^2}) \sqrt{1-\zeta^2} \frac{g(\zeta)}{\zeta-z} d\zeta \\ & - \sin(\mu\sqrt{1-z^2}) H(\sin(\mu\sqrt{1-\zeta^2}) g); \end{aligned}$$

and (d) $H_\mu^{-1} : L^p \rightarrow L^p$ given by (5.8) is continuous.

For $p \in (2, \infty)$: (a) the null-space of H_μ is trivial; (b) the range of H_μ has co-dimension one and consists of all $g \in L_p$ such that $\int_{-1}^1 g(\zeta) v_\mu(\zeta) d\zeta = 0$; (c) the pseudo-inverse of H_μ on its range is given by

$$(5.9) \quad \begin{aligned} H_\mu^{-1} g = & -\frac{1}{\pi} \cos(\mu\sqrt{1-z^2}) \sqrt{1-z^2} \int_{-1}^1 \frac{\cos(\mu\sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}} \frac{g(\zeta)}{\zeta-z} d\zeta \\ & - \sin(\mu\sqrt{1-z^2}) H(\sin(\mu\sqrt{1-\zeta^2}) g); \end{aligned}$$

and (d) $H_\mu^{-1} : L^p \rightarrow L^p$ given by (5.9) is continuous.

6. THE RANGE OF H_μ IN L^2

In this section we provide a characterization of the range of H_μ and, thus, the domain of its inverse. Our range description is the generalization of Theorem 3.2 in [OE91] to the H_μ transform. Similarly to Lemma 3.1 of [OE91], we see that H_μ is not Fredholm on L^2 .

Lemma 6.1. *The operator $H_\mu : L^2 \rightarrow L^2$ is injective, and its range is a proper dense subspace of L^2 ; i.e. $H_\mu : L^2 \rightarrow L^2$ is not Fredholm.*

Proof. The injectivity of H_μ follows from the fact that $v_\mu \notin L^2$. Since $H_\mu^* = -H_\mu$ and the dual space of L^2 is identified with itself, we see that $H_\mu(L^2)$ is dense. Next, let $g \in C_0^\infty([-1, 1])$ be an even nonnegative function which is not identically zero. From formula (5.8), $(H_\mu^{-1}g)(y) \sim \pm c_1/\sqrt{1-y^2}$ as $y \rightarrow \pm 1$, where $c_1 \neq 0$ is some constant. Clearly, there does not exist c_2 such that $H_\mu^{-1}g - c_2 v_\mu \in L^2$. Hence $g \notin H_\mu(L^2)$. $\square$

To expedite the computation and shorten the notation we introduce

$$(6.1) \quad \begin{aligned} \mathbf{w} &:= \sqrt{1-z^2}, \quad \mathbf{c} := \cos(\mu\sqrt{1-z^2}), \quad \mathbf{s} := \sin(\mu\sqrt{1-z^2}), \\ \mathbf{C} &:= \cosh(\mu z), \quad \mathbf{S} := \sinh(\mu z). \end{aligned}$$

Then the transform H_μ and formula (5.8) read

$$(6.2) \quad H_\mu[f] = \mathbf{C}H[\mathbf{C}f] - \mathbf{S}H[\mathbf{S}f], \quad H_\mu^{-1}[g] = -\frac{\mathbf{c}}{\mathbf{w}}H[\mathbf{c}wg] - \mathbf{s}H[\mathbf{s}g]$$

respectively.

Theorem 6.2. *For any $f \in L^p$, $p \in (1, 2]$, we have*

$$(6.3) \quad H_\mu^{-1}[H_\mu[f]] = f + \varkappa v_\mu, \quad \text{where } \varkappa := -\frac{1}{\pi} \int_{-1}^1 \cosh(\mu\zeta) f(\zeta) d\zeta,$$

and H_μ^{-1} is the expression (5.8).

Proof. The proof is a direct but somewhat complicated computation. We shall use Lemmas A.1 and A.2 in the appendix. Another result we need is the Poincaré–Bertrand theorem [Tri57], Theorem 4.2.IV: if $\phi \in L^p, \psi \in L^q$ are in conjugate spaces, then

$$(6.4) \quad H[\phi H[\psi]] = H[\phi] H[\psi] - \psi\phi - H[\psi H[\phi]].$$

Suppose that g is in the image of H_μ ; that is, $g = H_\mu[f] = CH[Cf] - SH[Sf]$. Then

$$(6.5) \quad \begin{aligned} -H_\mu^{-1}[g] &= \frac{\mathbf{c}}{\mathbf{w}} H[\mathbf{cw}g] + \mathbf{s}H[sg] \\ &= \frac{\mathbf{c}}{\mathbf{w}} H[\mathbf{cw}(CH[Cf] - SH[Sf])] + \mathbf{s}H[\mathbf{s}(CH[Cf] - SH[Sf])]. \end{aligned}$$

We apply the Poincaré–Bertrand theorem to each term, and we get the following two expressions:

$$(6.6) \quad \begin{aligned} H[\mathbf{cw}(CH[Cf] - SH[Sf])] &= H[\mathbf{cwC}] H[Cf] - \mathbf{cwC}^2 f - H[CfH[\mathbf{cwC}]] \\ &\quad - H[\mathbf{cwS}] H[Sf] + \mathbf{cwS}^2 f + H[SfH[\mathbf{cwS}]] \\ &= -\mathbf{cw}f + H\left[Cf\left(H[\mathbf{cwC}](z) - H[\mathbf{cwC}]\right)\right] - H\left[Sf\left(H[\mathbf{cwS}](z) - H[\mathbf{cwS}]\right)\right] \end{aligned}$$

and

$$(6.7) \quad \begin{aligned} H[\mathbf{s}(CH[Cf] - SH[Sf])] &= H[\mathbf{sC}] H[Cf] - \mathbf{sC}^2 f - H[CfH[\mathbf{sC}]] \\ &\quad - H[\mathbf{sS}] H[Sf] + \mathbf{sS}^2 f + H[SfH[\mathbf{sS}]] \\ &= -\mathbf{s}f + H\left[Cf\left(H[\mathbf{sC}](z) - H[\mathbf{sC}]\right)\right] - H\left[Sf\left(H[\mathbf{sS}](z) - H[\mathbf{sS}]\right)\right]. \end{aligned}$$

Plugging these expressions into (6.5) and using Lemmas A.1, A.2 we get

$$(6.8) \quad \begin{aligned} -H_\mu^{-1}[g] &= -f + \frac{\mathbf{c}}{\mathbf{w}} \left(H\left[f(\xi)\left(\mathbf{C}(\xi)H[\mathbf{cwC}](z) - \mathbf{S}(\xi)H[\mathbf{cwS}](z)\right)\right] \right. \\ &\quad \left. - f(\xi)\left(\mathbf{C}(\xi)H[\mathbf{cwC}](\xi) - \mathbf{S}(\xi)H[\mathbf{cwS}](\xi)\right)\right) + \mathbf{s}H\left[f(\xi)\left(\mathbf{C}(\xi)H[\mathbf{sC}](z) \right. \right. \\ &\quad \left. \left. - \mathbf{S}(\xi)H[\mathbf{sS}](z)\right)\right] - \mathbf{s}H\left[f(\xi)\left(\mathbf{C}(\xi)H[\mathbf{sC}](\xi) - \mathbf{S}(\xi)H[\mathbf{sS}](\xi)\right)\right]. \end{aligned}$$

We now apply Lemma A.1 to the second term in the last expression of (6.8) and Lemma A.2 to the last two terms. Note that in doing so, each lemma is applied once with $z = z$ and once with $z = \xi$. We obtain

$$(6.9) \quad \begin{aligned} -H_\mu^{-1}[g] &= -f + \frac{\mathbf{c}(z)}{\mathbf{w}(z)} H\left[f(\xi)\left(-\mathbf{S}(\xi - z)\mathbf{s}(z)\mathbf{w}(z) + (\xi - z)\mathbf{C}(\xi)\right)\right] \\ &\quad + \mathbf{s}(z)H\left[f(\xi)\left(\mathbf{S}(\xi - z)\mathbf{c}(z)\right)\right] \\ &= -f - \mathbf{c}(z)\mathbf{s}(z)H[\mathbf{S}(\xi - z)f(\xi)] + \frac{\mathbf{c}(z)}{\mathbf{w}(z)} H[(\xi - z)\mathbf{C}(\xi)f(\xi)] \\ &\quad + \mathbf{s}(z)\mathbf{c}(z)H[\mathbf{S}(\xi - z)f(\xi)] = -f + \frac{v_\mu}{\pi} \int_{-1}^1 \cosh(\mu\xi)f(\xi) \, d\xi. \end{aligned}$$

□

Theorem 6.3 (See Theorem 3.2 in [OE91]). *The function $g \in L^2$ is in the range of $H_\mu : L^2 \rightarrow L^2$ if and only if there is a constant $\varkappa$ such that*

$$(6.10) \quad H_\mu^{-1}[g](z) - \varkappa v_\mu(z) \in L^2,$$

where H_μ^{-1} is the expression (5.8). Then, if $g = H_\mu[f]$, the constant $\varkappa$ is given by (6.3).

Proof. Suppose that the constant $\varkappa$ exists. Since $g \in L^2$, then $g \in L^{2-\epsilon}$, where ϵ is a small positive number. Using the fact that $H_\mu : L^p \rightarrow L^p$ is onto for $p \in (1, 2)$, we have

$$(6.11) \quad H_\mu [H_\mu^{-1}[g] - \varkappa v_\mu] = g.$$

This means that $g \in H_\mu(L^2)$, being the H_μ -image of $H_\mu^{-1}[g] - \varkappa v_\mu \in L^2$. Vice versa, suppose that $g = H_\mu[f]$, with $g, f \in L^2$. But then $f, g \in L^{2-\epsilon}$ as well, so it must be that $H_\mu^{-1}[g] - f$ is in the null-space, namely

$$(6.12) \quad H_\mu^{-1}[g] - f = \varkappa v_\mu \Rightarrow H_\mu^{-1}[g] - \varkappa v_\mu = f \in L^2.$$

The value of the constant is provided by Theorem 6.2. $\square$

The constant in Theorems 6.2 and 6.3 is expressed in terms of the preimage of g , which might be inconvenient sometimes. It is interesting to note that in SPECT with constant attenuation, the value of the constant $\varkappa$ is easily accessible from the data [NDPC07]. By arguing analogously to [OE91], we can give an expression which does not use the preimage but resorts to a limit.

We need some additional notation:

$$(6.13) \quad I_n := \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad v_n(z) := v_\mu \chi_n(z), \quad g_n(z) := \chi_n(z) H_\mu^{-1}[g](z).$$

Here $(\cdot, \cdot)$ is the L^2 inner product, $\|\cdot\|$ is the L^2 norm, and χ_n is the characteristic function of I_n . It is clear that the sequence of v_n 's belongs to L^2 but their norms diverge. Let us denote

$$(6.14) \quad \mu_n := \frac{(g_n, v_n)}{\|v_n\|^2}.$$

Proposition 6.4. *The vector $g \in L^2$ belongs to the image of $H_\mu : L^2 \rightarrow L^2$ if and only if*

$$(6.15) \quad \sup_{n \in \mathbb{N}} \|g_n - \mu_n v_n\|^2 < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|v_n\|^2 |\mu_n - \varkappa|^2 < \infty$$

for some constant $\varkappa$. In this case $\varkappa = \lim_{n \rightarrow \infty} \mu_n$.

Proof. We have

$$(6.16) \quad \begin{aligned} |\mu_n - \varkappa|^2 + \frac{\|g_n - \mu_n v_n\|^2}{\|v_n\|^2} &= |\mu_n|^2 + |\varkappa|^2 - 2\Re(\mu_n, \overline{\varkappa}) + \frac{\|g_n\|^2}{\|v_n\|^2} - \mu_n \frac{(v_n, g_n)}{\|v_n\|^2} \\ &- \overline{\mu_n} \frac{(g_n, v_n)}{\|v_n\|^2} + |\mu_n|^2 = |\mu_n|^2 + |\varkappa|^2 - 2\Re(\mu_n, \overline{\varkappa}) + \frac{\|g_n\|^2}{\|v_n\|^2} - \mu_n \overline{\mu_n} - \overline{\mu_n} \mu_n + |\mu_n|^2 \\ &= |\varkappa|^2 - 2\Re(\mu_n, \overline{\varkappa}) + \frac{\|g_n\|^2}{\|v_n\|^2} = \frac{1}{\|v_n\|^2} (|\varkappa|^2 \|v_n\|^2 - 2\Re(\overline{\varkappa} v_n, g_n)) + \|g_n\|^2 \\ &= \frac{1}{\|v_n\|^2} \|g_n - \varkappa v_n\|^2, \end{aligned}$$

that is

$$(6.17) \quad \|v_n\|^2 |\mu_n - \varkappa|^2 + \|g_n - \mu_n v_n\|^2 = \|g_n - \varkappa v_n\|^2.$$

On the right-hand side there is the norm of the projection of the vector $h - \varkappa v_\mu := H_\mu^{-1}[g] - \varkappa v_\mu$ on the interval I_n ; these projections tend strongly to the identity in L^2 . If $h - \varkappa v_\mu \in L^2$, then the right-hand side of (6.17) has a limit, and thus it must be that

$$(6.18) \quad \sup_{n \in \mathbb{N}} \|v_n\|^2 |\mu_n - \varkappa|^2 < \infty, \quad \sup_{n \in \mathbb{N}} \|g_n - \mu_n v_n\|^2 < \infty.$$

In particular, since $\|v_n\| \rightarrow +\infty$, we must have $\mu_n \rightarrow \varkappa$.

Vice versa, if the second inequality in (6.18) holds and there is a constant $\varkappa$ such that the first inequality in (6.18) holds as well, then the $\sup_n$ of the right-hand side of (6.17) is necessarily finite. Since

$$(6.19) \quad \|h - \varkappa v_\mu\|^2 = \sup_n \|g_n - \varkappa v_n\|^2,$$

it follows that $h - \varkappa v_\mu \in L^2$. The assertion of the proposition then follows from Theorem 6.3. $\square$

Remark 6.5. Replacing the constant $\varkappa$ in (6.3) with its expression provided in Proposition 6.4 gives an alternative formula for inverting $H_\mu : L^2 \rightarrow L^2$.

APPENDIX A. TECHNICAL LEMMAS

Lemma A.1. *We have*

$$(A.1) \quad \mathbf{C}(\xi)H[\mathbf{cwC}](z) - \mathbf{S}(\xi)H[\mathbf{cwS}](z) = -\mathbf{S}(\xi - z)\mathbf{s}(z)\mathbf{w}(z) - z\mathbf{C}(\xi) + \frac{\mu}{2}\mathbf{S}(\xi).$$

Proof. Clearly,

$$(A.2) \quad \mathbf{C}(\xi)H[\mathbf{cwC}](z) - \mathbf{S}(\xi)H[\mathbf{cwS}](z) = \frac{1}{\pi} \int \frac{\mathbf{c}(\zeta)\mathbf{w}(\zeta)\mathbf{C}(\xi - \zeta)}{\zeta - z} d\zeta.$$

Note that $\sqrt{1 - \zeta^2}_+ = -i\sqrt{\zeta^2 - 1}_+$ and define $\mathbf{r}(z) := z - \sqrt{z^2 - 1}$. We first compute the Cauchy transform (γ is a contour clockwise around $[-1, 1]$):

$$\begin{aligned} (A.3) \quad & \int_{-1}^1 \frac{\cos(\mu\sqrt{1 - \zeta^2}_+)\sqrt{1 - \zeta^2}_+ \cosh(\mu(\xi - \zeta))}{\zeta - z} \frac{dz}{2\pi i} \\ &= \frac{1}{4} \int_{-1}^1 \frac{\sqrt{1 - \zeta^2}_+ \left(e^{\mu\sqrt{\zeta^2 - 1}_+} + e^{-\mu\sqrt{\zeta^2 - 1}_+} \right) (e^{\mu(\xi - \zeta)} + e^{-\mu(\xi - \zeta)})}{\zeta - z} \frac{dz}{2\pi i} \\ &= -\frac{i}{4} \int_{-1}^1 \frac{\sqrt{\zeta^2 - 1}_+ (e^{\mu(\xi - \mathbf{r}_+)} + e^{\mu(\xi - \mathbf{r}_-)} + e^{-\mu(\xi - \mathbf{r}_-)} + e^{-\mu(\xi - \mathbf{r}_+)})}{\zeta - z} \frac{dz}{2\pi i} \\ &= -\frac{i}{2} \int_{-1}^1 \frac{\cosh(\mu(\xi - \mathbf{r}_+))\sqrt{\zeta^2 - 1}_+ - \cosh(\mu(\xi - \mathbf{r}_-))\sqrt{\zeta^2 - 1}_-}{\zeta - z} \frac{dz}{2\pi i} \\ &= -\frac{i}{2} \oint_\gamma \frac{\cosh(\mu(\xi - \mathbf{r}))\sqrt{\zeta^2 - 1}}{\zeta - z} \frac{d\zeta}{2\pi i} \\ &= -\frac{i}{2} \left(\cosh(\mu(\xi - \mathbf{r}(z)))\sqrt{z^2 - 1} - z \cosh(\mu\xi) + \frac{\mu}{2} \sinh(\mu\xi) \right). \end{aligned}$$

Now (we use $\mathbf{w} := \sqrt{1 - z^2}$)

$$\begin{aligned}
 & \left[\cosh(\mu(\xi - \mathbf{r})) \sqrt{z^2 - 1} \right]_{\pm} = \cosh(\mu(\xi - z)) \cosh(\mu \sqrt{z^2 - 1}) \sqrt{z^2 - 1}_{\pm} \\
 (A.4) \quad & + \sinh(\mu(\xi - z)) \sinh(\mu \sqrt{z^2 - 1}) \sqrt{z^2 - 1}_{\pm} \\
 & = \pm \cosh(\mu(\xi - z)) \cos(\mu \mathbf{w}_+) i \mathbf{w}_+ + \sinh(\mu(\xi - z)) i \sin(\mu \mathbf{w}_+) i \mathbf{w}_+ \\
 & = \pm \cosh(\mu(\xi - z)) \cos(\mu \mathbf{w}) i \mathbf{w} - \sinh(\mu(\xi - z)) \sin(\mu \mathbf{w}) \mathbf{w}.
 \end{aligned}$$

Therefore, since $H = i(C_+ + C_-)$,

$$(A.5) \quad \frac{1}{\pi} \int \frac{\mathbf{c}(\zeta) \mathbf{w}(\zeta) \mathbf{C}(\xi - \zeta)}{\zeta - z} d\zeta = -\mathbf{S}(\xi - z) \mathbf{s}(z) \mathbf{w}(z) - z \mathbf{C}(\xi) + \frac{\mu}{2} \mathbf{S}(\xi).$$

□

Lemma A.2. *We have*

$$(A.6) \quad \mathbf{C}(\xi) H[\mathbf{sC}](z) - \mathbf{S}(\xi) H[\mathbf{sS}](z) = \mathbf{S}(\xi - z) \mathbf{c}(z) - \mathbf{S}(\xi).$$

Proof. This is very similar to the proof of Lemma A.1, and some steps are skipped. We have

$$(A.7) \quad \mathbf{C}(\xi) H[\mathbf{sC}](z) - \mathbf{S}(\xi) H[\mathbf{sS}](z) = \frac{1}{\pi} \int_{-1}^1 \frac{\cosh(\mu(\xi - \zeta)) \sin(\mu \sqrt{1 - \zeta^2}_+)}{\zeta - z} d\zeta.$$

The Cauchy transform gives

$$\begin{aligned}
 (A.8) \quad & \frac{1}{i} \int_{-1}^1 \frac{\cosh(\mu(\xi - \zeta)) \sinh(\mu \sqrt{\zeta^2 - 1}_+)}{\zeta - z} \frac{d\zeta}{2\pi i} \\
 & = \frac{1}{4i} \int_{-1}^1 \frac{(e^{\mu(\xi - \mathbf{r}_+)} + e^{-\mu(\xi - \mathbf{r}_-)} - e^{\mu(\xi - \mathbf{r}_-)} - e^{-\mu(\xi - \mathbf{r}_+)})}{\zeta - z} \frac{d\zeta}{2\pi i} \\
 & = \frac{1}{2i} \int_{-1}^1 \frac{\sinh(\mu(\xi - \mathbf{r}_+)) - \sinh(\mu(\xi - \mathbf{r}_-))}{\zeta - z} \frac{d\zeta}{2\pi i} = \frac{1}{2i} \oint_{\gamma} \frac{\sinh(\mu(\xi - \mathbf{r}))}{\zeta - z} \frac{d\zeta}{2\pi i} \\
 & = \frac{1}{2i} (\sinh(\mu(\xi - \mathbf{r}(z))) - \sinh(\mu\xi)).
 \end{aligned}$$

Now

$$(A.9) \quad \sinh(\mu(\xi - \mathbf{r}_{\pm}(z))) = \sinh(\mu(\xi - z)) \cos(\mu \sqrt{1 - z^2}_+) \pm i \cosh(\mu(\xi - z)) \sin(\mu \sqrt{1 - z^2}_+).$$

Therefore ($H = i(C_+ + C_-)$)

$$(A.10) \quad \frac{1}{\pi} \int_{-1}^1 \frac{\mathbf{C}(\xi - \zeta) \mathbf{s}(\zeta)}{\zeta - z} d\zeta = \mathbf{S}(\xi - z) \mathbf{c}(z) - \mathbf{S}(\xi).$$

□

APPENDIX B. GENERAL SOLUTION OF THE RHP 2.3 AND AN ALTERNATIVE EXPRESSION FOR INVERSION FORMULAE

In this section we will give an alternative way of deriving the inversion formulae for $H_{\mu}(f)$, where f is a Hölder-continuous (on $[-1, 1]$) function. Our derivation is based on the general solution $\Gamma(z)$ to the RHP 2.3. Only brief outlines of proofs are included here. In this section we do not require that $\Gamma(z)$ be invertible.

Proposition B.1. *Any solution to Problem 2.3 has growth bounded by $\mathcal{O}(1/\sqrt{z \mp 1})$.*

This result follows from the fact that the jump matrix M is analytically reducible (similar) to σ_3 .

Lemma B.2. *The determinant of any solution to Problem 2.3 is of the form*

$$(B.1) \quad \det \Gamma(z) = t \sqrt{\frac{z-1}{z+1}} + (1-t) \sqrt{\frac{z+1}{z-1}} = \frac{z-2t+1}{\sqrt{z^2-1}}, \quad t \in \mathbb{C}.$$

A solution $\Gamma(z)$ is invertible in $\mathbb{C} \setminus \{-1, 1\}$ if and only if $t = 0$ or $t = 1$.

We state the RHP for $\det \Gamma(z)$ and give the general solution to this RHP. Proposition B.1 is used to set endpoint conditions as $\det \Gamma(z) = \mathcal{O}((z \mp 1)^{-1})$, $z \rightarrow \pm 1$.

Proposition B.3. *If Γ is a solution of Problem 2.3 with determinant $\det \Gamma$ as in Lemma B.2, then it is of the form*

$$(B.2) \quad \Gamma(z; t, \alpha) = \begin{bmatrix} \frac{z+1-2t}{\sqrt{z^2-1}} & 0 \\ \frac{\alpha}{\sqrt{z^2-1}} & 1 \end{bmatrix} \begin{bmatrix} \cosh(\mathbf{r}\mu) & \sinh(\mathbf{r}\mu) \\ \sinh(\mathbf{r}\mu) & \cosh(\mathbf{r}\mu) \end{bmatrix}.$$

The proof is similar to Proposition 3.2. In fact, it is still valid if we relax the endpoint conditions (2.12) by $\Gamma(z; t, \alpha) = o((z \mp 1)^{-1})$, $z \rightarrow \pm 1$.

Theorem B.4. *The formula for the inverse of H_μ is*

$$(B.3) \quad f = J_\mu^{(\alpha, t)} g = -\frac{(z+1-2t)\mathbf{c}(z)}{\sqrt{1-z^2}} H\left(\frac{\sqrt{1-\zeta^2}\mathbf{c}(\zeta)}{\zeta+1-2t} g\right) - \mathbf{s}(z) H(\mathbf{s}(\zeta)g) + 2\alpha \mathcal{A}[g] v_\mu(z),$$

where $t \in \mathbb{C} \setminus (0, 1)$, $\alpha \in \mathbb{C}$, and

$$(B.4) \quad \mathbf{c}(z) := \cos\left(\mu\sqrt{1-z^2}\right), \quad \mathbf{s}(z) := \sin\left(\mu\sqrt{1-z^2}\right), \quad \mathcal{A}[g] := \int_{-1}^1 \frac{g(\zeta)\mathbf{s}(\zeta)}{\zeta+1-2t} \frac{d\zeta}{2i\pi}.$$

By Proposition 2.4, we have $F = [G(\zeta)\Gamma_+^{-1}(\zeta; t, \alpha)]\Gamma(z; t, \alpha)$. Now the result follows from calculations that are similar to those from Section 4.

Remark B.5. Theorem B.4 provides a two-parameter family of inversion formulæ depending on α, t . Direct calculation

$$J_\mu^{(\alpha, t)}(g) - J_\mu^{(\beta, s)}(g) = v_\mu(z) \left[\int_{-1}^1 \frac{2(s-t)\sqrt{1-\zeta^2}\mathbf{c}(\zeta) + ((\alpha-\beta)(\zeta+1) - 2\alpha s + 2\beta t)\mathbf{s}(\zeta)}{(\zeta+1-2t)(\zeta+1-2s)} g(\zeta) d\zeta \right]$$

shows that, as expected, the difference of any two inversions is a multiple of $v_\mu \in \mathcal{N}_\mu$. Moreover, $(H_\mu)_{L,R}^{-1}$ from (4.7) are given by $(H_\mu)_L^{-1} = J_\mu^{(0,0)}$ and $(H_\mu)_R^{-1} = J_\mu^{(0,1)}$, and, sending $t \rightarrow \infty$, we have

$$(B.5) \quad \begin{aligned} J_\mu^{(0, \infty)}[g] &= H_\mu^{-1}[g] \\ &= -\frac{\mathbf{c}(z)}{\sqrt{1-z^2}} H\left[\sqrt{1-\zeta^2}\mathbf{c}(\zeta)g(\zeta)\right] - \mathbf{s}(z) H[\mathbf{s}(\zeta)g(\zeta)]. \end{aligned}$$

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