ON A GENERALIZATION OF BAER THEOREM

L. A. KURDACHENKO, J. OTAL, AND I. YA. SUBBOTIN

(Communicated by Ken Ono)

ABSTRACT. R. Baer has proved that if the factor-group $G/\zeta_n(G)$ of a group G by the member $\zeta_n(G)$ of its upper central series is finite (here n is a positive integer), then the member $\gamma_{n+1}(G)$ of the lower central series of G is also finite. In particular, in this case, the nilpotent residual of G is finite. This theorem admits the following simple generalization, which has been published very recently by M. de Falco, F. de Giovanni, C. Musella and Ya. P. Sysak: “If the factor-group G/Z of a group G modulo its upper hypercenter Z is finite, then G has a finite normal subgroup L such that G/L is hypercentral.” In the current article we offer a new, simpler, very short proof of this theorem and specify it substantially. In fact, we prove that if $|G/Z| = t$, then $|L| \leq t^k$, where $k = \frac{1}{2}(\log_p t + 1)$ and p is the least prime divisor of t .

1. INTRODUCTION

One of the important long-standing results in the Theory of Groups is a classical theorem due to I. Schur [8], which establishes a connection between the factor-group $G/\zeta(G)$ of a group G modulo its center $\zeta(G)$ and the derived subgroup $[G, G]$ of G . It follows from Schur’s theorem [8] that *if $G/\zeta(G)$ is finite, then $[G, G]$ is also finite*. A natural question related to this result appears here, namely the question regarding the relationship between the orders $|G/\zeta(G)|$ and $|[G, G]|$. J. Wiegold in the paper [9] obtained the following answer to this question. Let G be a group such that $|G/\zeta(G)| = t$ is finite. Wiegold proved that there exists a function w such that $|[G, G]| \leq w(t)$. He also was able to obtain for this function the value $w(t) = t^m$, where $m = \frac{1}{2}(\log_p t - 1)$ and p is the least prime divisor of t . Later on, Wiegold was able to show that this boundary value may be attained if and only if $t = p^n$ for some prime p ([10]). When t has more than one prime divisor, the picture becomes more complicated.

Various generalizations of Schur’s theorem can be found in the mathematical literature. One of the most interesting approaches would be to *study properties of the factor-group $G/\zeta(G)$ such that the derived subgroup $[G, G]$ satisfies the same property*. A class of groups $\mathfrak{X}$ is said to be a *Schur class* if for every group G such that $G/\zeta(G) \in \mathfrak{X}$ the derived subgroup $[G, G]$ also belongs to $\mathfrak{X}$. Schur’s classes were introduced in the paper [3]. Besides the obvious examples of the classes of finite and of locally finite groups, the class of polycyclic-by-finite groups and the class of Chernikov groups are also Schur’s classes (see, for example, [7, Theorem 3.9]). In the paper [3] other Schur’s classes were found as well.

Received by the editors October 31, 2011.

2010 *Mathematics Subject Classification*. Primary 20F14.

The authors were supported by Proyecto MTM2010-19938-C03-03 of MICINN (Spain), the Government of Aragón (Spain) and FEDER funds from the European Union.

©2013 American Mathematical Society
 Reverts to public domain 28 years from publication

In the paper [1] R. Baer generalized Schur's theorem in a different direction. We recall that *the upper central series of a group G* is the ascending series

$$\langle 1 \rangle = \zeta_0(G) \leq \zeta_1(G) \leq \cdots \leq \zeta_\alpha(G) \leq \zeta_{\alpha+1}(G) \leq \cdots \leq \zeta_\delta(G) = \zeta_\infty(G)$$

given by the fact that $\zeta_1(G) = \zeta(G)$ is the center of G , and recursively that $\zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G))$ for all ordinals α and $\zeta_\lambda(G) = \bigcup_{\mu < \lambda} \zeta_\mu(G)$ for every limit ordinal λ . The last term $\zeta_\infty(G)$ of this series is called *the upper hypercenter of G* . G itself is called *hypercentral* if $\zeta_\infty(G) = G$. In general, the length of the upper central series of G is denoted by $zl(G)$. On the other hand, *the lower central series of G* is the descending series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \cdots$$

given by $\gamma_2(G) = [G, G]$, and recursively $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$ for all ordinals α and $\gamma_\lambda(G) = \bigcap_{\mu < \lambda} \gamma_\mu(G)$ for every limit ordinal λ .

Baer proved that *if for some positive integer n the factor-group $G/\zeta_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite too* ([1]). In particular, in this case the nilpotent residual of G (that is, the intersection of all normal subgroups N of G such that G/N is nilpotent) is finite. Very recently, in the paper [2], M. de Falco, F. de Giovanni, C. Musella and Ya. P. Sysak obtained the following generalization of this result:

Theorem A. *Let G be a group and let Z be the upper hypercenter of G . If G/Z is finite, then G has a finite normal subgroup L such that G/L is hypercentral.*

In Section 2 we provide an elementary proof of this result, which is considerably shorter than the original one.

Just as in the theorem of Schur, the question on finding a relationship between the factor-group $G/\zeta_\infty(G)$ and the hypercentral residual of G (the intersection of all normal subgroups N of G such that G/N is hypercentral) appears to be very natural. More specifically, *is there a function (depending on the order of $G/\zeta_\infty(G)$) that bounds the order of the hypercentral residual of G ?* In this note we show that Theorem A can be significantly improved. We prove that the order of the hypercentral residual of G is bounded by a function of the order of $G/\zeta_\infty(G)$, and moreover we are able to give an explicit form of this function. Thus the main result of the current note is the following.

Theorem B. *Let G be a group and let Z be the upper hypercenter of G . Suppose that G/Z is finite and put $|G/Z| = t$. Then G has a finite normal subgroup L such that G/L is hypercentral. Moreover, $|L| \leq t^k$, where $k = \frac{1}{2}(\log_p t + 1)$ and p is the least prime divisor of t .*

2. A SHORT PROOF OF THEOREM A

The proof makes use of an auxiliary result by N. S. Hekster [5, Lemma 2.4].

Lemma 2.1 ([5]). *Let G be a group, K a subgroup of G , and suppose that $G = K\zeta_n(G)$ for some positive integer n . Then the following properties hold:*

- (1) $\gamma_{n+1}(G) = \gamma_{n+1}(K)$.
- (2) $\zeta_n(K) = K \cap \zeta_n(G)$.
- (3) $\gamma_{n+1}(G) \cap \zeta_n(G) = \gamma_{n+1}(K) \cap \zeta_n(K)$.

Proof of Theorem A. We note that if $zl(G)$ is finite, the result follows from Baer's theorem [1]. Therefore we may suppose that $zl(G)$ is infinite. Let K be a finitely

generated subgroup with the property $G = ZK$. We have that K is nilpotent-by-finite (see [7, Proposition 3.19] for example). Since G/Z is not nilpotent, neither is K . Set $r = zl(K)$ and let C be the upper hypercenter of K . We claim that $C = C \cap Z$. Otherwise $CZ/Z \neq \langle 1 \rangle$, which means that the upper hypercenter of G/Z is not identic, a contradiction. Then $C = C \cap Z$ as claimed. By Baer's theorem [1], $\gamma_{r+1}(K)$ is finite. It follows that the nilpotent residual L of K is finite.

We now consider the local system $\mathcal{L}$ consisting of all finitely generated subgroups of G that contain K . Pick $V \in \mathcal{L}$ and let C_V be the upper hypercenter of V . Clearly we have $G = ZV$, and then $C_V = V \cap Z$. Since $V \leq KZ$ and $K \leq V$, we have $V = K(V \cap Z) = KC_V$. Put $n = zl(V)$. Since $V = KC_V$, we have that $\gamma_{n+1}(V) = \gamma_{n+1}(K)$ by Lemma 2.1. In particular, $\gamma_{n+1}(K)$ is normal in V . Since L is a characteristic subgroup of $\gamma_{n+1}(K)$, L is normal in V . Since this holds for each $V \in \mathcal{L}$, L is normal in $G = \bigcup_{V \in \mathcal{L}} V$. We have

$$\begin{aligned} G/ZL &\cong (G/L)(ZL/L) = (KZ/L)/(ZL/L) = (K/L)(ZL/L)/(ZL/L) \\ &\cong (K/L)/((K/L) \cap (ZL/L)). \end{aligned}$$

Since K/L is nilpotent, so is G/ZL . Since the hypercenter of G/L includes ZL/L , G/L has to be hypercentral. $\square$

3. PROOF OF THEOREM B

Let G be a group, R a ring and A an RG -module. We construct the upper RG -central series of A as the ascending chain of submodules

$$\{0\} = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \cdots A_\lambda,$$

where $A_1 = \zeta_{RG}(A) = \{a \in A \mid a(g-1) = 0\}$, $A_{\alpha+1}/A_\alpha = \zeta_{RG}(A/A_\alpha)$ for every ordinal $\alpha < \lambda$ and $\zeta_{RG}(A/A_\lambda) = \{0\}$. The last term A_λ of this series is called the upper RG -hypercenter of A and will be denoted by $\zeta_{RG}^\infty(A)$. If $A = A_\lambda$, then A is said to be RG -hypercentral. Moreover, if λ is finite, then A is said to be RG -nilpotent.

Let $B \leq C$ be RG -submodules of A . The factor C/B is called G -eccentric if $C_G(C/B) \neq G$. An RG -submodule C of A is said to be RG -hypereccentric if it has an ascending series of RG -submodules

$$\{0\} = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots \leq C_\lambda = C$$

such that every factor $C_{\alpha+1}/C_\alpha$ is a G -eccentric simple RG -module.

It is said that the RG -module A has the Z -decomposition if we can express

$$A = \zeta_{RG}^\infty(A) \oplus E_{RG}^\infty(A),$$

where $E_{RG}^\infty(A)$ is the maximal RG -hypereccentric RG -submodule of A (Zaitsev [11]). We note that if A has the Z -decomposition, then $E_{RG}^\infty(A)$ includes every RG -hypereccentric RG -submodule and, in particular, it is unique. Indeed, put $E = E_{RG}^\infty(A)$ and let B be a RG -hypereccentric RG -submodule of A . If $(B+E)/E$ is non-zero, then it has a non-zero simple RG -submodule U/E , say. Since $(B+E)/E \cong B/(B \cap E)$, U/E is RG -isomorphic to some simple RG -factor of B and then $G \neq C_G(U/E)$. But $(B+E)/E \leq A/E \cong \zeta_{RG}^\infty(A)$ and then $G = C_G(U/E)$, a contradiction that shows $B \leq E$. Hence E contains the RG -hypereccentric RG -submodules of A .

Lemma 3.1. *Let G be a finite nilpotent group and let A be a $\mathbb{Z}G$ -module. Suppose that the additive group of A is periodic. Then A has the Z -decomposition.*

Proof. Since G is finite, A has a local family $\mathcal{L}$ consisting of finite $\mathbb{Z}G$ -submodules. If $B \in \mathcal{L}$, applying the results of [11], B has the Z -decomposition. Now pick $C \in \mathcal{L}$ such that $B \leq C$. Then we have

$$B = \zeta_{\mathbb{Z}G}^\infty(B) \oplus E_{\mathbb{Z}G}^\infty(B), \quad C = \zeta_{\mathbb{Z}G}^\infty(C) \oplus E_{\mathbb{Z}G}^\infty(C).$$

Clearly $\zeta_{\mathbb{Z}G}^\infty(B) \leq \zeta_{\mathbb{Z}G}^\infty(C)$ and, since $E_{\mathbb{Z}G}^\infty(C)$ includes every $\mathbb{Z}G$ -hypercenric $\mathbb{Z}G$ -submodule, $E_{\mathbb{Z}G}^\infty(B) \leq E_{\mathbb{Z}G}^\infty(C)$. It follows that

$$\zeta_{\mathbb{Z}G}^\infty(A) = \bigcup_{B \in \mathcal{L}} \zeta_{\mathbb{Z}G}^\infty(B), \quad E_{\mathbb{Z}G}^\infty(A) = \bigcup_{B \in \mathcal{L}} E_{\mathbb{Z}G}^\infty(B).$$

Therefore $A = \zeta_{\mathbb{Z}G}^\infty(A) \oplus E_{\mathbb{Z}G}^\infty(A)$. $\square$

Lemma 3.2. *Let G be a finite group and Z a G -invariant subgroup of the hypercenter of G . Put $|G/Z| = t$. Then there exists a function f_1 such that the nilpotent residual of G has order at most $f_1(t)$.*

Proof. The subgroup Z has a series of G -invariant subgroups

$$\langle 1 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_n \leq Z_{n+1} = Z$$

whose factors Z_{j+1}/Z_j are G -central. Applying a result due to L. A. Kaloujnine [6], the factor-group $G/C_G(Z)$ is nilpotent. Put $C = C_G(Z)$ so that $Z \leq C_G(C)$. In particular, $|G/C_G(C)| \leq t$. Clearly $C \cap Z \leq \zeta(C)$, and so $C/(Z \cap C) \cong CZ/Z$ is a finite group of order at most t . By Wiegold's theorem [9], the derived subgroup $D = [C, C]$ has order at most $w(t)$. We note that D is G -invariant and C/D is abelian. By the facts proved above, the factor-group $(G/D)/C_{G/D}(C/D)$ is nilpotent. By Lemma 3.1, the $\mathbb{Z}G$ -module C/D has the Z -decomposition, that is, $C/D = \zeta_{\mathbb{Z}G}^\infty(C/D) \oplus E_{\mathbb{Z}G}^\infty(C/D)$. Clearly, $(C \cap Z)D/D \leq \zeta_{\mathbb{Z}G}^\infty(C/D)$, and then $L/D = E_{\mathbb{Z}G}^\infty(C/D)$ has order at most t . Hence $(C/D)/(L/D)$ is $\mathbb{Z}G$ -hypercentral. In other words, the hypercenter of G/L contains C/L . Since G/C is nilpotent, so is G/L . Finally, $|L| = |D||L/D| \leq tw(t) = tt^m = t^{m+1}$, where $m = \frac{1}{2}(\log_p t - 1)$ and p is the least prime divisor of t , so that $m+1 = \frac{1}{2}(\log_p t - 1) + 1 = \frac{1}{2}(\log_p t + 1)$. Therefore, it suffices to put $f_1(t) = t^k$, where $k = \frac{1}{2}(\log_p t + 1)$ and p is the least prime divisor of t . $\square$

If G is a group, we denote by $Tor(G)$ the maximal periodic normal subgroup of G . $Tor(G)$ is a characteristic subgroup of G and, if G is locally nilpotent, $G/Tor(G)$ is torsion-free.

Lemma 3.3. *Let G be a finitely generated group and Z a G -invariant subgroup of the hypercenter of G . Suppose that $|G/Z| = t$ is finite. Then G has a finite normal subgroup L such that G/L is nilpotent. Moreover, $|L| \leq f_1(t)$.*

Proof. Since G/Z is finite, Z is finitely generated. It follows that Z is nilpotent. Moreover, $z_l(G)$ is finite. By Baer's theorem [1], G has a finite normal subgroup F such that G/F is nilpotent. Being finitely generated, G/F has a finite periodic part $Tor(G/F) = K/F$. As we remarked above, the factor-group $(G/F)/(K/F) \cong G/K = B$ is torsion-free and nilpotent. We have that the subgroup Z is nilpotent and $T = Tor(G)$ is finite. Then Z has a torsion-free normal subgroup U such that

the orders of the elements of Z/U are the divisors of some positive integer k (see [4, Proposition 2] for example). Put $V = Z^k$ so that $V \leq U$ and V is also torsion-free. By construction, V is G -invariant and G/V is periodic. Being finitely generated nilpotent-by-finite, $C = G/V$ is finite. By Lemma 3.2, the nilpotent residual D of C has order at most $f_1(t)$.

Clearly $V \cap K = \langle 1 \rangle$. Applying Remak's theorem, we obtain an embedding $G \leq G/V \times G/K = C \times B = H$. Since B is torsion-free nilpotent, the nilpotent residual of H is exactly D . It follows that $G/(G \cap D) \cong GD/D \leq H/D$ is nilpotent. This shows that $G \cap D$ includes the nilpotent residual L of G . In particular, L is finite and, moreover, $|L| \leq |G \leq D| \leq |D| \leq f_1(t)$. $\square$

We are now in a position to show the main result of this paper.

Proof of Theorem B. Since G/Z is finite, there exists a finitely generated subgroup K such that $G = KZ$. We pick the family Σ of all finitely generated subgroups of G that contains K . Clearly G is FC -hypercentral, and then every finitely generated subgroup of G is nilpotent-by-finite (see [7, Proposition 3.19] for example). If $U \in \Sigma$, then the hypercenter of U includes a U -invariant subgroup $U \cap Z = Z_U$ such that U/Z_U is nilpotent and has order at most t . By Lemma 3.3, U has a finite normal subgroup H_U such that U/H_U is nilpotent and $|H_U| \leq f_1(t)$. Being finite-by-nilpotent, the nilpotent residual L_U of U is finite and L_U has order at most $f_1(t)$.

Pick $Y \in \Sigma$ such that $|L_Y|$ is maximal and let Σ_1 be the family of all finitely generated subgroups of G that contains Y . Pick $U \in \Sigma_1$. Then $Y \leq U$. The factor-group U/L_U is nilpotent and, since $Y/(Y \cap L_U) \cong YL_U/L_U \leq U/L_U$, $Y/(Y \cap L_U)$ is nilpotent. It follows that $L_Y \leq Y \cap L_U$ and then $L_Y \leq L_U$. But $|L_Y|$ is maximal, so that $L_U = L_Y$. In particular, L_Y is normal in U for every $U \in \Sigma_1$. Then L_Y is normal in $\bigcup_{U \in \Sigma_1} U = G$ and U/L_Y is nilpotent. Thus G/L_Y has a local family of nilpotent subgroups; that is, G/L_Y is locally nilpotent. Then $(G/L_Y)/(ZL_Y/L_Y)$ is nilpotent since it is finite. It follows that G/L_Y is hypercentral, because the upper hypercenter of G/L_Y includes ZL_Y/L_Y . $\square$

REFERENCES

- [1] R. Baer, *Endlichkeitskriterien für Kommutatorgruppen*, Math. Ann. **124** (1952), 161-177. MR0045720 (13:622a)
- [2] M. De Falco, F. de Giovanni, C. Musella, Ya. P. Sysak, *On the upper central series of infinite groups*, Proc. Amer. Math. Soc. **139** (2011), 385-389. MR2736322 (2011j:20063)
- [3] S. Franciosi, F. de Giovanni, L. A. Kurdachenko, *The Schur property and groups with uniform conjugacy classes*, J. Algebra **174** (1995), 823-847. MR1337173 (96e:20054)
- [4] H. Heineken, L. A. Kurdachenko, *Groups with subnormality for all subgroups that are not finitely generated*, Ann. Mat. Pura Appl. **169** (1995), 203-232. MR1378475 (97a:20048)
- [5] N. S. Hekster, *On the structure of n -isoclinism classes of groups*, J. Pure Appl. Algebra **40** (1986), 63-85. MR825181 (87k:20063)
- [6] L. A. Kaloujnine, *Über gewisse Beziehungen zwischen eine Gruppe und ihren Automorphismen*, Bericht über die Mathematiker-Tagung in Berlin, 1953, 164-172. MR0073599 (17:456f)
- [7] L. A. Kurdachenko, J. Otal, I. Ya. Subbotin, *Artinian modules over group rings*, Birkhäuser, Basel, 2007. MR2270897 (2007m:20006)
- [8] I. Schur, *Über die Darstellungen der endlichen Gruppen durch gebrochene lineare substitutionen*, J. reine angew. Math. **127** (1904), 20-50.

- [9] J. Wiegold, *Groups with boundedly finite classes of conjugate elements*, Proc. Roy. Soc. London, Series A **238** (1957), 389-401. MR0083488 (18:716a)
- [10] J. Wiegold, *Multiplicators and groups with finite central factor-groups*, Math. Z. **89** (1965), 345-347. MR0179262 (31:3510)
- [11] D. I. Zaitsev, *The hypercyclic extensions of abelian groups*, in *The groups defined by the properties of systems of subgroups*, Math. Inst. Kiev, 1979, 16-37. MR573739 (81i:20037)

DEPARTMENT OF ALGEBRA, NATIONAL UNIVERSITY OF DNEPROPETROVSK, VUL. NAUKOVA 13, DNEPROPETROVSK 50, UKRAINE 49050

Current address: Department of Algebra, School of Mathematics and Mechanics, National University of Dnepropetrovsk, Gagarin Prospect 72, Dnepropetrovsk 10, 49010 Ukraine

E-mail address: `lkurdachenko@gmail.com`

DEPARTMENT OF MATHEMATICS - IUUA, UNIVERSITY OF ZARAGOZA, PEDRO CERBUNA 12, 50009 ZARAGOZA, SPAIN

E-mail address: `otal@unizar.es`

DEPARTMENT OF MATHEMATICS AND NATURAL SCIENCES, NATIONAL UNIVERSITY, 5245 PACIFIC CONCOURSE DRIVE, LOS ANGELES, CALIFORNIA 90045

E-mail address: `isubboti@nu.edu`