ON THE WEIGHTED FORWARD REDUCED VOLUME OF RICCI FLOW

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ABSTRACT. In this paper, we introduce the weighted forward reduced volume of Ricci flow. The weighted forward reduced volume, which is related to expanders of Ricci flow, is well-defined on noncompact manifolds and monotone nonincreasing under Ricci flow. Moreover, we show that, just the same as Perelman's reduced volume, the weighted reduced volume entropy has the value $(4\pi)^{\frac{n}{2}}$ if and only if the Ricci flow is the trivial flow on flat Euclidean space.

1. INTRODUCTION

In [11], G. Perelman introduced the reduced entropy (i.e. reduced distance and reduced volume), which becomes one of the powerful tools for studying Ricci flow. The reduced entropy enjoys very nice analytic and geometric properties, including in particular the monotonicity of the reduced volume. These properties can be used, as demonstrated by Perelman, to show that the limit of the suitable rescaled Ricci flows is a gradient shrinking soliton.

Then M. Feldman, T. Ilmanen, and L. Ni [3] made an interesting observation that there is a dual version of Perelman's reduced entropy which is related to the expanders of Ricci flow. Let $g(t)$ solve the Ricci flow

$$(1.1) \quad \frac{\partial g}{\partial t} = -2Rc$$

on $M \times [0, T]$. Fix $x \in M^n$ and let γ be a path $(x(\eta), \eta)$ joining $(x, 0)$ and (y, t) . They define the forward $\mathcal{L}_+$ -length as

$$(1.2) \quad \mathcal{L}_+(\gamma) = \int_0^t \sqrt{\eta}(R(\gamma(\eta)) + |\gamma'(\eta)|^2)d\eta.$$

Denote $L_+(y, t)$ to be the length of a shortest forward $\mathcal{L}_+$ -length joining $(x, 0)$ and (y, t) . Let

$$(1.3) \quad l_+(y, t) = \frac{L_+(y, t)}{2\sqrt{t}}$$

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be the forward l_+ -length. Note that the forward reduced distance (1.3) is defined under the forward Ricci flow (1.1), which is the only difference from Perelman's reduced distance defined under the backward Ricci flow. The forward reduced volume is defined in [3] as

$$(1.4) \quad \theta_+(t) = \int_M (t)^{-\frac{n}{2}} e^{l_+(y,t)} d\text{vol}(y).$$

They also proved that the forward reduced volume defined in (1.4) is monotone nonincreasing along the Ricci flow (1.1).

Note that the forward reduced volume defined in (1.4) may not be well-defined on noncompact manifolds (see [3]). In this paper, we introduce the weighted forward reduced volume based on the work in [3] and [11]. We shall prove that the weighted forward reduced volume, which is related to the expanders of the Ricci flow, is well-defined on noncompact manifolds and monotone nonincreasing under the Ricci flow (1.1).

We define the weighted forward reduced volume as follows. First, we define the forward $\mathcal{L}_+$ -exponential map $\mathcal{L}_+ \exp(V, t) : T_x M \rightarrow M$ at time $t \in [0, T)$ as follows. For $V \in T_x M$, let γ_V denote the $\mathcal{L}_+$ -geodesic such that $\gamma_V(0) = p$, $\lim_{t \rightarrow 0} \sqrt{t} \gamma'_V(t) = V$. If γ_V exists on $[0, t]$, we set

$$(1.5) \quad \mathcal{L}_+ \exp(V, t) = \gamma_V(t).$$

Denote τ_V to be the first time the $\mathcal{L}_+$ -geodesic γ_V stops minimizing. Define

$$\Omega(t) = \{V \in T_x M^n : \tau_V > t\}.$$

Obviously, $\Omega(t_2) \subset \Omega(t_1)$ if $t_1 < t_2$. Let $J_i^V(t), i = 1, \dots, n$, be $\mathcal{L}_+$ -Jacobi fields along $\gamma_V(t)$ with

$$(1.6) \quad J_i^V(0) = 0, (\nabla_V J_i^V)(0) = E_i^0,$$

where $\{E_i^0\}_{i=1}^n$ is an orthonormal basis for $T_x M$ with respect to $g(0)$. Then $D(\mathcal{L}_+ \exp(V, t))(E_i^0) = J_i^V(t)$. We define

$$(1.7) \quad \mathcal{L}_+ J_V(t) = \sqrt{\det(\langle J_i^V(t), J_j^V(t) \rangle)}$$

and the weighted forward reduced volume as

$$(1.8) \quad \tilde{\mathcal{V}}_+(t) = \int_{\Omega(t)} t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} e^{-2|V|_{g(0)}^2} \mathcal{L}_+ J_V(t) dx_{g(0)}(V),$$

where $dx_{g(0)}$ is the standard Euclidean volume form on $(T_x M, g(x, 0))$; i.e. we define the weighted forward reduced volume as

$$(1.9) \quad \tilde{\mathcal{V}}_+(t) = \int_M t^{-\frac{n}{2}} e^{l_+(y,t)} e^{-2|\mathcal{L}_+ \exp^{-1}(y,t)|_{g(0)}^2} d\text{vol}(y),$$

where we set $V = \mathcal{L}_+ \exp^{-1}(y, t)$. We use the convention

$$\mathcal{L}_+ J_V(t) \doteq 0 \text{ for } t \geq \tau_V.$$

Then we can write the weighted forward reduced volume as

$$(1.10) \quad \tilde{\mathcal{V}}_+(t) = \int_{T_x M^n} t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) e^{-2|V|_{g(0)}^2} dx_{g(0)}(V).$$

We remark that the density of the forward reduced volume (1.4) (i.e. $(t)^{-\frac{n}{2}} e^{l_+(y,t)} d\text{vol}(y)$) is not pointwise monotone nonincreasing under the Ricci flow

(1.1). So it is not easy for us to add the weighted term to the forward reduced volume such that it could be defined on noncompact manifolds. In order to overcome this problem, we employ Perelman's technique [11] where we pull the density of the forward reduced volume back to the tangent space by the $\mathcal{L}_+$ -exponential map (here we define the $\mathcal{L}_+$ -exponential map in a similar way to [11]). Then we prove that the forward reduced volume density

$$d\mathcal{V}_+ = t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) dx_{g(0)}(V)$$

is pointwise monotone nonincreasing under the Ricci flow (1.1) with respect to $V \in T_x M$ (see Theorem 2.3). Notice that (see the proof of Theorem 1.1)

$$\lim_{t \rightarrow 0^+} t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) = 2^n e^{|V|_{g(0)}^2}.$$

So we only need to add the weighted term $e^{-2|V|_{g(0)}^2}$ to the density of the forward reduced volume, which guarantees that the weighted forward reduced volume we defined in (1.8) is well-defined on noncompact manifolds at $t = 0$. Moreover, the weighted forward reduced volume is monotone nonincreasing under the Ricci flow (1.1) since we have

$$\frac{d}{dt} (t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) e^{-2|V|_{g(0)}^2}) = e^{-2|V|_{g(0)}^2} \frac{d}{dt} (t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t)) \leq 0$$

for $V \in \Omega(t)$.

We have the following exact properties for the weighted forward reduced volume.

Theorem 1.1. *The weighted forward reduced volume defined in (1.8) is monotone non-increasing under the Ricci flow (1.1) and $\tilde{\mathcal{V}}_+(t) \leq \lim_{t \rightarrow 0^+} \tilde{\mathcal{V}}_+(t) \leq (4\pi)^{\frac{n}{2}}$. If $\tilde{\mathcal{V}}_+(t_1) = \tilde{\mathcal{V}}_+(t_2)$ for some $0 < t_1 < t_2$, then this flow is a gradient expanding soliton on $0 \leq t < \infty$ and hence is the trivial flow on flat Euclidean space. In particular, if $\tilde{\mathcal{V}}_+(\bar{t}) = (4\pi)^{\frac{n}{2}}$ for some time $\bar{t} > 0$, then this flow is the trivial flow on flat Euclidean space.*

We also have the following rescaling property for the weighted forward reduced volume.

Theorem 1.2. *We have $\tilde{\mathcal{V}}_+^j(t) = \tilde{\mathcal{V}}_+(\lambda_j^{-1}t)$ under the rescaling $g_j(t) = \lambda_j g(\lambda_j^{-1}t)$, where $\tilde{\mathcal{V}}_+^j$ and $\tilde{\mathcal{V}}_+$ denote the weighted forward reduced volume with respect to the metrics g_j and g respectively.*

The organization of the paper is as follows. In section 2, we first recall some basic formulas and properties about forward reduced entropy in [3]. Then we study the properties of forward reduced volume density which are defined by the forward $\mathcal{L}_+$ -exponential map. Finally, we give the proofs of Theorem 1.1 and Theorem 1.2. In the appendix of this paper, we give the proof of Theorem 2.1 and Theorem 2.2, which are proved in [3], for the sake of completeness.

2. WEIGHTED FORWARD REDUCED VOLUME AND EXPANDERS

Before we present the proofs of Theorem 1.1 and Theorem 1.2, we recall some basic formulas and properties about forward reduced entropy found in [3]. Clearly, one can show that the l_+ -length $l_+(y, t)$ is a locally Lipschitz function and the cut locus of $\mathcal{L}_+ \exp(V, t)$ is a closed set of measure zero by using methods similar to those in [14].

We need the following two theorems due to Feldman, Ilmanen, and Ni [3], which state the following adapted form. We will give the proof of Theorem 2.1 and Theorem 2.2 in the appendix of this paper for the sake of completeness.

Theorem 2.1 ([3]). *Let γ be a path $(x(\eta), \eta)$ joining $(x, 0)$ and (y, t) . Denote $L_+ \doteq L_+(y, t)$ to be the forward $\mathcal{L}_+$ -length joining $(x, 0)$ and (y, t) . Set $X = \gamma'(t)$ and let Y be a variational vector along γ such that $Y(0) = 0$. The first variation of $\mathcal{L}_+$ is that*

$$(2.1) \quad \delta \mathcal{L}_+ = 2\sqrt{t}\langle X, Y \rangle(t) + \int_0^t \sqrt{\eta} \langle Y, \nabla R - 2\nabla_X X + 4Rc(X, \cdot) - \frac{1}{\eta}X \rangle d\eta.$$

If $\gamma(t)$ is the minimal $\mathcal{L}_+$ -geodesic, then

$$(2.2) \quad \nabla L_+ = 2\sqrt{t}X,$$

$$(2.3) \quad t^{\frac{3}{2}}(R + |X|^2) = K + \frac{1}{2}L_+,$$

where $K = \int_0^t \eta^{\frac{3}{2}} H(X) d\eta$, and $H(X) = \frac{\partial R}{\partial t} + 2\langle \nabla R, X \rangle + 2Rc(X, X) + \frac{R}{t}$. The second variation of $\mathcal{L}_+$ is that

$$(2.4) \quad \begin{aligned} \delta_Y^2 \mathcal{L}_+ = & 2\sqrt{t}\langle X, Y \rangle(t) + \int_0^t \sqrt{\eta} (Hess R(Y, Y) - 2R(X, Y, X, Y) \\ & + 2|\nabla_X Y|^2 + 4\nabla_Y Rc(Y, X) - 2\nabla_X Rc(Y, Y)) d\eta. \end{aligned}$$

Let $\tilde{Y}$ be a vector field along γ that satisfies the ODE

$$(2.5) \quad \begin{cases} \nabla_X \tilde{Y}(\eta) = Rc(\tilde{Y}(\eta), \cdot) + \frac{1}{2\eta} \tilde{Y}(\eta), \eta \in [0, t], \\ \tilde{Y}(0) = Y(0) = 0. \end{cases}$$

Then

$$(2.6) \quad Hess L_+(\tilde{Y}, \tilde{Y}) \leq \frac{|\tilde{Y}|^2}{\sqrt{t}} + 2\sqrt{t}Rc(\tilde{Y}, \tilde{Y}) - \int_0^t \sqrt{\eta} H(X, \tilde{Y}) d\eta,$$

where $H(X, \tilde{Y}) = -Hess R(\tilde{Y}, \tilde{Y}) + 2R(X, \tilde{Y}, X, \tilde{Y}) + 2|Rc(X, \tilde{Y})|^2 + \frac{Rc(\tilde{Y}, \tilde{Y})}{t} + 2\frac{\partial Rc}{\partial t}(\tilde{Y}, \tilde{Y}) - 4\nabla_{\tilde{Y}} Rc(\tilde{Y}, X) + 4\nabla_X Rc(\tilde{Y}, \tilde{Y})$. The equality holds in (2.6) if and only if the vector field $\tilde{Y}$ satisfying (2.5) is an $\mathcal{L}_+$ -Jacobi field.

Theorem 2.2 ([3]). *Let $l_+ \doteq l_+(y, t) = \frac{L_+(y, t)}{2\sqrt{t}}$ be the l_+ -length from $(x, 0)$ to (y, t) . If (y, t) is not in the cut locus of $\mathcal{L}_+exp$, then at (y, t)*

$$(2.7) \quad \frac{\partial l_+}{\partial t} = R - \frac{l_+}{t} - \frac{K}{2t^{\frac{3}{2}}},$$

$$(2.8) \quad |\nabla l_+|^2 = \frac{l_+}{t} - R + \frac{K}{t^{\frac{3}{2}}},$$

$$(2.9) \quad \Delta l_+ \leq R + \frac{n}{2t} - \frac{K}{2t^{\frac{3}{2}}},$$

$$(2.10) \quad \frac{\partial l_+}{\partial t} + \Delta l_+ + |\nabla l_+|^2 - R - \frac{n}{2t} \leq 0,$$

$$(2.11) \quad 2\Delta l_+ + |\nabla l_+|^2 - R - \frac{l_+ + n}{t} \leq 0.$$

We first study properties of the forward reduced volume density defined as

$$(2.12) \quad d\mathcal{V}_+ = t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) dx_{g(0)}(V),$$

where $\mathcal{L}_+ J_V(t)$ is defined in (1.7).

Note that the weighted forward reduced volume is

$$\tilde{\mathcal{V}}_+(t) = \int_{T_x M^n} e^{-2|V|_{g(0)}^2} d\mathcal{V}_+.$$

Analogously to [11], we have the following theorem.

Theorem 2.3. *The forward reduced volume density $d\mathcal{V}_+$ defined in (2.12) is monotone nonincreasing along the Ricci flow (1.1). Moreover, if $d\mathcal{V}_+(t_1) = d\mathcal{V}_+(t_2)$ for some $0 < t_1 < t_2$, then this flow is a gradient expanding soliton.*

Proof. Let $\gamma_V(t)$ be the minimal $\mathcal{L}_+$ -geodesic defined in (1.5) and $y = \gamma_V(t)$. We consider (y, t) in the cut locus of $\mathcal{L}_+ \exp(V, t)$. Recall that $\nabla l_+(y, t) = \gamma'_V(t) = X(t)$. Then by (2.3) and (2.7), we get

$$\begin{aligned} \frac{dl_+(\gamma_V(t), t)}{dt} &= \frac{\partial l_+(y, t)}{\partial t} + \nabla l \cdot X \\ &= R - \frac{l_+(y, t)}{t} - \frac{K}{2t^{\frac{3}{2}}} + |X|^2 \\ (2.13) \quad &= \frac{1}{2} t^{-\frac{3}{2}} K. \end{aligned}$$

For any fixed t , we choose an orthonormal basis $\{E_i(t)\}$ of $T_{\gamma_V(t)}M$. We extend $E_i(\eta)$, $\eta \in [0, t]$, to an $\mathcal{L}_+$ -Jacobi field along γ_V with $E_i(0) = 0$. We write $J_i^V(t) = \sum_i^n A_i^j E_j(t)$ for some matrix $(A_i^j) \in GL(n, \mathbb{R})$. Then $J_i^V(\eta) = \sum_i^n A_i^j E_j(\eta)$ for all $\eta \in [0, t]$.

Hence, by (2.6), we calculate at time t :

$$\begin{aligned} \frac{d}{d\eta}|_{\eta=t} \ln \mathcal{L}_+ J_V &= \frac{d}{d\eta}|_{\eta=t} \ln \sqrt{\det(\langle \sum_{k=1}^n A_i^k E_k, \sum_{l=1}^n A_j^l E_l \rangle)} \\ &= \frac{1}{2} \frac{d}{d\eta}|_{\eta=t} \sum_i |E_i|^2 \\ &= \sum_i (-Rc(E_i, E_i) + \langle \nabla_{E_i} X, E_i \rangle) \\ (2.14) \quad &= \sum_i (-Rc(E_i, E_i) + \frac{1}{2\sqrt{t}} \text{Hess} L_+(E_i, E_i)) \end{aligned}$$

$$(2.15) \quad \leq \sum_i \left(\frac{1}{2t} - \frac{1}{2\sqrt{t}} \int_0^t \sqrt{\eta} H(X, \tilde{E}_i) d\eta \right),$$

where $\tilde{E}_i(\eta)$ are the vector fields along γ_V satisfying

$$(2.16) \quad \begin{cases} \nabla_X \tilde{E}_i(\eta) = Rc(\tilde{E}_i(\eta), \cdot) + \frac{1}{2\eta} \tilde{E}_i(\eta), \eta \in [0, t], \\ \tilde{E}_i(t) = E_i(t), \end{cases}$$

which in particular implies that

$$(2.17) \quad \langle \tilde{E}_i, \tilde{E}_j \rangle(\eta) = \frac{\eta}{t} \langle E_i, E_j \rangle(t) = \frac{\eta}{t} \delta_{ij}.$$

It follows that

$$H(X, \tilde{E}_i)(\eta) = \frac{\eta}{t} H(x).$$

Hence

$$\frac{d}{d\eta} \Big|_{\eta=t} \ln \mathcal{L}_+ J_V \leq \frac{n}{2t} - \frac{1}{2} t^{-\frac{3}{2}} K$$

and

$$(2.18) \quad \frac{d}{dt} \ln d\mathcal{V}_+ = -\frac{n}{2t} + \frac{\partial l_+}{\partial t} + \frac{d \ln \mathcal{L}_+ J_V}{dt} \leq 0.$$

If equality in (2.18) holds, then we have that equality in (2.15) holds. By Theorem 2.1, we conclude that each $\tilde{E}_i(\eta)$ is an $\mathcal{L}_+$ -Jacobi field. Hence

$$(2.19) \quad \frac{d}{d\eta} \Big|_{\eta=t} |E_i|^2 = \frac{d}{d\eta} \Big|_{\eta=t} |\tilde{E}_i|^2 = \frac{|E_i(t)|^2}{t}.$$

Combining this with (2.14) and (2.19), we get

$$Rc(E_i, E_i) - \frac{1}{2\sqrt{t}} \text{Hess} L_+(E_i, E_i) = -\frac{|E_i|^2}{2t}.$$

□

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 2.3, we know that

$$\frac{d}{dt} (t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t)) \leq 0$$

for $V \in \Omega(t)$. It follows that

$$\frac{d}{dt} (t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) e^{-2|V|_{g(0)}^2}) \leq 0$$

for $V \in \Omega(t)$. Moreover, since we have $\Omega(t_2) \subset \Omega(t_1)$,

$$\tilde{\mathcal{V}}_+(t_2) \leq \tilde{\mathcal{V}}_+(t_1)$$

for $t_1 < t_2$. We calculate that

$$\begin{aligned} \lim_{t \rightarrow 0+} l_+(\gamma_V(t), t) &= \lim_{t \rightarrow 0+} \frac{1}{2\sqrt{t}} \int_0^t \sqrt{\eta} (R(\gamma_V(\eta), \eta) + |\frac{d\gamma_V}{d\eta}|^2) d\eta \\ &= \lim_{t \rightarrow 0+} t (R(\gamma_V(t), t) + |\frac{d\gamma_V}{dt}|^2) \\ &= |V|_{g(0)}^2. \end{aligned}$$

Let $J_i^V(t), i = 1, \dots, n$, be $\mathcal{L}_+$ -Jacobi fields along $\gamma_V(t)$ with

$$(2.20) \quad J_i^V(0) = 0, (\nabla_V J_i^V)(0) = E_i^0,$$

where $\{E_i^0\}_{i=1}^n$ is an orthonormal basis for $T_x M$ with respect to $g(0)$. Since $(\nabla_V J_i^V)(0) = E_i^0$ and $V = \lim_{t \rightarrow 0} \sqrt{t} \gamma'_V(t)$, we get

$$\lim_{t \rightarrow 0^+} \frac{\mathcal{L}_+ J_V(t)}{t^{\frac{n}{2}}} = \lim_{t \rightarrow 0^+} \frac{\sqrt{\det(\langle 2\sqrt{t} E_i(t), 2\sqrt{t} E_j(t) \rangle_{g(0)})}}{t^{\frac{n}{2}}} = 2^n,$$

and we conclude that

$$\lim_{t \rightarrow 0^+} t^{-\frac{n}{2}} e^{l_+(\gamma_V(t), t)} \mathcal{L}_+ J_V(t) = 2^n e^{|V|_{g(0)}^2}.$$

Hence

$$\lim_{t \rightarrow 0^+} \tilde{\mathcal{V}}_+(t) \leq \int_{T_p M} 2^n e^{-|V|_{g(0)}^2} dx(V) = (4\pi)^{\frac{n}{2}}.$$

If $\tilde{\mathcal{V}}_+(t_1) = \tilde{\mathcal{V}}_+(t_2)$ for any $0 < t_1 < t_2$, then $d\mathcal{V}_+(t_1) = d\mathcal{V}_+(t_2)$ for any $0 < t_1 < t_2$. So $(M^n, g(t))$ must be a gradient expanding soliton by Theorem 2.3 and Theorem 3 in [4]; i.e. we have

$$Rc + Hess(-l_+) = -\frac{g}{2t}$$

for some smooth function l_+ on M^n . Let $\phi_t : M \rightarrow M, 0 < t \leq \bar{t}$ be the one-parameter family of diffeomorphisms obtained by

$$\frac{d\phi_t}{dt} = \nabla l_+ \quad \text{and} \quad \phi_{\bar{t}} = Id.$$

We consider $h(t) = \frac{\bar{t}}{t} \phi_t^* g(t)$ and calculate

$$\begin{aligned} \frac{dh}{dt} &= -\frac{\bar{t}}{t^2} \phi_t^* g(t) + \frac{\bar{t}}{t} \phi_t^* \mathcal{L}_{\frac{d\phi_t}{dt}}(g(t)) - 2\frac{\bar{t}}{t} \phi_t^* Rc(g(t)) \\ &= -\frac{\bar{t}}{t^2} \phi_t^* g(t) + \frac{\bar{t}}{t} 2Hess(l_+) + \frac{\bar{t}}{t} \phi_t^* \left(\frac{g}{t} - 2Hess(l_+)\right) = 0. \end{aligned}$$

It follows that

$$g(t) = \frac{t}{\bar{t}} (\phi_t^{-1})^* g(\bar{t}).$$

Suppose that there is some $(y, \bar{t})$ with $|Rm|(y, \bar{t}) = K > 0$. We have $|Rm|(\phi_t^{-1}(y), t) = \frac{K\bar{t}}{t}$, and these curvatures are not bounded as $t \rightarrow 0$, which is a contradiction. Then we have

$$Hess(l_+) = \frac{1}{2t} g.$$

Thus l_+ is a strictly convex function. Similar arguments to Lemma 2.3 in [14] can show that

$$l_+(y, t) \geq e^{-2ct} \frac{d_{g(0)}(x, y)}{4t} - \frac{nc}{3} t$$

if $Rc \geq -cg$ on $[0, t]$, so that $l_+(y, t)$ have only a minimum point in M^n . Hence M^n is diffeomorphic to $\mathbb{R}^n$.

Since $\tilde{\mathcal{V}}_+(t)$ is monotone nonincreasing, $\tilde{\mathcal{V}}_+(t)$ is independent of t if $\tilde{\mathcal{V}}_+(\bar{t}) = (4\pi)^{\frac{n}{2}}$ for some time $\bar{t} > 0$. Then we derive that $(M^n, g(\bar{t}))$ is isometric to $\mathbb{R}^n$. By Theorem 1 in [4], the Ricci flow $(M^n, g(t))_{t \in [0, T]}$ is in fact isometric to flat Euclidean space $\mathbb{R}^n$ for all $t \in [0, T]$. $\square$

Finally, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. We denote $\gamma_v^j(t)$ to be the minimal $\mathcal{L}_+$ -geodesic with respect to $g_j(t)$ which starts from $(x_j, 0)$ and satisfies $\lim_{t \rightarrow 0} \sqrt{t} \frac{d\gamma_v^j(t)}{dt} = V$.

We have $\gamma_{\sqrt{\lambda_j^{-1}V}}^j(t) = \gamma_V(\lambda_j^{-1}t)$, $l_+^j(y, t) = l_+(y, \lambda_j^{-1}t)$, and $\mathcal{L}_+ J_V^j(t) dx_{g_j(0)}(V) = (\lambda_j^{-1})^{-\frac{n}{2}} \mathcal{L}_+ J_{\sqrt{\lambda_j^{-1}V}}(\lambda_j^{-1}t) dx_{g(0)}(\sqrt{\lambda_j^{-1}V})$. Hence

$$\begin{aligned} \tilde{\mathcal{V}}_+^j(t) &= \int_{T_x M^n} (t)^{-\frac{n}{2}} e^{l_+^j(\gamma_V(t), t)} \mathcal{L}_+ J_V^j(t) e^{-2|V|_{g_j(0)}^2} dx_{g_j(0)}(V) \\ &= \int_{T_x M^n} (\lambda_j^{-1}t)^{-\frac{n}{2}} e^{l_+(\gamma_{\sqrt{\lambda_j^{-1}V}}(\lambda_j^{-1}t), \lambda_j^{-1}t)} \\ &\quad \times \mathcal{L}_+ J_{\sqrt{\lambda_j^{-1}V}}(\lambda_j^{-1}t) e^{-2|\sqrt{\lambda_j^{-1}V}|_{g(0)}^2} dx_{g(0)}(\sqrt{\lambda_j^{-1}V}) \\ &= \tilde{\mathcal{V}}_+(\lambda_j^{-1}t). \end{aligned}$$

Note that $\mathcal{L}_+^j \exp^{-1}(y, t) = \sqrt{\lambda_j^{-1}} \mathcal{L}_+ \exp^{-1}(y, \lambda_j^{-1}t)$. We can also prove Theorem 1.2 by using the form (1.9)

$$\begin{aligned} \tilde{\mathcal{V}}_+^j(t) &= \int_M t^{-\frac{n}{2}} e^{l_+^j(y, t)} e^{-2|\mathcal{L}_+^j \exp^{-1}(y, t)|_{g_j(0)}^2} d\text{vol}_{g_j(t)}(y) \\ &= \int_M (\lambda_j^{-1}t)^{-\frac{n}{2}} e^{l_+(y, \lambda_j^{-1}t)} e^{-2|\mathcal{L}_+ \exp^{-1}(y, \lambda_j^{-1}t)|_{g(0)}^2} d\text{vol}_{g(\lambda_j^{-1}t)}(y) \\ &= \tilde{\mathcal{V}}_+(\lambda_j^{-1}t). \end{aligned}$$

□

3. APPENDIX

In this section, we give the proofs of Theorem 2.1 and Theorem 2.2, which are proved in [3], for the sake of completeness.

Proof of Theorem 2.1 and Theorem 2.2. Note that (2.1) and (2.4) differ from the first and second variation of Perelman's $\mathcal{L}$ -length [11] only by the sign in front of Rc , since Perelman's $\mathcal{L}$ -length is defined under the backward Ricci flow (1.1). The calculations are almost the same as [11], and we omit the details here.

If $\gamma(t)$ is the minimal $\mathcal{L}_+$ -geodesic, it follows from (2.1) that

$$(3.1) \quad \nabla L_+ = 2\sqrt{t}X.$$

Now using the geodesic equation, we have

$$\begin{aligned} &\frac{d}{dt}(R(\gamma(t), t) + |X(t)|^2) \\ &= \frac{\partial}{\partial t} R + \nabla R \cdot X - 2Rc(X, X) + 2\langle \nabla_X X, X \rangle \\ &= \frac{\partial}{\partial t} R + 2\nabla R \cdot X + 2Rc(X, X) - \frac{1}{t}|X|^2. \end{aligned}$$

Hence

$$\frac{d}{dt}(R(\gamma(t), t) + |X(t)|^2) = H(x) - \frac{1}{t}(R + |X|^2),$$

where $H(X) = \frac{\partial R}{\partial t} + 2\nabla R \cdot X + 2Rc(X, X) + \frac{R}{t}$, and this gives (2.3) immediately.

Let $\tilde{Y}$ be a vector field along γ that satisfies the ODE

$$(3.2) \quad \begin{cases} \nabla_X \tilde{Y}(\eta) = Rc(\tilde{Y}(\eta), \cdot) + \frac{1}{2\eta} \tilde{Y}(\eta), \eta \in [0, t], \\ \tilde{Y}(0) = Y(0) = 0. \end{cases}$$

Then we have $|\tilde{Y}(\eta)|^2 = \frac{\eta}{t} |\tilde{Y}(t)|^2$. It follows that

$$(3.3) \quad \begin{aligned} Hess L_+(\tilde{Y}, \tilde{Y}) &\leq \frac{1}{\sqrt{t}} |\tilde{Y}(t)|^2 + \int_0^t \sqrt{\eta} (Hess R(\tilde{Y}, \tilde{Y}) - 2R(X, \tilde{Y}, X, \tilde{Y}) + 2|Rc(\cdot, \tilde{Y})|^2) \\ &+ \frac{2}{\eta} Rc(\tilde{Y}, Y) + (4\nabla_{\tilde{Y}} Rc(\tilde{Y}, X) - 2\nabla_X Rc(\tilde{Y}, \tilde{Y})) + 2\nabla_X Rc(\tilde{Y}, \tilde{Y}) d\eta. \end{aligned}$$

Since

$$\frac{d}{dt}(Rc(\tilde{Y}, \tilde{Y})) = \frac{\partial Rc}{\partial t}(\tilde{Y}, \tilde{Y}) + \nabla_X Rc(\tilde{Y}, \tilde{Y}) + 2|Rc(\cdot, \tilde{Y})|^2 + \frac{1}{t} Rc(\tilde{Y}, \tilde{Y}),$$

we get

$$(3.4) \quad 2 \int_0^t \sqrt{\eta} \nabla_X Rc(\tilde{Y}, \tilde{Y})$$

$$(3.5) \quad = 2\sqrt{t} Rc(\tilde{Y}, \tilde{Y}) - \int_0^t \sqrt{\eta} \left(3 \frac{Rc(\tilde{Y}, \tilde{Y})}{\eta} + 2 \frac{\partial Rc}{\partial t}(\tilde{Y}, \tilde{Y}) + 4|Rc(\cdot, \tilde{Y})|^2 \right) d\eta.$$

Substituting (3.4) into (3.3), we have

$$Hess L_+(\tilde{Y}, \tilde{Y}) \leq \frac{|\tilde{Y}|^2}{\sqrt{t}} + 2\sqrt{t} Rc(\tilde{Y}, \tilde{Y}) - \int_0^t \sqrt{\eta} H(X, \tilde{Y}) d\eta,$$

where $H(X, \tilde{Y}) = -Hess R(\tilde{Y}, \tilde{Y}) + 2R(X, \tilde{Y}, X, \tilde{Y}) + 2|Rc(\tilde{Y}, \cdot)|^2 + \frac{Rc(\tilde{Y}, \tilde{Y})}{t} + 2\frac{\partial Rc}{\partial t}(\tilde{Y}, \tilde{Y}) - 4\nabla_{\tilde{Y}} Rc(\tilde{Y}, X) + 4\nabla_X Rc(\tilde{Y}, \tilde{Y})$. The equality holds in (2.6) if and only if the vector field $\tilde{Y}$ satisfying (2.16) is an $\mathcal{L}_+$ -Jacobi field. Tracing (2.6) in Y yields

$$(3.6) \quad \Delta L_+ \leq \frac{n}{\sqrt{t}} + 2\sqrt{t} R - \frac{1}{t} K.$$

By (2.2), we have

$$(3.7) \quad |\nabla L_+|^2 = -4tR + 4t(R + |X|^2)$$

and

$$(3.8) \quad \begin{aligned} \frac{\partial L_+}{\partial t}(y, t) &= \frac{dL_+(\gamma(t), t)}{dt} - \nabla L_+ \cdot X \\ &= \sqrt{t}(R + |X|^2) - 2\sqrt{t}|X|^2 \\ &= 2\sqrt{t}R - \sqrt{t}(R + |X|^2). \end{aligned}$$

Substituting (2.3) into (3.7) and (3.8), we have

$$(3.9) \quad |\nabla L_+|^2 = -4tR + \frac{2}{\sqrt{t}} L_+ + \frac{4}{\sqrt{t}} K$$

and

$$(3.10) \quad \frac{\partial L_+}{\partial t} = 2\sqrt{t}R - \frac{1}{t}K - \frac{1}{2t}L_+.$$

Then (2.7)-(2.11) follow from (3.6), (3.9) and (3.10). $\square$

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