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INEQUIVALENT CANTOR SETS IN R^3 WHOSE COMPLEMENTS HAVE THE SAME FUNDAMENTAL GROUP

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ABSTRACT. For each Cantor set C in R^3, all points of which have bounded local genus, we show that there are infinitely many inequivalent Cantor sets in R^3 with the complement having the same fundamental group as the complement of C. This answers a question from *Open Problems in Topology* and has as an application a simple construction of nonhomeomorphic open 3-manifolds with the same fundamental group. The main techniques used are analysis of local genus of points of Cantor sets, a construction for producing rigid Cantor sets with simply connected complement, and manifold decomposition theory. The results presented give an argument that for certain groups G, there are uncountably many nonhomeomorphic open 3-manifolds with fundamental group G.

1. INTRODUCTION

The following question was asked in *Open Problems in Topology II* (see Question 14 in [GR07]):

Question 1.1. Can two different (rigid) Cantor sets have complements with the same fundamental group?

There are two parts to the question. First, are there any Cantor sets satisfying the condition? Second, are there rigid such Cantor sets? Answering for rigid Cantor sets seems more difficult. We focus our attention on answering these questions in R^3. The same techniques apply to embeddings in S^3. Here, *different* Cantor sets means Cantor sets that are inequivalently embedded in the ambient space. The following theorem from [GRŽ06], together with results on local genus, $g_x(X)$, of points x in a Cantor set X (see Section 3) give a positive answer to the above question when the fundamental group of the complement is trivial.

Theorem 1.2 ([GRŽ06]). *For each increasing sequence $S = (n_1, n_2, \ldots)$ of integers such that $n_1 > 2$, there exists a wild Cantor set, $X = C(S)$, in R^3 and a countable dense subset $A = \{a_1, a_2, \ldots\} \subset X$ such that the following conditions hold:*

(1) $g_x(X) \leq 2$ *for every* $x \in X \setminus A$,

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(2) *$g_{a_i}(X) = n_i$ for every $a_i \in A$, and*
(3) *$R^3 \setminus X$ is simply connected.*

Remark 1.3. For later reference, we note that the construction in [GRŽ06] actually works for any sequence $S = (n_1, n_2, \ldots)$ of integers where each $n_i \geq 3$. The condition that the sequence is increasing was used in this earlier paper to prove rigidity of the Cantor sets.

Since local genus is preserved by equivalence, any Cantor sets $X_1 = C(S_1)$ and $X_2 = C(S_2)$ corresponding to distinct increasing sequences S_1 and S_2 as above are inequivalent. Since there are uncountably many such sequences, there are uncountably many inequivalent Cantor sets with simply connected complement. Since all of these Cantor sets are rigid, this answers both of the above questions in the case when the complement is simply connected.

The more interesting case is when a Cantor set C in R^3 has nonsimply connected complement. Question 1.1 asks whether there is an inequivalent Cantor set D with complement having the same fundamental group. We answer this question affirmatively for Cantor sets of bounded local genus and for a large class of other Cantor sets. The essential ingredient is the class of Cantor sets provided by Theorem 1.2 with genus of points taking values in carefully chosen sequences of positive integers and with simply connected complement.

Theorem 1.4 (Main Theorem). *Let C be a Cantor set in R^3. Suppose there is some integer $N \geq 3$ such that there are only finitely many points in C of local genus N. Then there are uncountably many inequivalent Cantor sets C_α in R^3 with complement having the same fundamental group as the complement of C.*

Corollary 1.5 (Bounded genus). *Let C be a Cantor set in R^3 of bounded local genus, or more generally, a Cantor set where the local genus of points never takes on a specific integer value $N \geq 3$. Then there are uncountably many inequivalent Cantor sets C_α in R^3 with complement having the same fundamental group as the complement of C.*

Proof. If C is a Cantor set as described in the corollary, then there is an integer $N \geq 3$ as in the statement of Theorem 1.4. □

By considering the open 3-manifolds that are the complements of the Cantor sets in the above results, we are able to show in Section 7 that these open 3-manifolds have uncountably many associated nonhomeomorphic open 3-manifolds with the same fundamental group. In particular, we show:

Theorem 1.6. *Let M be an open 3-manifold with end point (Freudenthal) compactification M^* such that $M^* \cong S^3$ and such that $M^* \setminus M$ is a Cantor set C. If the Cantor set C has an associated integer $N \geq 3$ such that only finitely many points of C have local genus N, then there are uncountably many open 3-manifolds not homeomorphic to M that have the same fundamental group as M.*

2. Terminology

A subset $A \subset R^n$ is said to be *rigid* if whenever $f\colon R^n \to R^n$ is a homeomorphism with $f(A) = A$, it follows that $f|_A = id_A$. There are many examples in R^3 of wild Cantor sets that are either rigid or have simply connected complement. In [GRŽ06], examples were constructed having both properties.

See the bibliography for more results on embeddings of Cantor sets. In particular, see Kirkor [Ki58], DeGryse and Osborne [DO86], Ancel and Starbird [AS89], and Wright [Wr89] for further discussion of wild Cantor sets with simply connected complement.

Two Cantor sets X and Y in R^3 are said to be *topologically distinct* or *inequivalent* if there is no homeomorphism of R^3 to itself taking X to Y. Sher proved in [Sh68] that there exist uncountably many inequivalent Cantor sets in R^3. He showed that varying the number of components in the Antoine construction leads to these inequivalent Cantor sets.

A simple way of producing new Cantor sets in R^3 is to take the union of two disjoint Cantor sets in R^3. This leads to the following definition. A Cantor set in R^3 is said to be *splittable* into Cantor sets C_1 and C_2 if

- $C = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$ and
- there are disjoint tame closed 3-cells D_1 and D_2 in R^3 with $C_1 \subset D_1$ and $C_2 \subset D_2$.

Given a Cantor set X in R^3, we denote the fundamental group of its complement by $\pi_1(X^c)$. As in the following remark, Theorem 1.2 could be used to produce splittable examples of Cantor sets $X \cup C$ as the union of disjoint sets with $\pi_1(X^c) \simeq \pi_1((X \cup C)^c)$. The examples we produce in the present paper require a more careful construction and are not splittable in this fashion.

Remark 2.1. Note that if C is splittable into C_1 and C_2, then a Seifert-Van Kampen argument shows that $\pi_1(C^c) \simeq \pi_1(C_1^c) * \pi_1(C_2^c)$, where $*$ represents the free product. Thus if $\pi_1(C_1^c)$ is trivial, then $\pi_1(C^c) \simeq \pi_1(C_2^c)$.

3. Defining sequences and local genus

The following definitions about genus are from [Že05].

A *defining sequence* for a Cantor set $X \subset R^3$ is a sequence (M_i) of compact 3-manifolds with boundary such that

(a) each M_i consists of pairwise disjoint cubes with handles,
(b) $M_{i+1} \subset \operatorname{Int} M_i$ for each i, and
(c) $X = \bigcap_i M_i$.

Let $\mathcal{D}(X)$ be the set of all defining sequences for X. It is known (see [Ar66]) that every Cantor set in R^3 has a defining sequence, but the sequence is not uniquely determined. In fact, every Cantor set has many inequivalent (see [Sh68] for the definition) defining sequences.

Let M be a handlebody. We denote the genus of M by $g(M)$. For a disjoint union of handlebodies $M = \bigsqcup_{\lambda \in \Lambda} M_\lambda$, we define $g(M) = \sup\{g(M_\lambda);\ \lambda \in \Lambda\}$.

Let $(M_i) \in \mathcal{D}(X)$ be a defining sequence for a Cantor set $X \subset R^3$. For any subset $A \subset X$ we denote by M_i^A the union of those components of M_i which intersect A. Define

$$\begin{aligned} g_A(X; (M_i)) &= \sup\{g(M_i^A);\ i \geq 0\} \quad \text{and} \\ g_A(X) &= \inf\{g_A(X; (M_i));\ (M_i) \in \mathcal{D}(X)\}. \end{aligned}$$

The number $g_A(X)$ is either a nonnegative integer or ∞ and is called *the genus of the Cantor set X with respect to the subset A*. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$.

For $A = X$ we call the number $g_X(X)$ *the genus of the Cantor set* X and denote it by $g(X)$.

Let x be an arbitrary point of a Cantor set X and $h\colon R^3 \to R^3$ a homeomorphism. Then any defining sequence for X is mapped by h onto a defining sequence for $h(X)$. Hence the local genus $g_x(X)$ is the same as the local genus $g_{h(x)}(h(X))$. Therefore local genus is an embedding invariant.

Determining the (local) genus of a given Cantor set using the definition is not easy. If a Cantor set is given by a defining sequence one can determine an upper bound.

A direct consequence of the definitions is the following result.

Lemma 3.1. *The local genus of x in C is a nonnegative integer k if and only if:*

(1) *for each defining sequence (M_i) for C, there exists a natural number N such that if $n \geq N$, then $g(M_n^{\{x\}}) \geq k$; and*
(2) *there exists a defining sequence (N_i) for C and a natural number M so that if $i \geq M$, then $g(N_i^{\{x\}}) = k$.*

The local genus of x in C is ∞ if and only if:

for each defining sequence (M_i) for C, and for every pair of natural numbers (j, k), there exists an interger $\ell \geq j$ with $g(M_\ell^{\{x\}}) \geq k$.

4. Wedges of Cantor sets

Every Cantor set in R^3 is contained in a closed round 3-cell. By shrinking the radius of this cell, one can find a 3-cell that contains the Cantor set and which has a point (or points) of the Cantor set in its boundary.

Definition 4.1. Let C be a Cantor set in R^3, and B a tame 3-cell in R^3 with $C \subset B$. If $C \cap \mathrm{Bd}(B) \neq \emptyset$, C is said to be *supported* by B.

The following result is a consequence of the definition of tameness.

Lemma 4.2 (See Figure 1)**.** *Suppose C is a Cantor set in R^3 supported by B, and $x \in C \cap \mathrm{Bd}(B)$. Then if (D, p) is a pair consisting of a tame ball of some radius in R^3 together with a point in the boundary of this ball, then there is a homeomorphism from R^3 to itself taking (B, x) to (D, p).*

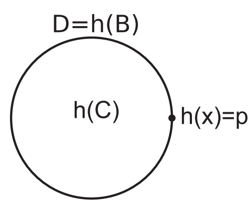

FIGURE 1. Image of B and C

Definition 4.3 (See Figure 2)**.** Let C_i be a Cantor set in R^3 supported by B_i with $x_i \in \mathrm{Bd}(B_i) \cap C_i, i \in \{1, 2\}$. The wedge of C_1 and C_2 at x_1 and x_2, $(C_1, x_1) \vee (C_2, x_2)$, is defined as follows. Choose orientation preserving self-homeomorphisms

h_i of R^3 taking (B_i, x_i) to $(D_i, \mathbf{0})$, where D_i is the ball of radius 1 about the point $\left(2 * (i - \frac{3}{2}), 0, 0\right)$ in R^3. Then

$$(C_1, x_1) \vee (C_2, x_2) \equiv h_1(C_1) \cup h_2(C_2).$$

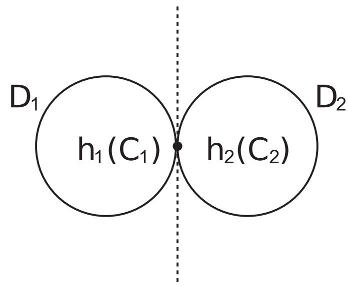

FIGURE 2. Wedge of Cantor sets

Remark 4.4. Note that $(C_1, x_1) \vee (C_2, x_2)$ is a Cantor set with sub-Cantor sets $h_i(C_i)$ embedded in R^3 in a manner equivalent to C_i.

To make the proof of the next theorem and a result in the next section easier, we provide an alternate definition of the wedge of two Cantor sets that is equivalent to the definition above.

Definition 4.5 (Alternate definition of wedge; see Figure 3)**.** Let C_i be a Cantor set in R^3 supported by B_i with $x_i \in \mathrm{Bd}(B_i) \cap C_i, i \in \{1, 2\}$. The wedge of C_1 and C_2 at x_1 and x_2, $(C_1, x_1) \vee' (C_2, x_2)$, is defined as follows. Choose orientation preserving self-homeomorphisms k_i of R^3 taking (B_i, x_i) to $(D_i', (2 * (i - \frac{3}{2}), 0, 0)$, where D_i' is the ball of radius 1 about the point $(4*(i-\frac{3}{2}), 0, 0)$ in R^3. Let A be the straight arc from $(-1, 0, 0)$ to $(1, 0, 0)$ in R^3. Let $p : R^3 \to R^3/A$ be the quotient map. Then

$$(C_1, x_1) \vee' (C_2, x_2) \equiv p\left(k_1(C_1) \cup k_2(C_2)\right) \equiv p\left(k_1(C_1) \cup A \cup k_2(C_2)\right).$$

Since $R^3/A \cong R^3$ and since $p|_{k_i(C_i)}$ is 1-1, it follows that $(C_1, x_1) \vee' (C_2, x_2)$ is homeomorphic to $(C_1, x_1) \vee (C_2, x_2)$. It remains to check that the embeddings of these homeomorphic spaces in R^3 are equivalent. This follows from the next lemma.

Lemma 4.6. $\left(R^3, D_1, D_2, \mathbf{0}\right)$ *is homeomorphic to* $\left(R^3/A, p(D_1'), p(D_2'), p((1, 0, 0))\right)$.

Proof. There is a closed map $h : (R^3, D_1', D_2', A) \to (R^3, D_1, D_2, \mathbf{0})$ with the only nondegenerate point inverse being $h^{-1}(\mathbf{0}) = A$. By standard topological results about quotient spaces, one can now establish the claim. $\square$

Theorem 4.7. *Suppose* C_1 *and* C_2 *are as in Definition* 4.5 *and that* $\pi_1(C_2^c)$ *is trivial. Then* $\pi_1\left(\left((C_1, x_1) \vee (C_2, x_2)\right)^c\right)$ *is isomorphic to* $\pi_1(C_1^c)$.

Proof. $\left((C_1, x_1) \vee (C_2, x_2)\right)^c$ is homeomorphic to $\left((C_1, x_1) \vee' (C_2, x_2)\right)^c$ by Lemma 4.6. Also, $p : R^3 \to R^3/A$ takes $W = R^3 \setminus \left(k_1(C_1) \cup A \cup k_2(C_2)\right)$ homeomorphically onto $\left((C_1, x_1) \vee' (C_2, x_2)\right)^c$ since p is a quotient map and p restricted to W is 1-1. So it suffices to show that $\pi_1(W)$ is isomorphic to $\pi_1(C_1^c)$.

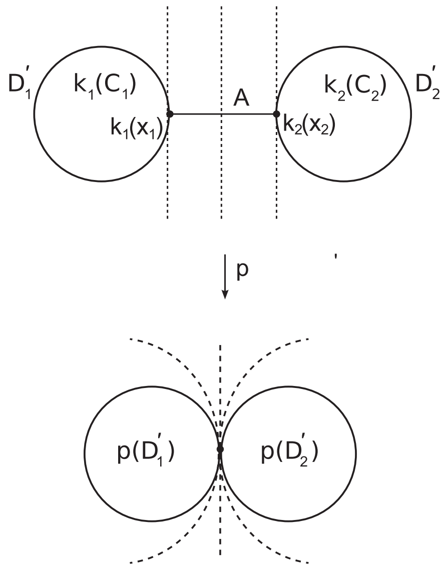

FIGURE 3. Alternate construction for wedge

Let $U = \{(x,y,z) \in W | x < 1\}$ and $V = \{(x,y,z) \in W | x > -1\}$. We will apply the Seifert-Van Kampen Theorem to $U, V, U \cap V$ and $W = U \cup V$. Note that $\pi_1(V) \simeq \pi_1(C_2^c)$, which is trivial. Next, $\pi_1(U) \simeq \pi_1(C_1^c)$. Also, $\pi_1(U \cap V) \simeq \mathbb{Z}$. Since the inclusion-induced homomorphism from $U \cap V$ to V is trivial, $\pi_1(W) \simeq \pi_1(C_1^c)$. □

5. LOCAL GENUS OF POINTS IN A WEDGE

The main result of this section, Theorem 5.4, states that local genus of points in Cantor sets is preserved by taking wedges, except at the wedge point. This allows us in the next section to distinguish between various wedges with complements having the same fundamental group.

For the first two results below, we view the wedge of Cantor sets C_1 and C_2, $(C_1, x_1) \vee (C_2, x_2)$, as in Definition 4.3 and Figure 2, and for ease of notation, view C_1 and C_2 as subspaces of the wedge.

Lemma 5.1. *Let* $W = (C_1, x_1) \vee (C_2, x_2)$ *be as in Definition* 4.3*. Then* $g_{\{x_1 = x_2\}}(W) = g_{\{x_1\}}(C_1) + g_{\{x_2\}}(C_2)$.

Proof. This follows directly from Theorem 13 in [Že05] since there is a 3-cell containing $x_1 = x_2$ in its interior satisfying the conditions needed for that theorem. □

Lemma 5.2. *Let* $W = (C_1, x_1) \vee (C_2, x_2)$ *be as in Definition* 4.3 *and let* p *be a point in* $C_1 \setminus \{x_1\}$*. Then* $g_{\{x\}}(C_1) \leq g_{\{x\}}(W)$.

Proof. The result is obvious if $g_{\{x\}}(W) = \infty$. Assume $g_{\{x\}}(W) = k \in \mathbb{N}$. By Lemma 3.1, there is a defining sequence (M_i) for W such that for every i, $g(M_i^{\{x\}}) = k$. Form a defining sequence (N_i) for C_1 as follows. N_i consists of the components of M_i that intersect C_1. Then for every i, $g(N_i^{\{x\}}) = k$, and thus again by Lemma 3.1, $g_{\{x\}}(C_1) \leq k$. □

For the next result, we view the wedge of Cantor sets C_1 and C_2, $(C_1, x_1) \vee (C_2, x_2)$, as in Definition 4.5 and Figure 3. For ease of notation, we identify C_i with $k_i(C_i)$ so that

$$(C_1, x_1) \vee' (C_2, x_2) \equiv p\,(C_1 \cup C_2) \equiv p\,(C_1 \cup A \cup C_2)\,.$$

Lemma 5.3. *Let* $W = (C_1, x_1) \vee' (C_2, x_2)$ *be as in Definition* 4.5*, let* x *be a point in* $C_1 \setminus \{x_1\}$ *and let* $x' = p(x)$*. Then* $g_{\{x\}}(C_1) \geq g_{\{x'\}}(W)$*.*

Proof. Again, the result is obvious if $g_{\{x\}}(C_1) = \infty$. Assume $g_{\{x\}}(C_1) = k \in \mathbb{N}$. By Lemma 3.1, there is a defining sequence (M_i) for C_1 such that for every i, $g(M_i^{\{x\}}) = k$. By starting the defining sequence at a late enough stage, we may assume that each component of each stage of the defining sequence is in the half space $\{(x, y, z) | x < -\frac{1}{2}\}$ in R^3, and we may assume that the component of each M_i that contains x is distinct from the component of M_i that contains x_1. Choose a defining sequence N_i for C_2 so that each component of each stage of the defining sequence is in the half space $\{(x, y, z) | x > \frac{1}{2}\}$ in R^3.

We now adjust the defining sequence (M_i), replacing it by a defining sequence (M_i') so that the only component of M_i' that has nonempty intersection with A is the component containing x_1 and so that for every i, $g({M'}_i^{\{x\}})$ is still k. Let $M_{(1,1)}$ be the component of $M_1 = M_{n_1}$ containing x_1. Let $C_{(1,2)}$ be the Cantor set $C_1 \setminus M_{(1,1)}$ and $C_{(1,1)}$ be the Cantor set $C_1 \cap M_{(1,1)}$. Let d_1 be the minimum of

$$\left\{d(C_{(1,2)}, A), d(C_{(1,2)}, M_{(1,1)}), d(C_{(1,1)}, \mathrm{Bd}(M_{(1,1)}))\right\}.$$

Choose a stage n_2 such that all components of M_{n_2} have diameter less than $\dfrac{d_1}{2}$. Let M_1' consist of $M_{(1,1)}$ together with the components of M_{n_2} that do not intersect $M_{(1,1)}$.

For the second step, repeat the above procedure on M_{n_2}, letting $M_{(2,1)}$ be the component of M_{n_2} containing x_1, $C_{(2,2)}$ be the Cantor set $C_1 \setminus M_{(2,1)}$ and $C_{(2,1)}$ be the Cantor set $C_1 \cap M_{(2,1)}$. Let d_2 be the minimum of

$$\left\{d(C_{(2,2)}, A), d(C_{(2,2)}, M_{(2,1)}), d(C_{(2,1)}, \mathrm{Bd}(M_{(2,1)}))\right\}.$$

Choose a stage n_3 such that all components of M_{n_3} have diameter less than $\dfrac{d_2}{2}$. Let M_2' consist of $M_{(2,1)}$ together with the components of M_{n_2} that do not intersect $M_{(2,1)}$.

Continuing inductively produces the desired defining sequence (M_i'). Similarly, construct a defining sequence (N_i') for C_2 so that the only component of N_i' intersecting A is the component containing x_2. Finally, choose a sequence of regular neighborhoods of A, (P_i), converging to A so that for each i, $W_{(i,1)} = M(i,1) \cup P_i \cup N_{(i,1)}$ form a manifold neighborhood of A and converge to A.

The defining sequence for W is then produced as follows. W_i consists of $p(W_{(i,1)})$ together with $p(C)$ for all components C of M_i' distinct from $M_{(i,1)}$ and all components C of N_i' distinct from $N_{(i,1)}$. This defining sequence for W has the property that for each i, the component of W_i containing x' has genus k. It follows that $k \geq g_{\{x'\}}(W)$, as required. $\square$

The previous three lemmas together yield a proof of the following main theorem on genus of points in a wedge. Again, identify C_i with $h_i(C_i)$ for ease of notation.

Theorem 5.4. *Let* $W = (C_1, x_1) \vee (C_2, x_2)$ *be as in Definition* 4.3. *Then:*

- *if* $x \in C_i \setminus \{x_i\}$, *then* $g_x(C_i) = g_x(W)$; *and*
- $g_{x_1}(W) = g_{x_1}(C_1) + g_{x_2}(C_2)$.

6. Main result

We are now ready to prove the main result, Theorem 1.4.

Proof of Theorem 1.4. Let C be a Cantor set, and suppose there is some integer $N \geq 3$ such that there are only finitely many points in C of local genus N.

By Theorem 1.2 and the remark following that theorem, for each sequence $S = (n_1, n_2, \ldots)$ of integers in S_1 such that $n_i > 2$, there exists a wild Cantor set in R^3, $X_S = C(S)$, and a countable dense set $A = \{a_1, a_2, \ldots\} \subset X$ such that the following conditions hold:

(1) $g_x(X_S) \leq 2$ for every $x \in X \setminus A$,
(2) $g_{a_i}(X_S) = n_i$ for every $a_i \in A$, and
(3) $R^3 \setminus X_S$ is simply connected.

The construction in [GRŽ06] yields the fact that the sets $A_1 = \{a_i | i \text{ is odd}\}$ and $A_2 = \{a_i | i \text{ is even}\}$ are also dense in $X(S)$. Choose an increasing sequence of integers $(m_1, m_2, \ldots)$ such that $m_1 \geq 3$, and form a sequence of integers $S = (n_1, n_2, \ldots\}$ by specifying $n_{2i} = m_i$ and $n_{2i+1} = N$ for each i. The construction in [GRŽ06] also yields the fact that $X(S)$ is rigidly embedded.

Let $Y_S = (C, x_1) \vee (X_S, x_2)$ for some points $x_1 \in C$ and $x_2 \in X_S$ as in Definition 4.3. Condition (3) above together with Theorem 4.7 imply that $\pi_1(Y_S^c)$ is isomorphic to $\pi_1(C^c)$.

By Theorem 5.4, Y_S has countably many points of genus N and C has only finitely many points of genus N. So C and Y_S are inequivalent Cantor sets.

Next suppose that S and S' are formed from distinct increasing sequences of integers as above. Suppose there is a homeomorphism h of R^3 taking Y_S to $Y_{S'}$. By local genus considerations, a dense subset of the countable dense set of points in the copy of X_S in Y_S that have genus N must be taken by h into a dense subset of points of $X_{S'}$ in $Y_{S'}$ that have genus N. Also, a dense subset of the countable dense set of points in the copy of $X_{S'}$ in $Y_{S'}$ that have genus N must be taken by h^{-1} into a dense subset of the points of X_S in Y_S that have genus N. It follows that h takes the copy of X_S in Y_S onto the copy of $X_{S'}$ in $Y_{S'}$. But this contradicts the fact that there is either a genus that occurs among points of $X(S)$ that does not occur among points of $X(S')$ or vice versa.

Thus every increasing sequence of integers ≥ 3 yields a different Cantor set Y_S. Since there are uncountably many such sequences, there are uncountably many such examples for each Cantor set C as in the theorem. The result now follows. □

Corollary 6.1. *The Cantor sets constructed in the proof of Theorem* 1.4, $Y_S = (C, x_1) \vee' (X_S, x_2)$, *are* not *splittable as* $C \cup A$.

Proof. If these Cantor sets were splittable, C would be in a 3-cell D_1 disjoint from a 3-cell D_2 containing $Y_S \setminus C = X_S \setminus x_2$. This would be a contradiction. □

Corollary 6.2. *There are uncountably many rigid Cantor sets with complement nonsimply connected that have the same fundamental group of the complement.*

Proof. It suffices to take for C in the proof of Theorem 1.4 any of the rigid Antoine Cantor sets of local genus 1 everywhere. See [Wr86a] for a description of these Cantor sets. One can also take for C any of the rigid Cantor sets constructed in [GRŽ06] since they take on certain genera only once. □

Theorem 1.4 and Corollary 6.2 completely answer both parts of Question 1.1.

7. Application to 3-manifolds

If we work in S^3 instead of R^3, the following lemma is a consequence of results about Freudenthal compactifications and the theory of ends (see [Fr42], [Di68], and [Si65]). For completeness, we provide a proof based on defining sequences.

Lemma 7.1. *Let C and D be Cantor sets (or more generally, any compact 0-dimensional sets) in $\mathbb{R}^3$. Suppose there is a homeomorphism $h : \mathbb{R}^3 \setminus C \to \mathbb{R}^3 \setminus D$. Then h extends to a homeomorphism $\bar{h} : (\mathbb{R}^3, C) \to (\mathbb{R}^3, D)$. In particular, C is homeomorphic to D, and C and D are equivalently embedded.*

Proof. Let (M_i) be a defining sequence for C. Suppose $M_i = \{M_{(i,1)}, M_{(i,2)}, \ldots M_{(i,n(i))}\}$. Let $N_{(i,j)}$ be the bounded component of $\mathbb{R}^3 \setminus h(Bd(M_{(i,j)}))$, and let $N_i = \{N_{(i,1)}, N_{(i,2)}, \ldots, N_{(i,n(i))}\}$. The claim is that (N_i) is a defining sequence for D so that the nested sequence in M_i associated with a point $c \in C$ corresponds to a nested sequence in N_i corresponding to a point $d \in D$. This forces $\bar{h}(c)$ to be defined to be d. Now $\bar{h}$ defined in this way is continuous and 1-1 because of the definition of defining sequences. The fact that a similar construction can be done using h^{-1} shows that $\bar{h}$ takes C onto D.

It is clear that any nested sequence in (N_i) has intersection that is compact, connected and in D, so it consists of a single point of D. This establishes the claim at the beginning of the preceding paragraph and completes the proof. □

We now provide the proof of Theorem 1.6.

Proof. The 3-manifolds are the complements in R^3 of the Cantor sets constructed in the proof of Theorem 1.4 in Section 6. These are all nonhomeomorphic by Lemma 7.1. □

See [KM] and [Mc62] for earlier results on nonhomeomorphic 3-manifolds with the same fundamental group.

8. Related questions

The techniques in this paper lead to a number of open questions.

Question 8.1. Are there examples of inequivalent Cantor sets with the same fundamental group of the complement for which the genus of all Cantor sets involved is bounded? The constructions described in this paper yield wedges that have points or arbitrarily large genus.

Question 8.2. Given a Cantor set that does not meet the criteria of Theorem 1.4, are there inequivalent Cantor sets with the same fundamental group of the complement? Note that a Cantor set not covered by these results would have an infinite number of points of every genus greater than or equal to three.

Question 8.3. Do the results of Theorem 1.6 remain true if the restriction on a local genus of points in the Cantor set is removed?

Question 8.4. Does Theorem 1.4 remain true if the Cantor set C has only a finite number of points of genus 1 or of genus 2, or only a finite number of points of genus ∞?

Question 8.5. Can the construction in [GRŽ06] be modified to produce specific points in a countable dense subset that have infinite genus rather than certain specified finite genera?

A Cantor set C is said to be *strongly homogeneously embedded* in R^3 if every self-homeomorphism of C extends to a self-homeomorphism of R^3. Define the *embedding homogeneity group* of the Cantor set to be the group of self homeomorphisms that extend to homeomorphisms of R^3. Rigid Cantor sets have a trivial embedding homogeneity group.

Question 8.6. Given a finitely generated abelian group G, is there a Cantor set C in R^3 with embedding homogeneity group G?

Question 8.7. Given a finite abelian group G, is there a Cantor set C in R^3 with embedding homogeneity group G?

Question 8.8. What kinds of groups arise as embedding homogeneity groups of Cantor sets?

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