NOVEL CRITERIA FOR EXPONENTIAL STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We first give explicit criteria for exponential stability of general linear nonautonomous functional differential equations. Then the obtained results are extended to nonlinear functional differential equations. Two examples are given to illustrate the results. To the best of our knowledge, the results of this note are new.

1. INTRODUCTION AND PRELIMINARIES

Functional differential equations have numerous applications in science and engineering. They are used as models for a variety of phenomena in the life sciences, physics and technology, chemistry and economics; see e.g. [11], [12], [18].

In particular, problems of stability of functional differential equations have been studied intensively during the past decades; see e.g. [4]-[18], [20], [21] and the references therein. Recently, problems of exponential stability of functional differential equations have attracted much attention from researchers; see e.g. [9]-[10], [14]-[17], [20]-[21]. The traditional approaches to stability analysis of nonautonomous functional differential equations are Lyapunov's method and its variants (Razumikhin-type theorems and Lyapunov-Krasovskii functional techniques); see e.g. [6], [2], [8], [11]. By a novel approach, in this note we present some new explicit criteria for exponential stability of general linear nonautonomous functional differential equations. Furthermore, the obtained results are extended to nonlinear functional differential equations. Two examples are given to illustrate the obtained results. Finally, proofs of the main theorems are given in the Appendix.

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let us denote $\underline{m} := \{1, 2, \dots, m\}$. For integers $l, q \geq 1$, $\mathbb{R}^l$ denotes the l -dimensional vector space over $\mathbb{R}$ and $\mathbb{R}^{l \times q}$ stands for the set of all $l \times q$ -matrices with entries in $\mathbb{R}$. Inequalities between real matrices or vectors will be understood componentwise; i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l \times q}$, we write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1, \dots, l, j = 1, \dots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \dots, l, j = 1, \dots, q$, then we write $A \gg B$ instead of $A \geq B$. We denote by $\mathbb{R}_+^{l \times q}$ the set of all nonnegative matrices $A \geq 0$. Similar notation is adopted for vectors. For $x \in \mathbb{R}^n$ and $P \in \mathbb{R}^{l \times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{R}^n, |x| \leq |y|$. Every p -norm on

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$\mathbb{R}^n (\|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}, 1 \leq p < \infty, \text{ and } \|x\|_\infty = \max_{i=1,2,\dots,n} |x_i|)$ is monotonic. In what follows, a norm of vectors on $\mathbb{R}^n$ is monotonic.

For any matrix $M \in \mathbb{R}^{n \times n}$ the *spectral abscissa* of M is denoted by $\mu(M) = \max\{\Re \lambda : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of M .

A matrix $M \in \mathbb{R}^{n \times n}$ is called a *Metzler matrix* if all off-diagonal elements of M are nonnegative. We now summarize some properties of Metzler matrices in the following theorem.

Theorem 1.1 ([19]; see also [1]). *Suppose that $M \in \mathbb{R}^{n \times n}$ is a Metzler matrix. Then:*

- (i) (Perron-Frobenius) $\mu(M)$ is an eigenvalue of M , and there exists a non-negative eigenvector $x \neq 0$ such that $Mx = \mu(M)x$.
- (ii) Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Mx \geq \alpha x$ if and only if $\mu(M) \geq \alpha$.
- (iii) $(tI_n - M)^{-1}$ exists and is nonnegative if and only if $t > \mu(M)$.
- (iv) Given $B \in \mathbb{R}_+^{n \times n}$, $C \in \mathbb{C}^{n \times n}$. Then

$$|C| \leq B \implies \mu(M + C) \leq \mu(M + B).$$

The following is immediate from Theorem 1.1 and is used in what follows.

Theorem 1.2. *Let $M \in \mathbb{R}^{n \times n}$ be a Metzler matrix. Then the following statements are equivalent:*

- (i) $\mu(M) < 0$.
- (ii) $Mp \ll 0$ for some $p \in \mathbb{R}_+^n$.
- (iii) M is invertible and $M^{-1} \leq 0$.
- (iv) For given $b \in \mathbb{R}^n, b \gg 0$, there exists $x \in \mathbb{R}_+^n$ such that $Mx + b = 0$.
- (v) For any $x \in \mathbb{R}_+^n \setminus \{0\}$, the row vector $x^T M$ has at least one negative entry.

Let $\mathbb{R}^n$ be endowed with the norm $\|\cdot\|$ and let $C([\alpha, \beta], \mathbb{R}^n)$ be the Banach space of all continuous functions on $[\alpha, \beta]$ with values in $\mathbb{R}^n$ normed by the maximum norm $\|\phi\| = \max_{\theta \in [\alpha, \beta]} \|\phi(\theta)\|$. In particular, we write $\mathcal{C}$ instead of $C([-h, 0], \mathbb{R}^n)$ and denote $\mathcal{C}_r := \{\phi \in \mathcal{C} : \|\phi\| \leq r\}$, for given $r > 0$. As usual, $L_{loc}^p(\mathbb{R}, \mathbb{R}^m)$ denotes the space of functions from $\mathbb{R}$ into $\mathbb{R}^m$ that are Lebesgue p -integrable on each compact subset of $\mathbb{R}$ and $L_{loc}^\infty(\mathbb{R}, \mathbb{R}^m)$ denotes the space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^m$ that are locally essentially bounded on $\mathbb{R}$.

2. EXPONENTIAL STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Consider a linear functional differential equation of the form

$$(2.1) \quad \dot{x}(t) = A(t)x(t) + \int_{-h}^0 d[\eta(t, \theta)]x(t + \theta), \quad t \geq \sigma,$$

where $A(\cdot) \in L_{loc}^\infty(\mathbb{R}, \mathbb{R}^{n \times n})$ and $\eta(\cdot, \cdot) : \mathbb{R} \times [-h, 0] \rightarrow \mathbb{R}^{n \times n}$ is measurable in (t, θ) and has bounded variation in θ for each $t \in \mathbb{R}$ such that $\text{Var}_{[-h, 0]} \eta(t, \cdot) \leq m(t), t \in \mathbb{R}$, for some $m \in L_{loc}^1(\mathbb{R}, \mathbb{R})$. Furthermore, η is normalized so that η is continuous from the left in θ on $[-h, 0]$ and $\eta(t, \theta) = 0$ for $\theta \geq 0$ and $\eta(t, \theta) = \eta(t, -h)$ for $\theta \leq -h$.

For a fixed $\sigma \in \mathbb{R}$ and a given $\phi \in \mathcal{C}$, there exists a unique solution of (2.1) satisfying the initial condition

$$(2.2) \quad x(\sigma + s) = \phi(s), \quad s \in [-h, 0].$$

It is denoted by $x(\cdot; \sigma, \phi)$. Recall that $x(\cdot; \sigma, \phi)$ is continuous on $[\sigma - h, \infty)$ and absolutely continuous on each compact interval of $[\sigma, \infty)$ and satisfies (2.1) almost everywhere on $[\sigma, \infty)$; see e.g. [8, Theorem 1.1, page 168].

Then (2.1) is said to be *globally exponentially stable* if there exist positive numbers K, β such that for any $\sigma \in \mathbb{R}$ and any $\phi \in \mathcal{C}$, the solution $x(\cdot; \sigma, \phi)$ of (2.1)-(2.2) satisfies

$$\|x(t; \sigma, \phi)\| \leq K e^{-\beta(t-\sigma)} \|\phi\|, \quad \forall t \geq \sigma.$$

We are now in the position to state the first result of this note.

Theorem 2.1. *Let $A(t) := (a_{ij}(t)), t \in \mathbb{R}$, and $\eta(t, \theta) := (\eta_{ij}(t, \theta)), t \in \mathbb{R}, \theta \in [-h, 0]$. Suppose there exist $A_0 := (a_{ij}^{(0)}) \in \mathbb{R}^{n \times n}$ and $A_1 := (a_{ij}^{(1)}) \in \mathbb{R}_+^{n \times n}$ so that*

$$(2.3) \quad a_{ii}(t) \leq a_{ii}^{(0)}, \quad \forall t \in \mathbb{R}, \quad i \in \underline{n}, \quad |a_{ij}(t)| \leq a_{ij}^{(0)}, \quad \forall t \in \mathbb{R}, \quad \forall i \neq j, \quad i, j \in \underline{n};$$

and

$$(2.4) \quad \text{Var}_{[-h, 0]} \eta_{ij}(t, \theta) \leq a_{ij}^{(1)}, \quad \forall t \in \mathbb{R}, \quad \forall i, j \in \underline{n}.$$

If $M := A_0 + A_1$ satisfies one of the equivalent conditions (i)-(v) of Theorem 1.2, then (2.1) is globally exponentially stable.

To state the next result, we now consider a linear differential system with time-varying delay of the form

$$(2.5) \quad \dot{x}(t) = A_0(t)x(t) + \sum_{k=1}^m A_k(t)x(t - h_k(t)) + \int_{-h}^0 B(t, \theta)x(t + \theta) d\theta,$$

where $h_k(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($k \in \underline{m}$) are measurable functions such that

$$(2.6) \quad 0 \leq h_k(t) \leq h, \quad \forall t \in \mathbb{R}, \quad \forall k \in \underline{m},$$

for some $h > 0$ and $A_0(\cdot) \in L_{loc}^\infty(\mathbb{R}, \mathbb{R}^{n \times n})$, $A_k(\cdot) \in L_{loc}^1(\mathbb{R}, \mathbb{R}^{n \times n})$ for $k \in \underline{m}$ and $B(\cdot; \cdot) : \mathbb{R} \times [-h, 0] \rightarrow \mathbb{R}^{n \times n}$ is measurable in t for each θ in $[-h, 0]$ and is integrable in θ for each $t \in \mathbb{R}$ so that

$$\int_{-h}^0 \|B(t, \theta)\| d\theta \leq b(t), \quad \forall t \in \mathbb{R},$$

for some $b(\cdot) \in L_{loc}^1(\mathbb{R}, \mathbb{R})$.

Theorem 2.2. *Let $A_0(t) := (a_{ij}^{(0)}(t)), t \in \mathbb{R}$. Suppose there exist $A_0 := (a_{ij}^{(0)}) \in \mathbb{R}^{n \times n}$ and $B, A_k \in \mathbb{R}_+^{n \times n}, k \in \underline{m}$, so that*

$$(2.7) \quad a_{ii}^{(0)}(t) \leq a_{ii}^{(0)}, \quad \forall t \in \mathbb{R}, \quad i \in \underline{n}; \quad |a_{ij}^{(0)}(t)| \leq a_{ij}^{(0)}, \quad \forall t \in \mathbb{R}, \quad \forall i \neq j, \quad i, j \in \underline{n};$$

and

$$(2.8) \quad |A_k(t)| \leq A_k, \quad \forall t \in \mathbb{R}; \quad \int_{-h}^0 |B(t, \theta)| d\theta \leq B, \quad \forall t \in \mathbb{R}.$$

If $M := A_0 + \sum_{k=1}^m A_k + B$ satisfies one of the equivalent conditions (i)-(v) of Theorem 1.2, then (2.5) is globally exponentially stable.

Remark 2.3. In particular, if $A_0(\cdot) \equiv A_0 \in \mathbb{R}^{n \times n}$ is a Metzler matrix and $A_k(\cdot) \equiv A_k \in \mathbb{R}_+^{n \times n}, k \in \underline{m}$, are nonnegative matrices and $B(\cdot; \cdot) \equiv 0$, then Theorem 2.2 reduces to a very recent result in [14, Theorem 1].

Example 2.4. In the standard book [8, page 154], it has been shown that the equation

$$\dot{x}(t) = -a(t)x(t) - \sum_{k=1}^m b_k(t)x(t - h_k(t))$$

is globally exponentially stable for all bounded continuous functions $a, b_k, h_k, k \in \underline{m}$, provided $a(t) \geq \delta > 0, \sum_{k=1}^m |b_k(t)| \leq r\delta, 0 < r < 1, 0 \leq h_k(t) \leq h$, for all $t \in \mathbb{R}$. The proof given in [8] relies completely upon a Razumikhin-type theorem. However, this is immediate from Theorem 2.1.

A similar example has been found in [11, Example 5.1, page 74]. By applying a Razumikhin-type theorem to the equation

$$\dot{x}(t) = -ax(t) + b(t)x(t - h), \quad t \geq t_0,$$

where $a > 0, h > 0$ and $b(t)$ is a continuous function, the authors showed that the equation under consideration is globally exponentially stable if $\sup_{t \geq t_0} |b(t)| < a$. One again, this assertion is clear by Theorem 2.2.

We now extend Theorem 2.1 to nonlinear functional differential equations. Consider a nonlinear functional differential equation of the form

$$(2.9) \quad \dot{x}(t) = A(t)x(t) + F(t; x(t), \int_{-h}^0 d[\eta(t, \theta)]x(t + \theta)), \quad t \geq \sigma,$$

where $A(\cdot), \eta(\cdot; \cdot)$ are as in equation (2.1) and $F(\cdot; \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given continuous function satisfying $F(t; 0, 0) = 0, \forall t \in \mathbb{R}$, and $F(t; u_0, u_1)$ is (locally) Lipschitz continuous with respect to u_0, u_1 on each compact subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$.

These assumptions ensure that for a fixed $\sigma \in \mathbb{R}$ and a given $\phi \in \mathcal{C}$, there exists a unique local solution of (2.9) satisfying the initial condition (2.2). This solution is defined and continuous on $[\sigma - h, \gamma)$ for some $\gamma > \sigma$ and satisfies (2.9) almost everywhere on $[\sigma, \gamma)$; see e.g. [4], [8]. It is denoted by $x(\cdot; \sigma, \phi)$. Furthermore, if the interval $[\sigma - h, \gamma)$ is the maximum interval of existence of the solution $x(\cdot; \sigma, \phi)$, then $x(\cdot; \sigma, \phi)$ is said to be *noncontinuable*. The existence of a noncontinuable solution follows from Zorn's lemma, and the maximum interval of existence must be open.

Definition 2.5. (i) Equation (2.9) is said to be *locally exponentially stable* if there exist positive numbers r, K, β such that for each $\sigma \in \mathbb{R}$ and each $\phi \in \mathcal{C}_r$, the solution $x(\cdot; \sigma, \phi)$ of (2.9)-(2.2) exists on $[\sigma - h, \infty)$ and furthermore satisfies

$$\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}, \quad \forall t \geq \sigma.$$

(ii) Equation (2.9) is said to be *globally exponentially stable* if there exist positive numbers K, β such that for each $\sigma \in \mathbb{R}$ and each $\phi \in \mathcal{C}$, the solution $x(\cdot; \sigma, \phi)$ of (2.9)-(2.2) exists on $[\sigma - h, \infty)$ and furthermore satisfies

$$\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}\|\phi\|, \quad \forall t \geq \sigma.$$

The following is an extension of Theorem 2.1 to nonlinear equations of the form (2.9).

Theorem 2.6. Suppose that (2.3)-(2.4) hold and there exist $B_0, B_1 \in \mathbb{R}_+^{n \times n}$ such that

$$(2.10) \quad |F(t; u_0, u_1)| \leq B_0|u_0| + B_1|u_1|, \quad \forall t \in \mathbb{R}, \quad \forall u_0, u_1 \in \mathbb{R}^n.$$

If $M := A_0 + B_0 + B_1 A_1$ satisfies one of the equivalent conditions (i)-(v) of Theorem 1.2, then (2.9) is locally exponentially stable. In addition, if the function F is positive homogeneous of degree one with respect to u_0, u_1 (that is, $F(t; \alpha u_0, \alpha u_1) = \alpha F(t; u_0, u_1)$, $\forall t \in \mathbb{R}; \forall u_1, u_2 \in \mathbb{R}^n; \forall \alpha \in \mathbb{R}_+$), then (2.9) is globally exponentially stable.

Remark 2.7. It is important to note that if $F(t; u_0, u_1)$ is (globally) Lipschitz continuous with respect to u_0, u_1 on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $F(t; 0, 0) = 0, \forall t \in \mathbb{R}$, then (2.10) holds automatically for some $B_0, B_1 \in \mathbb{R}_+^{n \times n}$.

We illustrate Theorem 2.1 by a simple example.

Example 2.8. Consider the nonlinear functional differential equation

$$(2.11) \quad \dot{x}(t) = (-p + \sin t)x(t) + \sqrt{qe^{-t^2}x(t)^2 + r\left(\int_{-h}^0 e^s x(t+s)ds\right)^2},$$

where $p, q, r \geq 0$ are parameters and $h > 0$ is a given positive number.

Clearly (2.11) is of the form (2.9) with $a(t) := -p + \sin t$, $t \in \mathbb{R}$, and $F(t; u_0, u_1) := \sqrt{qe^{-t^2}u_0^2 + ru_1^2}$, $t \in \mathbb{R}, u_0, u_1 \in \mathbb{R}$. Furthermore, it is easy to see that $a(t) := -p + \sin t \leq -p + 1, \forall t \in \mathbb{R}$, and F is positive homogeneous of degree one with respect to u_0, u_1 and satisfies

$$F(t; u_0, u_1) = \sqrt{qe^{-t^2}u_0^2 + ru_1^2} \leq \sqrt{q}|u_0| + \sqrt{r}|u_1|$$

and

$$|F(t; u_0, u_1) - F(t; v_0, v_1)| \leq \sqrt{q}|u_0 - v_0| + \sqrt{r}|u_1 - v_1|,$$

for any $t \in \mathbb{R}$ and any $u_i, v_i \in \mathbb{R}, i \in \{0, 1\}$. Therefore all hypotheses of Theorem 2.6 hold. Thus, (2.11) is globally exponentially stable if

$$-p + 1 + \sqrt{q} + \sqrt{r} \int_{-h}^0 e^s ds < 0$$

or equivalently

$$-p + 1 + \sqrt{q} + \sqrt{r}(1 - e^{-h}) < 0.$$

3. APPENDIX: PROOFS OF THEOREMS 2.1, 2.2, 2.6

It is clear that (2.1) is a particular case of (2.9) and Theorem 2.1 is a particular case of Theorem 2.6. Moreover, (2.5) is just a particular case of (2.1) with

$$\eta(t, \theta) := \sum_{k=1}^m \eta_k(t, \theta) + \int_{\theta}^0 -B(t, s)ds, \quad (t, \theta) \in \mathbb{R} \times [-h, 0],$$

$$\eta_k(t, \theta) = \begin{cases} -A_k(t) & \text{if } \theta \in [-h, -h_k(t)], \\ 0 & \text{if } \theta \in (-h_k(t), 0], \end{cases} \quad k \in \underline{m}.$$

Furthermore, (2.7) and (2.8) imply that (2.3) and (2.4) hold. Thus, Theorem 2.2 is a particular case of Theorem 2.1. It remains to prove Theorem 2.6.

Since M is a Metzler matrix, any two of (i)-(v) of Theorem 1.2 are equivalent. We first show that (2.9) is locally exponentially stable provided (iv) of Theorem 1.2 holds. Let $\phi \in C$ be given and let $x(t) := x(t; \sigma, \phi), t \in [\sigma - h, \gamma]$, be a noncontinuable solution of (2.9)-(2.2). We divide the proof into two steps.

Step I. There exists $\beta > 0$ such that for any $\sigma \in \mathbb{R}$, any $r > 0$ and any $\phi \in \mathcal{C}_r$,

$$(3.1) \quad \|x(t; \sigma, \phi)\| \leq K e^{-\beta(t-\sigma)}, \quad \forall t \in [\sigma, \gamma],$$

where K depends on β, r .

By (iv) of Theorem 1.2, there exists $p \in \mathbb{R}_+^n$ such that

$$(3.2) \quad (A_0 + B_0 + B_1 A_1)p \ll 0.$$

By continuity, (3.2) also holds for some $p := (\alpha_1, \alpha_2, \dots, \alpha_n)^T, \alpha_i > 0, \forall i \in \underline{n}$. Furthermore, (3.2) implies that

$$(3.3) \quad (A_0 + B_0 + e^{\beta h} B_1 A_1)p \ll -\beta(\alpha_1, \dots, \alpha_n)^T,$$

for some sufficiently small $\beta > 0$. Fix $r > 0$ and choose $K > 0$ such that $|\phi(t)| \ll K e^{-\beta t} p$ for any $t \in [-h, 0]$ and for any $\phi \in \mathcal{C}_r$. Set $u(t) := K e^{-\beta(t-\sigma)} p, t \in [\sigma - h, \infty)$, and $x(t) := x(t; \sigma, \phi), t \in [\sigma - h, \gamma]$. Then we have $|x(t)| \ll u(t), \forall t \in [\sigma - h, \sigma]$. We claim that $|x(t)| \leq u(t)$ for any $t \in [\sigma, \gamma]$.

Assume on the contrary that there exists $t_0 > \sigma$ such that $|x(t_0)| \not\leq u(t_0)$. Set $t_1 := \inf\{t \in (\sigma, \gamma) : |x(t)| \not\leq u(t)\}$. By continuity, $t_1 > \sigma$ and there is $i_0 \in \underline{n}$ such that

$$(3.4) \quad |x(t)| \leq u(t), \quad \forall t \in [\sigma, t_1]; \quad |x_{i_0}(t_1)| = u_{i_0}(t_1), \quad |x_{i_0}(t)| > u_{i_0}(t), \quad \forall t \in (t_1, t_1 + \epsilon),$$

for some $\epsilon > 0$. For every $i \in \underline{n}$, we have

$$\begin{aligned} \frac{d}{dt}|x_i(t)| &= \text{sgn}(x_i(t))\dot{x}_i(t) \leq a_{ii}(t)|x_i(t)| + \sum_{j=1, j \neq i}^n |a_{ij}(t)||x_j(t)| \\ &\quad + |F_i(t; x(t), \int_{-h}^0 d[\eta(t, \theta)]x(t + \theta))| \end{aligned}$$

for almost any $t \in [\sigma, \gamma]$. Let $A_0 := (a_{ij}^{(0)})$, $B_0 := (b_{ij}^{(0)})$, $B_1 := (b_{ij}^{(1)}) \in \mathbb{R}^{n \times n}$. Then (2.3) implies

$$\frac{d}{dt}|x_i(t)| \leq a_{ii}^{(0)}|x_i(t)| + \sum_{j=1, j \neq i}^n a_{ij}^{(0)}|x_j(t)| + |F_i(t; x(t), \int_{-h}^0 d[\eta(t, \theta)]x(t + \theta))|,$$

for almost any $t \in [\sigma, \gamma]$. It follows from (2.4) and (2.10) that

$$\begin{aligned} \frac{d}{dt}|x_i(t)| &\stackrel{(2.10)}{\leq} a_{ii}^{(0)}|x_i(t)| + \sum_{j=1, j \neq i}^n a_{ij}^{(0)}|x_j(t)| + \sum_{j=1}^n b_{ij}^{(0)}|x_j(t)| + \sum_{j=1}^n b_{ij}^{(1)} \\ &\quad \left| \sum_{k=1}^n \int_{-h}^0 d[\eta_{jk}(t, \theta)]x_k(t + \theta) \right| \leq a_{ii}^{(0)}|x_i(t)| + \sum_{j=1, j \neq i}^n a_{ij}^{(0)}|x_j(t)| + \sum_{j=1}^n b_{ij}^{(0)}|x_j(t)| \\ &\quad + \sum_{j=1}^n b_{ij}^{(1)} \sum_{k=1}^n \text{Var}_{[-h, 0]} \eta_{jk}(t, \cdot) \max_{\theta \in [-h, 0]} |x_k(t + \theta)| \stackrel{(2.4)}{\leq} a_{ii}^{(0)}|x_i(t)| + \sum_{j=1, j \neq i}^n a_{ij}^{(0)}|x_j(t)| \\ &\quad + \sum_{j=1}^n b_{ij}^{(0)}|x_j(t)| + \sum_{j=1}^n b_{ij}^{(1)} \sum_{k=1}^n a_{jk}^{(1)} \max_{\theta \in [-h, 0]} |x_k(t + \theta)|, \end{aligned}$$

for almost any $t \in [\sigma, \gamma)$. Thus, we have for any $t \in [\sigma, \gamma)$,

$$\begin{aligned} D^+ |x_i(t)| &:= \limsup_{h \rightarrow 0^+} \frac{|x_i(t+h)| - |x_i(t)|}{h} = \limsup_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} \frac{d}{ds} |x_i(s)| ds \\ &\leq a_{ii}^{(0)} |x_i(t)| + \sum_{j=1, j \neq i}^n a_{ij}^{(0)} |x_j(t)| + \sum_{j=1}^n b_{ij}^{(0)} |x_j(t)| \\ &\quad + \sum_{j=1}^n b_{ij}^{(1)} \sum_{k=1}^n a_{jk}^{(1)} \max_{\theta \in [-h, 0]} |x_k(t+\theta)|, \end{aligned}$$

where D^+ denotes the Dini upper-right derivative.

In particular, it follows from (3.3) and (3.4) that

$$\begin{aligned} D^+ |x_{i_0}(t_1)| &\stackrel{(3.4)}{\leq} a_{i_0 i_0}^{(0)} K e^{-\beta(t_1-\sigma)} \alpha_{i_0} + \sum_{j=1, j \neq i_0}^n a_{i_0 j}^{(0)} K e^{-\beta(t_1-\sigma)} \alpha_j \\ &\quad + \sum_{j=1}^n b_{i_0 j}^{(0)} K e^{-\beta(t_1-\sigma)} \alpha_j + \sum_{j=1}^n b_{i_0 j}^{(1)} \sum_{k=1}^n a_{jk}^{(1)} K e^{-\beta(t_1-\sigma)} e^{\beta h} \alpha_k \\ &= K e^{-\beta(t_1-\sigma)} \left(\sum_{j=1}^n a_{i_0 j}^{(0)} \alpha_j + \sum_{j=1}^n b_{i_0 j}^{(0)} \alpha_j + \sum_{j=1}^n b_{i_0 j}^{(1)} \sum_{k=1}^n a_{jk}^{(1)} e^{\beta h} \alpha_k \right) \\ &\stackrel{(3.3)}{<} K e^{-\beta(t_1-\sigma)} (-\beta \alpha_{i_0}) = D^+ u_{i_0}(t_1). \end{aligned}$$

However, this conflicts with (3.4). Therefore

$$|x(t; \sigma, \phi)| \leq u(t) = K e^{-\beta(t-\sigma)} p, \quad \forall \sigma \geq 0; \quad \forall \phi \in \mathcal{C}_r; \quad \forall t \in [\sigma, \gamma).$$

By the monotonicity of vector norms, this yields

$$\|x(t; \sigma, \phi)\| \leq K_1 e^{-\beta(t-\sigma)}, \quad \forall \sigma \geq 0; \quad \forall \phi \in \mathcal{C}_r; \quad \forall t \in [\sigma, \gamma),$$

for some $K_1 > 0$.

Step II. We claim that $\gamma = \infty$, and so (2.9) is locally exponentially stable.

Seeking a contradiction, we assume that $\gamma < \infty$. Then it follows from (3.1) that $x(\cdot; \sigma, \phi)$ is bounded on $[\sigma, \gamma)$. Furthermore, this together with (2.9) and (2.10) implies that $\dot{x}(\cdot)$ is bounded almost everywhere on $[\sigma, \gamma)$. Thus $x(\cdot)$ is uniformly continuous on $[\sigma, \gamma)$. Therefore, $\lim_{t \rightarrow \gamma^-} x(t)$ exists and $x(\cdot)$ can be extended to a continuous function on $[\sigma, \gamma]$. Moreover, the closure of $\{x_t : t \in [\sigma, \gamma)\}$ is a compact set in $\mathcal{C}$ by the Arzela-Ascoli theorem [3]. Note that

$$\{(t, x_t) : t \in [\sigma, \gamma)\} \subset [\sigma, \gamma] \times \text{the closure of } \{x_t : t \in [\sigma, \gamma)\}.$$

Thus, the closure of $\{(t, x_t) : t \in [\sigma, \gamma)\}$ is a compact set in $\mathbb{R} \times \mathcal{C}$. Since (γ, x_γ) belongs to this compact set, one can find a solution of (2.9) through this point to the right of γ . This contradicts the noncontinuability hypothesis on $x(\cdot)$. Thus γ must be equal to ∞ .

Finally, we show that (2.9) is globally exponentially stable provided F is positive homogeneous of degree one with respect to u_0, u_1 . Let $\phi \in \mathcal{C}$ be given. Since F is positive homogeneous of degree one with respect to u_0, u_1 , it follows

that $\frac{1}{\|\phi\|}x(\cdot; \sigma, \phi)$ is the solution of (2.9) satisfying the initial condition $x(t + \sigma) = \frac{1}{\|\phi\|}\phi(t)$, $t \in [-h, 0]$. Since $\frac{1}{\|\phi\|}\phi \in \mathcal{C}_1$, we have

$$\left\| \frac{1}{\|\phi\|}x(t; \sigma, \phi) \right\| \leq Ke^{-\beta(t-\sigma)}, \quad \forall t \geq \sigma,$$

or, equivalently, $\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}\|\phi\|$, $\forall t \geq \sigma$. Here K, β are independent with ϕ , and thus (2.9) is globally exponentially stable. This completes the proof.

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