A FAMILY OF NUMBER FIELDS WITH UNIT RANK AT LEAST 4 THAT HAS EUCLIDEAN IDEALS

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(Communicated by Matthew A. Papanikolas)

ABSTRACT. We will prove that if the unit rank of a number field with cyclic class group is large enough and if the Galois group of its Hilbert class field over $\mathbb{Q}$ is abelian, then every generator of its class group is a Euclidean ideal class. We use this to prove the existence of a non-principal Euclidean ideal class that is not norm-Euclidean by showing that $\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})$ has such an ideal class.

1. INTRODUCTION

Euclidean ideals, introduced by Lenstra, generalize Euclidean algorithms in that the existence of a Euclidean algorithm for a domain R implies that R has trivial class group, while the existence of a Euclidean ideal in a domain R implies that R has cyclic class group. If an ideal is Euclidean, then so is every other ideal in its ideal class, and therefore we say the ideal class is Euclidean. If a domain R has a Euclidean ideal class $[C]$, then $[C]$ generates the class group of R .

Lenstra showed ([9]), assuming the generalized Riemann hypothesis (henceforth abbreviated GRH), that if K is a number field with ring of integers $\mathcal{O}_K$ and class group Cl_K , and if $|\mathcal{O}_K^\times| = \infty$, then

$$\text{Cl}_K = \langle [C] \rangle \text{ if and only if } [C] \text{ is a Euclidean ideal class.}$$

Using techniques used by Harper and Murty ([7], [8]), we will prove the following weaker result without assuming the GRH.

Theorem 1. *Let K be a number field, Galois over $\mathbb{Q}$, with ring of integers $\mathcal{O}_K$ and cyclic class group Cl_K . If its Hilbert class field, $H(K)$, has an abelian Galois group over $\mathbb{Q}$ and if $\text{rank}(\mathcal{O}_K^\times) \geq 4$, then*

$$\text{Cl}_K = \langle [C] \rangle \text{ if and only if } [C] \text{ is a Euclidean ideal class.}$$

2. EUCLIDEAN IDEAL CLASSES

The following is equivalent to Lenstra's definition [9] but is stated differently [4].

Definition 1. Suppose R is a Dedekind domain and that $\mathbb{I}$ is the set of its non-zero integral ideals. If C is an ideal of R , then it is called *Euclidean* if there exists a function $\psi : \mathbb{I} \rightarrow W$, W a well-ordered set, such that for all integral ideals I and all $x \in I^{-1}C \setminus C$, there exists some $y \in C$ such that

$$\psi((x - y)IC^{-1}) < \psi(I).$$

Received by the editors May 27, 2011 and, in revised form, October 5, 2011 and November 16, 2011.

2010 *Mathematics Subject Classification.* Primary 11-XX, 13F07.

We say ψ is a *Euclidean algorithm* for C and C is a *Euclidean ideal*.

Generalizing the work of Malcolm Harper ([7]), the first author showed the following growth result ([3]).

Theorem 2. *Let K be a number field with ring of integers $\mathcal{O}_K^\times$ and cyclic class group Cl_K . Fix an ideal class $[C]$ in Cl_K . If $|\mathcal{O}_K^\times| = \infty$ and if*

$$\left| \left\{ \begin{array}{l} \text{prime ideals} \\ \mathfrak{p} \subset \mathcal{O}_K \end{array} : \text{Nm}(\mathfrak{p}) \leq x, [\mathfrak{p}] = [C], \mathcal{O}_K^\times \twoheadrightarrow (\mathcal{O}_K/\mathfrak{p})^\times \right\} \right| \gg \frac{x}{\log^2 x},$$

then $[C]$ is a Euclidean ideal class.

3. PRIMES AND HILBERT CLASS FIELDS

Suppose that K is a number field Galois over $\mathbb{Q}$ and that its Hilbert class field $H(K)$ has abelian Galois group over $\mathbb{Q}$. Let $f(K)$ be the conductor of K , which is also the conductor of $H(K)$, so that both fields are contained in $\mathbb{Q}(\zeta_{f(K)})$, where $\zeta_{f(K)}$ is a primitive $f(K)$ -th root of unity. Note that $H(K)$ lies in $\mathbb{Q}(\zeta_{f(K)})$ because $\text{Gal}(H(K)/\mathbb{Q})$ is abelian. We define d to be the smallest even number such that every root of unity in K is a d -th root of unity.

Given a prime p in $\mathbb{Q}$, we choose a prime $\mathfrak{p}$ in K that lies above p , a prime $\mathfrak{P}$ in $H(K)$ that lies above $\mathfrak{p}$, and a prime P in $\mathbb{Q}(\zeta_{f(K)})$ that lies above $\mathfrak{P}$:

$$\begin{array}{ccc} \mathbb{Q}(\zeta_{f(K)}) & & P \\ | & & | \\ H(K) & & \mathfrak{P} \\ | & & | \\ K & & \mathfrak{p} \\ | & & | \\ \mathbb{Q}(\zeta_d) & & | \\ | & & | \\ \mathbb{Q} & & p \end{array}$$

If $[C]$ generates the class group of K , then the Artin map maps all primes $\mathfrak{q}$ such that $[\mathfrak{q}] = [C]$ to a particular element σ of $\text{Gal}(H(K)/K)$ because $\text{Cl}_K \cong \text{Gal}(H(K)/K)$. The Galois group $\text{Gal}(H(K)/K)$ is isomorphic to

$$\text{Gal}(\mathbb{Q}(\zeta_{f(K)})/K)/\text{Gal}(\mathbb{Q}(\zeta_{f(K)})/H(K)).$$

We can therefore identify $\text{Gal}(H(K)/K)$ (and thus Cl_K) with a set of elements in $\text{Gal}(\mathbb{Q}(\zeta_{f(K)})/\mathbb{Q})$, as $\text{Gal}(\mathbb{Q}(\zeta_{f(K)})/K)$ is a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_{f(K)})/\mathbb{Q})$. By the isomorphism

$$\tau : \text{Gal}(\mathbb{Q}(\zeta_{f(K)})/\mathbb{Q}) \rightarrow (\mathbb{Z}/f(K)\mathbb{Z})^\times,$$

we can see that there exists some $0 < a < f(K)$, $(a, f(K)) = 1$, such that if $p \equiv a \pmod{f(K)}$, then $\mathfrak{p}$ is of first degree and $[\mathfrak{p}] = [C]$.

This, along with Theorem 2, implies that in order to prove Theorem 1, it suffices to show that

$$(1) \quad \left| \{ \text{primes } p \leq x : p \equiv a \pmod{f(K)}, \mathcal{O}_K^\times \twoheadrightarrow (\mathcal{O}_K/\mathfrak{p})^\times \} \right| \gg \frac{x}{\log^2 x},$$

where the implied constant depends only on K . Using the linear sieve, we will show this when $\text{rank}(\mathcal{O}_K^\times) \geq 4$.

4. THE LINEAR SIEVE

The following notation is taken from Halberstam and Richert [6]. Suppose that $\mathfrak{A}$ is a finite set of integers, that P is a collection of primes, and let $z \in \mathbb{R}, z \geq 2$. We define $S(\mathfrak{A}; P, z)$ to be the number of elements of $\mathfrak{A}$ that are not divisible by any prime p in P such that $p \leq z$. It is a generalization of $\pi(y; q, a)$, the number of primes less than or equal to y which are congruent to $a \pmod{q}$. In order to bound $S(\mathfrak{A}; P, z)$, we need to have a decent estimate for the size of $\mathfrak{A}$, which we denote by X .

For q square-free, we define $\mathfrak{A}_q := \{a \in \mathfrak{A} : a \equiv 0 \pmod{q}\}$ and we choose a function ω_0 ; we will be using $\frac{\omega_0(p)}{p}X$ to estimate $|\mathfrak{A}_p|$ for p prime. The definition of ω_0 is extended to all square-free q by defining $\omega_0(1) = 1$ and $\omega_0(q) = \prod_{p|q} \omega_0(p)$. In sieve theory we begin with this data, where we use approximations of the sizes of sets $\mathfrak{A}_q$ and keep track of the error terms that emanate from this calculation.

We now relate these definitions to the set of primes P . For ease of notation, we define the set of all primes not in P to be $\overline{P}$, so that $P \cap \overline{P} = \emptyset$ and $P \cup \overline{P}$ is the set of all primes. For p a prime, we define

$$\omega(p) = \begin{cases} \omega_0(p) & \text{if } p \in P, \\ 0 & \text{if } p \in \overline{P}. \end{cases}$$

For q square-free, we define $\omega(1) = 1$, $\omega(q) = \prod_{p|q} \omega(p)$, and

$$R_q := |\mathfrak{A}_q| - \frac{\omega(q)}{q}X \text{ if } \mu(q) \neq 0,$$

where μ is the Möbius function. The linear sieve can only be applied if the sets $\mathfrak{A}$ and P satisfy certain conditions, enumerated below.

Condition 1. There exists a constant $A_1 \geq 1$ such that

$$0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1}.$$

Condition 2. There exist constants L and A_2 , both at least one, independent of z and w , such that if $2 \leq w \leq z$, then

$$-L \leq \sum_{w \leq p \leq z} \frac{\omega(p) \log p}{p} - \log \left(\frac{z}{w} \right) \leq A_2.$$

In order to state Condition 3, we define $(q, \overline{P}) = 1$ if every prime dividing q is in P .

Condition 3. There exists an $\alpha, 0 < \alpha \leq 1$, such that

$$\sum_{\substack{q < \frac{X^\alpha}{(\log X)^{A_4}} \\ (q, \overline{P}) = 1}} \mu^2(q) 3^{\nu(q)} |R_q| \leq A_5 \frac{X}{\log^2 X} \quad (X \geq 2)$$

for some constants $A_4, A_5 \geq 1$.

Theorem 3 (The linear sieve lower bound). *If $\mathfrak{A}$ and P satisfy Conditions 1, 2, and 3, and if $z \leq X$, then*

$$S(\mathfrak{A}; P, z) \geq X \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right) \left\{ f\left(\alpha \frac{\log X}{\log z}\right) - B \frac{L}{(\log X)^{\frac{1}{14}}} \right\},$$

where B is an absolute constant and f is a classical function defined in [6]. When $2 \leq u \leq 4$, $f(u) := \frac{2e^\gamma \log(u-1)}{u}$, where γ is the Euler-Mascheroni constant.

5. APPLYING THE LINEAR SIEVE

In order to prove that (1) holds in any number field K satisfying the conditions in Theorem 1, we will show that for any given $x \in \mathbb{R}^{>0}$, $\mathfrak{A}$, P , and z satisfy Conditions 1, 2, and 3, where

$$\mathfrak{A} = \left\{ \frac{p-1}{d} : p \leq x, p \equiv a \pmod{f(K)}, \left(\frac{p-1}{d}, 2f(K) \right) = 1 \right\}$$

and P is the set of primes $\leq z$. The a in the definition of $\mathfrak{A}$ is chosen so that if $p \equiv a \pmod{f(K)}$, then $[\mathfrak{p}] = [C]$ and $\mathfrak{p}$ is of degree one. Note that $p \equiv 1 \pmod{d}$. Since

$$\left| \left\{ \frac{p-1}{d} : p \leq x, p \equiv a \pmod{f(K)} \right\} \right| \sim \frac{\text{li}(x)}{\phi(f(K))},$$

$|\mathfrak{A}| \sim \frac{C\text{li}(x)}{\phi(f(K))}$ by the Eratosthenes sieve for some constant $0 < C < 1$ that depends only on $f(K)$.

If p' is a prime, $p' \leq x$, and $p' \nmid 2f(K)$, then $|\mathfrak{A}_{p'}| \sim \frac{C\text{li}(x)}{\phi(f(K))\phi(p')}$, so

$$\frac{\omega_0(p')}{p'} = \frac{1}{\phi(p')}.$$

If p' is a prime, $p' \leq x$, and $p' | 2f(K)$, then $|\mathfrak{A}_{p'}| = 0$, so $\frac{\omega_0(p')}{p'} = 0$. From this, we see that for all square-free q ,

$$\omega(q) = \begin{cases} \frac{q}{\phi(q)} & \text{if } (q, \overline{P}) = 1, (q, 2f(K)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *The sets $\mathfrak{A}$ and P satisfy Condition 1.*

Proof. Note that for $p > 2$, $\frac{\omega(p)}{p} = \frac{1}{\phi(p)}$ or 0 and

$$0 \leq \frac{1}{\phi(p)} \leq \frac{1}{2} = 1 - \frac{1}{2}.$$

As $2 | 2f(K)$, we can see that $0 \leq \frac{\omega(2)}{2} = 0 < 1 - \frac{1}{2}$. □

Lemma 2. *The set $\mathfrak{A}$, the set P , and the quantity z satisfy Condition 2.*

Proof. Suppose that $2 \leq w \leq z$. Then

$$\sum_{w \leq p \leq z} \frac{\omega(p) \log p}{p} = \sum_{w \leq p \leq z} \frac{\frac{p}{p-1} \log p}{p} - \sum_{w \leq p \leq z, p | 2f(K)} \frac{\frac{p}{p-1} \log p}{p},$$

and as

$$\sum_{w \leq p \leq z, p \nmid 2f(K)} \frac{\frac{p}{p-1} \log p}{p} \leq \sum_{p \mid 2f(K)} \frac{\frac{p}{p-1} \log p}{p} = \mathcal{O}(1),$$

we know that

$$(2) \quad \sum_{w \leq p \leq z} \frac{\omega(p) \log p}{p} - \log \left(\frac{z}{w} \right) = \sum_{w \leq p \leq z} \frac{\log p}{p} + \sum_{w \leq p \leq z} \frac{\log p}{p(p-1)} - \log \left(\frac{z}{w} \right) - \mathcal{O}(1).$$

The sequence $\sum_{p \leq x} \frac{\log p}{p(p-1)} = \mathcal{O}(1)$. By Chebyshev's Theorem (see [1], p. 6),

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + \mathcal{O}(1),$$

so (2) becomes

$$\log z - \log w - \log \left(\frac{z}{w} \right) + \mathcal{O}(1) = \mathcal{O}(1).$$

□

To prove that Condition 3 holds, we must first prove the following lemma.

Lemma 3. *If $c \in \mathbb{N}$, then*

$$\sum_{\substack{(q, \overline{P}) = 1 \\ q \leq z \\ q \text{ square-free}}} \frac{c^{\nu(q)}}{q} \ll \log^c z,$$

where $\nu(q)$ is the number of distinct prime factors of q .

Proof. Note that

$$\sum_{\substack{(q, \overline{P}) = 1 \\ q \leq z \\ q \text{ square-free}}} \frac{c^{\nu(q)}}{q} \leq \prod_{p \leq z} \left(1 + \frac{c}{p} \right) \leq \prod_{p \leq z} \left(1 + \frac{1}{p} \right)^c.$$

Now

$$\prod_{p \leq z} \left(1 + \frac{1}{p} \right) \leq \prod_{p \leq z} \left(\sum_{j=0}^{\infty} \frac{1}{p^j} \right) = \prod_{p \leq z} \left(1 - \frac{1}{p} \right)^{-1}.$$

Mertens' Theorem (see [10], p. 128) states that

$$\prod_{p \leq z} \left(1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log z} \left(1 + \mathcal{O} \left(\frac{1}{\log z} \right) \right),$$

where γ is the Euler constant. Thus

$$\prod_{p \leq z} \left(1 - \frac{1}{p} \right)^{-1} = e^{\gamma} \log z \left(1 + \mathcal{O} \left(\frac{1}{\log z} \right) \right)^{-1}.$$

Noting that

$$\left(1 + \mathcal{O} \left(\frac{1}{\log z} \right) \right)^{-1} = 1 + \mathcal{O} \left(\frac{1}{\log z} \right),$$

we see

$$\prod_{p \leq z} \left(1 - \frac{1}{p} \right)^{-1} = e^{\gamma} \log z + \mathcal{O}(1).$$

Thus

$$\prod_{p \leq z} \left(1 + \frac{1}{p}\right)^c = \mathcal{O}(\log^c(z)). \quad \square$$

Lemma 4. *The sets $\mathfrak{A}$ and P satisfy Condition 3.*

Proof. According to the Bombieri-Vinogradov inequality (see [1], p. 39), there exists some $B > 1$ such that

$$\sum_{q \leq \frac{x^{\frac{1}{2}}}{\log^{B-1} x}} \max_{y \leq x} \max_{(\alpha, q)=1} \left| \pi(y; q, \alpha) - \frac{\text{li}(y)}{\phi(q)} \right| \ll \frac{x}{\log^{13} x}.$$

By applying Cauchy-Schwarz (see [1], p. 27), we see that

$$\begin{aligned} \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1}} \mu^2(q) 3^{\nu(q)} |R_q| &\leq \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1}} 3^{\nu(q)} |R_q| \\ &\leq \sqrt{\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1}} 9^{\nu(q)} |R_q|} \sqrt{\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1}} |R_q|}. \end{aligned}$$

Let us examine the first term in the product. We can see that for q square-free,

$$|R_q| \sim \left(|\mathfrak{A}_q| - \frac{C \text{li}(x)}{\phi(qf(K))} \right) \text{ if } (q, \overline{P}) = (q, 2f(K)) = 1$$

and

$$|R_q| = 0 \text{ otherwise,}$$

so

$$|R_q| \leq |\mathfrak{A}_q| + \left| \frac{C \text{li}(x)}{\phi(qf(K))} \right| \leq \frac{2x}{q}.$$

Therefore

$$\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1}} 9^{\nu(q)} |R_q| \leq 2x \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1 \\ q \text{ square-free}}} \frac{9^{\nu(q)}}{q} \leq 2x \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P})=1 \\ q \text{ square-free}}} \frac{9^{\nu(q)}}{q} \ll 2x \log^9 x$$

by Lemma 3.

We shall now examine the second term in the product. By definition, if $(q, 2f(K)) \neq 1$, then $|R_q| = 0$. If $(q, 2f(K)) = 1 = (q, \overline{P})$, then

$$\begin{aligned}
 |R_q| &= \left| \mathfrak{A}_q - \frac{1}{\phi(q)} |\mathfrak{A}| \right| \\
 &= \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{f(K)} \\ p \equiv 1 \pmod{q} \\ \left(\frac{p-1}{d}, 2f(K)\right) = 1}} 1 - \frac{1}{\phi(q)} \sum_{\substack{p \leq x \\ p \equiv a \pmod{f(K)} \\ \left(\frac{p-1}{d}, 2f(K)\right) = 1}} 1 \right| \\
 &= \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{f(K)} \\ p \equiv 1 \pmod{q}}} \sum_{l \mid \left(\frac{p-1}{d}, 2f(K)\right)} \mu(l) - \frac{1}{\phi(q)} \sum_{\substack{p \leq x \\ p \equiv a \pmod{f(K)}}} \sum_{l \mid \left(\frac{p-1}{d}, 2f(K)\right)} \mu(l) \right| \\
 &= \left| \sum_{l \mid 2f(K)} \mu(l) \left(\sum_{\substack{p \leq x \\ p \equiv a \pmod{f(K)} \\ p \equiv 1 \pmod{q} \\ p \equiv 1 \pmod{dl}}} 1 - \frac{1}{\phi(q)} \sum_{\substack{p \leq x \\ p \equiv a \pmod{f(K)} \\ p \equiv 1 \pmod{dl}}} 1 \right) \right| \\
 &\leq \sum_{l \mid 2f(K)} \left| \pi(x, q[f(K), dl], a') - \frac{1}{\phi(q)} \pi(x, [f(K), dl], a^*) \right|,
 \end{aligned}$$

where a' is the smallest positive solution to $a' \equiv a \pmod{f(K)}$, $a' \equiv 1 \pmod{q}$, $a' \equiv 1 \pmod{dl}$; where a^* is the smallest positive solution to $a^* \equiv a \pmod{f(K)}$, $a^* \equiv 1 \pmod{dl}$; and where $[c_1, \dots, c_k] = \text{lcm}(c_1, \dots, c_k)$. This means that

$$\begin{aligned}
 \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P}) = 1}} |R_q| &\leq \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P}) = 1}} \left| \pi(x, q[f(K), dl], a') - \frac{\text{li}(x)}{\phi(q[f(K), dl])} \right| \\
 &+ \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P}) = 1}} \left| \frac{\text{li}(x)}{\phi(q[f(K), dl])} - \frac{1}{\phi(q)} \pi(x, [f(K), dl], a^*) \right|.
 \end{aligned}$$

Note that

$$\begin{aligned}
 &\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, \overline{P}) = 1}} \left| \pi(x, q[f(K), dl], a') - \frac{\text{li}(x)}{\phi(q[f(K), dl])} \right| \\
 &\leq \sum_{q \leq \frac{[f(K), dl] x^{\frac{1}{2}}}{\log^B x}} \left| \pi(x, q, a') - \frac{\text{li}(x)}{\phi(q)} \right| \leq \sum_{q \leq \frac{[f(K), dl] x^{\frac{1}{2}}}{\log^B x}} \max_{(r, q)=1} \left| \pi(x, q, r) - \frac{\text{li}(x)}{\phi(q)} \right|.
 \end{aligned}$$

By definition, as $(q, [f(K), dl]) = 1$,

$$\frac{1}{\phi(q)}\pi(x, [f(K), dl], a^*) = \frac{1}{\phi(q)} \sum_{r=0}^{q-1} \pi(x, q[f(K), dl], a_r^*),$$

where a_r^* is the smallest positive solution to

$$t \equiv a^* \pmod{[f(K), dl]}, t \equiv r \pmod{q}.$$

The sum $\sum_{(r,q) \neq 1} \pi(x, [f(K), dl], a_r^*)$ counts all the primes less than or equal to x that are equivalent to $a^* \pmod{[f(K), dl]}$ and $r \pmod{q}$, where r is not relatively prime to q . If $p \equiv r \pmod{q}$ and $(r, q) \neq 1$, then p must divide q . Our sum, therefore, is bounded above by $\sum_{p|q} 1 = \nu(q) \leq \frac{\log q}{\log 2}$, so

$$\begin{aligned} \frac{1}{\phi(q)} \sum_{r=0}^{q-1} \pi(x, q[f(K), dl], a_r^*) &= \frac{1}{\phi(q)} \nu(q) + \frac{1}{\phi(q)} \sum_{(r,q)=1} \pi(x, q[f(K), dl], a_r^*) \\ &= \frac{1}{\phi(q)} \sum_{(r,q)=1} \pi(x, q[f(K), dl], r) + \frac{1}{\phi(q)} \nu(q). \end{aligned}$$

This is used to rewrite

$$\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, P) = 1}} \left| \frac{\text{li}(x)}{\phi(q[f(K), dl])} - \frac{1}{\phi(q)} \pi(x, [f(K), dl], a^*) \right|$$

as

$$\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, P) = 1}} \left| \frac{\nu(q)}{\phi(q)} + \frac{1}{\phi(q)} \sum_{(r,q)=1} \pi(x, q[f(K), dl], r) - \frac{\text{li}(x)}{\phi(q[f(K), dl])} \right|,$$

which is bounded above by

$$\begin{aligned} &\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, P) = 1}} \frac{\nu(q)}{\phi(q)} + \left| \sum_{(r,q)=1} \frac{\pi(x, q[f(K), dl], r)}{\phi(q)} - \frac{\text{li}(x)}{\phi(q)\phi(q[f(K), dl])} \right| \\ &\leq \sum_{q \leq \frac{x^{\frac{1}{2}}}{\log^B x}} \frac{\log q}{\log 2} + \max_{(r,q)=1} \left| \pi(x, q[f(K), dl], r) - \frac{\text{li}(x)}{\phi(q[f(K), dl])} \right|. \end{aligned}$$

Putting the pieces together, we see that

$$\begin{aligned}
 \sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, P) = 1}} |R_q| &\leq \sum_{l|2f(K)} \left(\sum_{q \leq \frac{x^{\frac{1}{2}} [f(K), dl]}{\log^B x}} 2 \max_{(r, q)=1} \left| \pi(x, q, r) - \frac{\text{li}(x)}{\phi(q)} \right| + \sum_{q \leq \frac{x^{\frac{1}{2}}}{\log^B x}} \frac{\log q}{\log 2} \right) \\
 &\ll \frac{x^{\frac{1}{2}}}{\log^{B-1} x} + \sum_{q \leq \frac{x^{\frac{1}{2}} [f(K), dl]}{\log^B x}} \max_{(r, q)=1} \left| \pi(x, q, r) - \frac{\text{li}(x)}{\phi(q)} \right| \\
 &\ll \frac{x^{\frac{1}{2}}}{\log^{B-1} x} + \sum_{q \leq \frac{x^{\frac{1}{2}}}{\log^{B-1} x}} \max_{(r, q)=1} \left| \pi(x, q, r) - \frac{\text{li}(x)}{\phi(q)} \right| \\
 &\ll \frac{x^{\frac{1}{2}}}{\log^{B-1} x} + \frac{x}{\log^{13} x} \ll \frac{x}{\log^{13} x}.
 \end{aligned}$$

In conclusion,

$$\sum_{\substack{q \leq \frac{x^{\frac{1}{2}}}{\log^B x} \\ (q, P) = 1}} \mu^2(q) 3^{\nu(q)} |R_q| \ll \sqrt{2x \log^9 x} \sqrt{\frac{x}{\log^{13} x}} \ll \frac{x}{\log^2 x}. \quad \square$$

We may now apply the linear sieve to $\mathfrak{A}$.

Lemma 5. *For any small $\epsilon > 0$, there exists a positive constant k such that if x is large enough, then*

$$S(\mathfrak{A}, P, x^{\frac{1-\epsilon}{4}}) \geq k \frac{x}{\log^2 x}.$$

Proof. By Lemmas 1, 2, and 4 we may apply the linear sieve to $\mathfrak{A}$, implying that

$$\begin{aligned}
 &S(\mathfrak{A}, P, x^{\frac{1-\epsilon}{4}}) \\
 &\geq \frac{C \text{li}(x)}{\phi(f(K))} \prod_{\substack{p < x^{\frac{1-\epsilon}{4}} \\ (p, 2f(K)) = 1}} \left(1 - \frac{1}{\phi(p)} \right) \left\{ f \left(\frac{1}{2} \frac{\log \left(\frac{Cx}{\phi(f(K)) \log x} \right)}{\log x^{\frac{1-\epsilon}{4}}} \right) - \frac{BL}{(\log x)^{\frac{1}{14}}} \right\} \\
 &\gg \frac{Cx}{\phi(f(K)) \log x} \prod_{\substack{p < x^{\frac{1-\epsilon}{4}} \\ (p, 2f(K)) = 1}} \left(1 - \frac{1}{\phi(p)} \right) \\
 &\quad \times \left\{ f \left(\frac{1}{2} \frac{\log \left(\frac{Cx}{\phi(f(K)) \log x} \right)}{\log x^{\frac{1-\epsilon}{4}}} \right) - \frac{BL}{(\log x)^{\frac{1}{14}}} \right\}.
 \end{aligned}$$

For $0 < \epsilon' < \epsilon$ and for x large enough,

$$\frac{1}{2} \left(\frac{\log x - \log \log x + \log \left(\frac{C}{\phi(f(K))} \right)}{\left(\frac{1-\epsilon}{4} \right) \log x} \right) \leq \frac{1}{2} \left(\frac{\log x}{\left(\frac{1-\epsilon}{4} \right) \log x} \right) = \frac{2}{1-\epsilon} < 4 \text{ and}$$

$$\frac{1}{2} \left(\frac{\log x - \log \log x + \log \left(\frac{C}{\phi(f(K))} \right)}{\left(\frac{1-\epsilon}{4} \right) \log x} \right) \geq \frac{1}{2} \left(\frac{\log x(1-\epsilon')}{\left(\frac{1-\epsilon}{4} \right) \log x} \right) = \frac{2(1-\epsilon')}{1-\epsilon} > 2,$$

so that $f \left(\frac{1}{2} \frac{\log \left(\frac{Cx}{\phi(f(K)) \log x} \right)}{\log x^{\frac{1-\epsilon}{4}}} \right)$ is bounded below by a positive constant.

Thus there exist constants $C_1, C_2 > 0$ such that for large enough x ,

$$S(\mathfrak{A}, P, x^{\frac{1-\epsilon}{4}}) \geq \left(C_1 \frac{x}{\log x} - C_2 \frac{x}{(\log x)^{\frac{15}{14}}} \right) \prod_{p \leq x^{\frac{1-\epsilon}{4}}} \left(1 - \frac{1}{p} \right)$$

$$\geq \left(C_1 \frac{x}{\log x} - C_2 \frac{x}{(\log x)^{\frac{15}{14}}} \right) \frac{e^{-\gamma}}{\log(x^{\frac{1-\epsilon}{4}})} \left(1 + \mathcal{O} \left(\frac{1}{\log(x^{\frac{1-\epsilon}{4}})} \right) \right)$$

by Mertens' Theorem (see [10], p. 128), and so there exists some $k > 0$ such that for large enough x ,

$$S(\mathfrak{A}, P, x^{\frac{1-\epsilon}{4}}) \geq k \frac{x}{\log^2 x}. \quad \square$$

6. PROOF OF THEOREM 1

In order to prove Theorem 1, we first need to state the following definition and the Gupta-Murty bound [5].

Definition 2. If $\mathfrak{M}$ is a monoid in $\mathcal{O}_K$ such that its elements are relatively prime to an ideal I , then we define $f_{\mathfrak{M}}(I)$ to be the size of the image of $\mathfrak{M}$ in $(\mathcal{O}_K/\mathfrak{p})^\times$. We define $f(I)$ to be $f_{\mathcal{O}_K^\times}(I)$.

Proposition 1 (The Gupta-Murty bound [5]). *If $\mathfrak{M}$ is a monoid in $\mathcal{O}_K^\times$ containing t multiplicatively independent elements, then*

$$|\{\mathfrak{p} : f_{\mathfrak{M}}(\mathfrak{p}) \leq x\}| \ll x^{\frac{t+1}{t}}.$$

We can now prove Theorem 1.

Proof of Theorem 1. Recall that, by Lemma 5,

$$\left| \left\{ p \leq x : \begin{array}{l} p \equiv a \pmod{f(K)} \\ \left(\frac{p-1}{d}, 2f(K) \right) = 1 \\ l \mid \frac{p-1}{d} \Rightarrow l = 1 \text{ or } l > x^{\frac{1-\epsilon}{4}} \end{array} \right\} \right| \geq k \frac{x}{\log^2 x} \text{ for some } k > 0.$$

For the following, suppose that p is one of the primes in the above set and that $\mathfrak{p}$ lies above it in K . Since $p \equiv a \pmod{f(K)}$, $\text{Nm}(\mathfrak{p}) = p$ and $f(\mathfrak{p}) \mid (p-1)$. Note that $\mathcal{O}_K^\times \twoheadrightarrow (\mathcal{O}_K^\times/\mathfrak{p})^\times$ if and only if $f(\mathfrak{p}) = p-1$.

As $d \nmid f(\mathfrak{p})$ if and only if $p \nmid \text{Nm}(1 - \zeta_d^{r-1})$ for some r such that $(r, d) = 1$, $d \nmid f(\mathfrak{p})$ implies that $p \nmid d$. This is a contraction as $p \equiv a \equiv 1 \pmod{d}$, so $d \mid f(\mathfrak{p})$, and we can see that if $l \mid \frac{p-1}{f(\mathfrak{p})}$, then $l = 1$ or $l > x^{\frac{1-\epsilon}{4}}$. Thus either $\frac{p-1}{f(\mathfrak{p})} = 1$ or $\frac{p-1}{f(\mathfrak{p})} > x^{\frac{1-\epsilon}{4}}$.

If $\frac{p-1}{f(\mathfrak{p})} \neq 1$, we can see that $x^{\frac{3+\epsilon}{4}} > f(\mathfrak{p})$. The Gupta-Murty bound implies that $\left| \{ \mathfrak{p} : f(\mathfrak{p}) \leq x^{\frac{3+\epsilon}{4}} \} \right| \ll x^{\frac{3+\epsilon}{4} \cdot \frac{5}{4}} = x^{\frac{15+5\epsilon}{16}}$ since we assumed that $\text{rank}(\mathcal{O}_K^\times) \geq 4$.

This implies that

$$\left| \left\{ p \leq x : \begin{array}{l} p \equiv a \pmod{f(K)} \\ \left(\frac{p-1}{d}, 2f(K)\right) = 1 \\ f(\mathfrak{p}) = p-1 \end{array} \right\} \right| \gg \frac{x}{\log^2 x},$$

so

$$\left| \left\{ p \leq x : \begin{array}{l} [\mathfrak{p}] = [C] \\ \mathcal{O}_K^\times \rightarrow (\mathcal{O}_K/\mathfrak{p})^\times \end{array} \right\} \right| \gg \frac{x}{\log^2 x},$$

and thus $[C]$ is a Euclidean ideal class by Theorem 2. $\square$

7. APPLICATION

When Lenstra defined Euclidean ideals, he was initially inspired by rings for which the algebraic norm of its elements is a Euclidean algorithm, leading him to define norm-Euclidean ideals [9].

Definition 3. If K is a number field and C is a fractional ideal of $\mathcal{O}_K$, then C is norm-Euclidean if for all $x \in K$, there exists some $y \in C$ such that

$$\text{Nm}(x - y) < \text{Nm}(C).$$

One can check that this is equivalent to $\psi = \text{Nm}$ in Definition 1. If C is norm-Euclidean, then we say that $[C]$ is a norm-Euclidean ideal class. A ring can have at most one norm-Euclidean ideal class [9].

In the same paper, Lenstra showed that K has a non-principal Euclidean ideal if $K = \mathbb{Q}(\sqrt{d})$, for $d = -20, -15, 40, 60$ and 85 [9]. In each of these situations, the class number is two and the generating ideal is norm-Euclidean. These examples were found without assuming GRH [9], [2]. The only other example in the literature that does not assume GRH is $\mathbb{Q}(\sqrt{2}, \sqrt{35})$, which has class number two [4]. It is unknown whether the generating ideal is norm-Euclidean.

Proposition 2. *The field $\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})$ has a non-principal Euclidean ideal class that is not norm-Euclidean.*

Proof. If one enters the commands

```
1 sage: z=sqrt(5) + sqrt(22) + sqrt(21);
2 sage: f=z.minpoly();
3 sage: L.<a>=NumberField(f, 'x');
4 sage: C=L.class_group();C
```

into SAGE [11], then the output is

```
1 Class group of order 4 with structure C4 of
2 Number Field in a with defining polynomial
3 x^8 - 192*x^6 + 8408*x^4 - 70272*x^2 + 163216
```

Thus the class group of $\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})$ is cyclic and of size four. The field $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})$ is an unramified, degree four extension of $\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})$ and is therefore its Hilbert class field.

As

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11})/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^5$$

and $\mathrm{rank}(\mathcal{O}_{\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})}) = 7 > 4$, both generators of the class group of $\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})$ are Euclidean ideal classes by Theorem 1. At most one generator can be norm-Euclidean, so $\mathbb{Q}(\sqrt{5}, \sqrt{21}, \sqrt{22})$ has a non-principal Euclidean ideal class that is not norm-Euclidean. $\square$

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