A NOTE ON THE SPACES OF VARIABLE INTEGRABILITY AND SUMMABILITY OF ALMEIDA AND HÄSTÖ

HENNING KEMPKA AND JAN VYBÍRAL

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ABSTRACT. We address an open problem posed recently by Almeida and Hästö. They defined the spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ of variable integrability and summability and showed that $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is a norm if $q \geq 1$ is constant almost everywhere or if $1/p(x) + 1/q(x) \leq 1$ for almost every $x \in \mathbb{R}^n$. Nevertheless, the natural conjecture (expressed also by Almeida and Hästö) is that the expression is a norm if $p(x), q(x) \geq 1$ almost everywhere. We show that $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is a norm if $1 \leq q(x) \leq p(x)$ for almost every $x \in \mathbb{R}^n$. Furthermore, we construct an example of $p(x)$ and $q(x)$ with $\min(p(x), q(x)) \geq 1$ for every $x \in \mathbb{R}^n$ such that the triangle inequality does not hold for $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$.

1. INTRODUCTION

For the definition of the spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ we closely follow [1]. Spaces of variable integrability $L_{p(\cdot)}$ and variable sequence spaces $\ell_{q(\cdot)}$ were first considered in 1931 by Orlicz [5], but the modern development started with the paper [4]. We refer to [3] for an excellent overview of the vastly growing literature on the subject.

First of all we recall the definition of the variable Lebesgue spaces $L_{p(\cdot)}(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$. A measurable function $p : \Omega \rightarrow (0, \infty]$ is called a variable exponent function if it is bounded away from zero. For a set $A \subset \Omega$ we denote $p_A^+ = \text{ess-sup}_{x \in A} p(x)$ and $p_A^- = \text{ess-inf}_{x \in A} p(x)$; we use the abbreviations $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. The variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all measurable functions f such that there exists an $\lambda > 0$ such that the modular

$$\varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda) = \int_{\Omega} \varphi_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) dx$$

is finite, where

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

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This definition is nowadays standard and was also used in [1, Section 2.2] and [3, Definition 3.2.1].

If we define $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$ and $\Omega_0 = \Omega \setminus \Omega_\infty$, then the Luxemburg norm of a function $f \in L_{p(\cdot)}(\Omega)$ is given by

$$\begin{aligned} \|f\|_{L_{p(\cdot)}(\Omega)} &= \inf\{\lambda > 0 : \varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda) \leq 1\} \\ &= \inf\left\{\lambda > 0 : \int_{\Omega_0} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1 \text{ and } |f(x)| \leq \lambda \text{ for a.e. } x \in \Omega_\infty\right\}. \end{aligned}$$

If $p(\cdot) \geq 1$, then it is a norm, but it is always a quasi-norm if at least $p^- > 0$; see [4] for details. We denote the class of all measurable functions $p : \mathbb{R}^n \rightarrow (0, \infty]$ such that $p^- > 0$ by $\mathcal{P}(\mathbb{R}^n)$ and the corresponding modular is denoted by $\varrho_{p(\cdot)}$ instead of $\varrho_{L_{p(\cdot)}(\mathbb{R}^n)}$.

To define the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ we have to define another modular. For $p, q \in \mathcal{P}(\mathbb{R}^n)$ and a sequence $(f_\nu)_{\nu \in \mathbb{N}_0}$ of $L_{p(\cdot)}(\mathbb{R}^n)$ functions we define

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf\left\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \leq 1\right\},$$

where we put $\lambda^{1/\infty} := 1$. The (quasi-) norm in the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces is defined as usual by

$$\|f_\nu\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} = \inf\{\mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \leq 1\}.$$

This (quasi-)norm was used in [1] to define the spaces of Besov type with variable integrability and summability. Spaces of Triebel-Lizorkin type with variable indices have recently been considered in [2]. The appropriate $L_{p(\cdot)}(\ell_{q(\cdot)})$ space is a normed space whenever $\text{ess-inf}_{x \in \mathbb{R}^n} \min(p(x), q(x)) \geq 1$. This was the expected result and coincides with the case of constant exponents.

As pointed out in the remark after Theorem 3.8 in [1], the same question is still open for the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces.

2. WHEN DOES $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ DEFINE A NORM?

In Theorem 3.6 of [1] the authors proved that if the condition $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ holds for almost every $x \in \mathbb{R}^n$, then $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ defines a norm. They also proved in Theorem 3.8 that $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is a quasi-norm for all $p, q \in \mathcal{P}(\mathbb{R}^n)$. Furthermore, the authors of [1] posed a question if the (rather natural) condition $p(x), q(x) \geq 1$ for almost every $x \in \mathbb{R}^n$ ensures that $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is a norm.

We give (in Theorem 1) a positive answer if $1 \leq q(x) \leq p(x) \leq \infty$ almost everywhere on $\mathbb{R}^n$. Furthermore, in Theorem 2, we construct two functions $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\inf_{x \in \mathbb{R}^n} \min(p(x), q(x)) \geq 1$, but the triangle inequality does not hold for $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$.

2.1. Positive results. We summarize in the following theorem all the cases when the expression $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is known to be a norm. We include the proof of the case discussed already in [1] for the sake of completeness.

Theorem 1. *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ such that either $p(x) \geq 1$ and $q \geq 1$ is constant almost everywhere or $1 \leq q(x) \leq p(x) \leq \infty$ for almost every $x \in \mathbb{R}^n$, or $1/p(x) + 1/q(x) \leq 1$ for almost every $x \in \mathbb{R}^n$. Then $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ defines a norm.*

Proof. If $p(x) \geq 1$ and $q \geq 1$ is constant almost everywhere, then the proof is trivial.

In the remaining cases, we want to show that

$$\|f_\nu + g_\nu\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \leq \|f_\nu\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} + \|g_\nu\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$$

for all sequences of measurable functions $\{f_\nu\}_{\nu \in \mathbb{N}_0}$ and $\{g_\nu\}_{\nu \in \mathbb{N}_0}$. Let $\mu_1 > 0$ and $\mu_2 > 0$ be given with

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})} \left(\frac{f_\nu}{\mu_1} \right) \leq 1 \quad \text{and} \quad \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})} \left(\frac{g_\nu}{\mu_2} \right) \leq 1.$$

We want to show that

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})} \left(\frac{f_\nu + g_\nu}{\mu_1 + \mu_2} \right) \leq 1.$$

For every $\varepsilon > 0$, there exist sequences of positive numbers $\{\lambda_\nu\}_{\nu \in \mathbb{N}_0}$ and $\{\Lambda_\nu\}_{\nu \in \mathbb{N}_0}$ such that

$$(1) \quad \varrho_{p(\cdot)} \left(\frac{f_\nu(x)}{\mu_1 \lambda_\nu^{1/q(x)}} \right) \leq 1 \quad \text{and} \quad \varrho_{p(\cdot)} \left(\frac{g_\nu(x)}{\mu_2 \Lambda_\nu^{1/q(x)}} \right) \leq 1,$$

together with

$$\sum_{\nu=0}^{\infty} \lambda_\nu \leq 1 + \varepsilon \quad \text{and} \quad \sum_{\nu=0}^{\infty} \Lambda_\nu \leq 1 + \varepsilon.$$

We set

$$A_\nu := \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2}, \quad \text{i.e.} \quad \sum_{\nu=0}^{\infty} A_\nu \leq 1 + \varepsilon.$$

We shall prove that

$$(2) \quad \varrho_{p(\cdot)} \left(\frac{f_\nu(x) + g_\nu(x)}{A_\nu^{1/q(x)} (\mu_1 + \mu_2)} \right) \leq 1 \quad \text{for all } \nu \in \mathbb{N}_0.$$

Let $\Omega_0 := \{x \in \mathbb{R}^n : p(x) < \infty\}$ and $\Omega_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\}$. We put for every $x \in \Omega_0$

$$F_\nu(x) := \left(\frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \right)^{p(x)} \quad \text{and} \quad G_\nu(x) := \left(\frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \right)^{p(x)}.$$

Then (1) may be reformulated as

$$(3) \quad \int_{\Omega_0} F_\nu(x) dx \leq 1 \quad \text{and} \quad \operatorname{ess-sup}_{x \in \Omega_\infty} \frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \leq 1$$

and

$$(4) \quad \int_{\Omega_0} G_\nu(x) dx \leq 1 \quad \text{and} \quad \operatorname{ess-sup}_{x \in \Omega_\infty} \frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \leq 1.$$

Our aim is to prove (2), which reads

$$(5) \quad \int_{\Omega_0} \left(\frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)} (\mu_1 + \mu_2)} \right)^{p(x)} dx \leq 1 \quad \text{and} \quad \operatorname{ess-sup}_{x \in \Omega_\infty} \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)} (\mu_1 + \mu_2)} \leq 1.$$

We first prove the second part of (5). First we observe that (3) and (4) imply that

$$|f_\nu(x)| \leq \mu_1 \lambda_\nu^{1/q(x)} \quad \text{and} \quad |g_\nu(x)| \leq \mu_2 \Lambda_\nu^{1/q(x)}$$

hold for almost every $x \in \Omega_\infty$. Using $q(x) \geq 1$ and Hölder's inequality in the form

$$\frac{\mu_1 \lambda_\nu^{1/q(x)} + \mu_2 \Lambda_\nu^{1/q(x)}}{\mu_1 + \mu_2} \leq \left(\frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2} \right)^{1/q(x)},$$

we get

$$\frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \leq 1.$$

If $q(x) = \infty$, only notational changes are necessary.

Next we prove the first part of (5). Let $1 \leq q(x) \leq p(x) < \infty$ for almost all $x \in \Omega_0$. Then we use Hölder's inequality in the form

$$\begin{aligned} (6) \quad & F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2 \\ & \leq (\mu_1 + \mu_2)^{1-1/q(x)} (\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu)^{1/q(x)-1/p(x)} (F_\nu(x) \lambda_\nu \mu_1 + G_\nu(x) \Lambda_\nu \mu_2)^{1/p(x)}. \end{aligned}$$

If $1/p(x) + 1/q(x) \leq 1$ for almost every $x \in \Omega_0$, then we replace (6) by

$$\begin{aligned} (7) \quad & F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2 \\ & \leq (\mu_1 + \mu_2)^{1-1/p(x)-1/q(x)} (\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu)^{1/q(x)} (F_\nu(x) \mu_1 + G_\nu(x) \mu_2)^{1/p(x)}. \end{aligned}$$

Using (6), we may continue:

$$\begin{aligned} & \int_{\Omega_0} \left(\frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} dx \\ &= \int_{\Omega_0} \left(\frac{F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2}{\mu_1 + \mu_2} \right)^{p(x)} \cdot \left(\frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2} \right)^{-\frac{p(x)}{q(x)}} dx \\ &\leq \int_{\Omega_0} \frac{F_\nu(x) \lambda_\nu \mu_1 + G_\nu(x) \Lambda_\nu \mu_2}{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu} dx \\ &= \frac{\mu_1 \lambda_\nu}{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu} \int_{\Omega_0} F_\nu(x) dx + \frac{\mu_2 \Lambda_\nu}{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu} \int_{\Omega_0} G_\nu(x) dx \leq 1, \end{aligned}$$

where we also used (3) and (4). If we start with (7) instead, we proceed in the following way:

$$\begin{aligned} & \int_{\Omega_0} \left(\frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} dx \\ &= \int_{\Omega_0} \left(\frac{F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2}{\mu_1 + \mu_2} \right)^{p(x)} \cdot \left(\frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2} \right)^{-\frac{p(x)}{q(x)}} dx \\ &\leq \int_{\Omega_0} \frac{F_\nu(x) \mu_1 + G_\nu(x) \mu_2}{\mu_1 + \mu_2} dx = \frac{\mu_1}{\mu_1 + \mu_2} \int_{\Omega_0} F_\nu(x) dx + \frac{\mu_2}{\mu_1 + \mu_2} \int_{\Omega_0} G_\nu(x) dx \leq 1. \end{aligned}$$

In both cases, this finishes the proof of (5). $\square$

Remark 1. (i) A simpler proof of Theorem 1 is possible (and was proposed to us by the referee) if $1 \leq q(x) \leq p(x) \leq \infty$. Namely, if $1 \leq q \leq p \leq \infty$, $\lambda > 0$ and $t \geq 0$, then

$$(8) \quad \varphi_p \left(\frac{t}{\lambda^{1/q}} \right) = \varphi_{\frac{p}{q}} \left(\frac{\varphi_q(t)}{\lambda} \right),$$

where we use the convention that $\frac{p}{q} = 1$ if $p = q = \infty$. This allows us to simplify the modular $\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ to

$$(9) \quad \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \left\| \varphi_{q(\cdot)}(|f_\nu|) \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

This shows that $\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu)$ is a composition of only convex functions. Hence, it is a convex modular, and therefore it induces a norm. Unfortunately, we were not able to find such a simplification for the case $1/p(x) + 1/q(x) \leq 1$. The advantage of our proof of Theorem 1 is that it proves both cases in a unified way.

- (ii) Let us observe that (8) loses its sense if $p < q = \infty$. This shows why (9) (which was already used in [1] for $q^+ < \infty$) has to be applied with certain care.
- (iii) The method of the proof of Theorem 1 can actually be used to show that under the conditions posed on $p(\cdot)$ and $q(\cdot)$ in Theorem 1, $\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ is a convex modular, which is a stronger result than the norm property.

2.2. Counterexample.

Theorem 2. *There exist functions $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $\inf_{x \in \mathbb{R}^n} p(x) \geq 1$ and $\inf_{x \in \mathbb{R}^n} q(x) \geq 1$ such that $\|\cdot\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$ does not satisfy the triangle inequality.*

Proof. Let $Q_0, Q_1 \subset \mathbb{R}^n$ be two disjoint unit cubes, let $p(x) := 1$ everywhere on $\mathbb{R}^n$, and put $q(x) := \infty$ for $x \in Q_1$ and $q(x) := 1$ for $x \notin Q_1$. Let $f_1 = \chi_{Q_0}$ and $f_2 = \chi_{Q_1}$. Finally, we put $f = (f_1, f_2, 0, \dots)$ and $g = (f_2, f_1, 0, \dots)$.

We calculate for every $L > 0$ fixed

$$\inf \left\{ \lambda_1 > 0 : \varrho_{p(\cdot)} \left(\frac{f_1(x)}{\lambda_1^{1/q(x)} L} \right) \leq 1 \right\} = \inf \left\{ \lambda_1 > 0 : \frac{1}{\lambda_1 L} \leq 1 \right\} = 1/L$$

and

$$\inf \left\{ \lambda_2 > 0 : \varrho_{p(\cdot)} \left(\frac{f_2(x)}{\lambda_2^{1/q(x)} L} \right) \leq 1 \right\} = \inf \left\{ \lambda_2 > 0 : \frac{1}{L} \leq 1 \right\}.$$

If $L \geq 1$, then the last expression is equal to zero; otherwise it is equal to ∞ .

We obtain

$$\|f\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} = \inf \{ L > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f/L) \leq 1 \} = \inf \{ L > 0 : 1/L + 0 \leq 1 \} = 1,$$

and the same is also true for $\|g\|_{\ell_{q(\cdot)}(L_{p(\cdot)})}$. It is therefore enough to show that $\|f + g\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} > 2$.

Using the calculation

$$\begin{aligned} \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left(\frac{f_1(x) + f_2(x)}{L \cdot \lambda^{1/q(x)}} \right) \leq 1 \right\} &= \inf \left\{ \lambda > 0 : \int_{Q_0} \frac{1}{L \cdot \lambda} + \int_{Q_1} \frac{1}{L} \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \frac{1}{L \cdot \lambda} + \frac{1}{L} \leq 1 \right\} = \frac{1}{L-1}, \end{aligned}$$

which holds for every $L > 1$ fixed, we get

$$\begin{aligned} \|f + g\|_{\ell_{q(\cdot)}(L_{p(\cdot)})} &= \inf \left\{ L > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})} \left(\frac{f+g}{L} \right) \leq 1 \right\} \\ &= \inf \left\{ L > 0 : 2 \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left(\frac{f_1(x) + f_2(x)}{L \cdot \lambda^{1/q(x)}} \right) \leq 1 \right\} \leq 1 \right\} \\ &= \inf \left\{ L > 1 : 2 \cdot \frac{1}{L-1} \leq 1 \right\} = 3. \end{aligned}$$

□

Remark 2. Let us observe that $1 \leq q(x) \leq p(x) \leq \infty$ holds for $x \in Q_0$ and that $1/p(x) + 1/q(x) \leq 1$ is true for $x \in Q_1$. It is therefore necessary to interpret the assumptions of Theorem 1 in a correct way, namely, that one of the conditions of Theorem 1 holds for (almost) all $x \in \mathbb{R}^n$. This is not to be confused with the statement that for (almost) every $x \in \mathbb{R}^n$ at least one of the conditions is satisfied, which is not sufficient.

Remark 3. A similar calculation (which we shall not repeat in detail) shows that one may also put $q(x) := q_0$ large enough for $x \in Q_1$ to obtain a counterexample. Hence there is nothing special about the infinite value of q , and the same counterexample may be reproduced with uniformly bounded exponents $p, q \in \mathcal{P}(\mathbb{R}^n)$.

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MATHEMATICAL INSTITUTE, FRIEDRICH-SCHILLER-UNIVERSITY, D-07737 JENA, GERMANY
E-mail address: henning.kempka@uni-jena.de

RICAM, AUSTRIAN ACADEMY OF SCIENCES, ALTENBERGSTRASSE 69, A-4040 LINZ, AUSTRIA
E-mail address: jan.vybiral@oeaw.ac.at