ON THE VOLUME SET OF POINT SETS IN VECTOR SPACES OVER FINITE FIELDS

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ABSTRACT. We show that if $\mathcal{E}$ is a subset of the d -dimensional vector space over a finite field $\mathbb{F}_q$ ($d \geq 3$) of cardinality $|\mathcal{E}| \geq (d-1)q^{d-1}$, then the set of volumes of d -dimensional parallelepipeds determined by $\mathcal{E}$ covers $\mathbb{F}_q$. This bound is sharp up to a factor of $(d-1)$, as taking $\mathcal{E}$ to be a $(d-1)$ -hyperplane through the origin shows.

1. INTRODUCTION

Let q be an odd prime power, and let $\mathbb{F}_q$ be a finite field of q elements. The distribution of the determinant of matrices with entries in a restricted subset of $\mathbb{F}_q$ has been studied recently by various researchers (see, for example, [1, 2, 5, 6] and the references therein). In particular, Covert et al. [2] studied this problem in a more general setting. They examined the distribution of volumes of d -dimensional parallelepipeds determined by a large subset of $\mathbb{F}_q^d$. More precisely, for any $\mathbf{x}^1, \dots, \mathbf{x}^d \in \mathbb{F}_q^d$, define $\text{vol}(\mathbf{x}^1, \dots, \mathbf{x}^d)$ as the determinant of the matrix whose rows are $\mathbf{x}^j$ s. The focus of [2] is to study the cardinality of the volume set

$$\text{vol}(\mathcal{E}) = \{\text{vol}(\mathbf{x}^1, \dots, \mathbf{x}^d) : \mathbf{x}^j \in \mathcal{E}\}.$$

A subset $\mathcal{E} \subseteq \mathbb{F}_q^d$ is called a product-like set if $|\mathcal{E} \cap \mathcal{H}_n| \ll |\mathcal{E}|^{n/d}$ for any n -dimensional subspace $\mathcal{H}_n \subset \mathbb{F}_q^d$. Covert et al. [2] showed that if $\mathcal{E} \subseteq \mathbb{F}_q^d$ is a product-like set of cardinality $|\mathcal{E}| \gg q^{15/8}$, then $\mathbb{F}_q^* \subseteq \text{vol}(\mathcal{E})$. When $\mathcal{E} \subseteq \mathbb{F}_q^3$ is an arbitrary set, they obtained the following result.

Theorem 1.1 ([2, Theorem 2.10]). *Suppose that $\mathcal{E} \subseteq \mathbb{F}_q^3$ of cardinality $|\mathcal{E}| \geq Cq^2$ for a sufficiently large constant $C > 0$. There exists $c > 0$ such that*

$$|\text{vol}(\mathcal{E})| \geq cq.$$

In this short note, we show that under the same condition, $\mathcal{E}$ determines all possible volumes. More precisely, we will prove the following general result.

Theorem 1.2. *When $\mathcal{E} \subseteq \mathbb{F}_q^d$ and $|\mathcal{E}| \geq (d-1)q^{d-1}$, $\text{vol}(\mathcal{E}) = \mathbb{F}_q$.*

Remark 1.3. The assumption $|\mathcal{E}| \geq (d-1)q^{d-1}$ is sharp up to a factor of $(d-1)$, as taking $\mathcal{E}$ to be a $(d-1)$ -hyperplane through the origin shows.

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Note that the implied constant in the symbol ‘ $\gg$ ’ may depend on the integer parameter d . We recall that the notation $U \gg V$ is equivalent to the assertion that $|U| \geq c|V|$ holds for some constant $c > 0$.

2. PREPARATIONS

Recall that

$$\text{vol}(\mathbf{x}^1, \dots, \mathbf{x}^d) = \mathbf{x}^1 \cdot (\mathbf{x}^2 \wedge \dots \wedge \mathbf{x}^d),$$

where the dot product is defined by the usual formula

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \dots + u_d v_d.$$

The generalized cross product, also called the wedge product, is given by the identity

$$\mathbf{u}^2 \wedge \dots \wedge \mathbf{u}^d = \det \begin{pmatrix} \mathbf{i} \\ \mathbf{u}^2 \\ \vdots \\ \mathbf{u}^d \end{pmatrix},$$

where $\mathbf{i} = (i^1, \dots, i^d)$ indicates the coordinate directions in $\mathbb{F}_q^d$.

2.1. Geometric incidence estimates. One of our main tools is a two-set version of the geometric incidence estimate developed by D. Hart, A. Iosevich, D. Koh, and M. Rudnev in [4] (see also [3] for an earlier version and [2] for a function version of this estimate). Note that going from one-set formulation in the proof of [4, Theorem 2.1] to a two-set formulation is just a matter of inserting a different letter into a couple of places.

Lemma 2.1 ([4, Theorem 2.1]). *Let $B(\cdot, \cdot)$ be a nondegenerate bilinear form in $\mathbb{F}_q^d$. For any $\mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^d$, let*

$$\nu_{t,B}(\mathcal{E}, \mathcal{F}) = \sum_{B(\mathbf{x}, \mathbf{y})=t} \mathcal{E}(\mathbf{x})\mathcal{F}(\mathbf{y}),$$

where here and throughout the paper $\mathcal{E}(\mathbf{x})$ denotes the characteristic function of $\mathcal{E}$. We have

$$\nu_{t,B}(\mathcal{E}, \mathcal{F}) = |\mathcal{E}||\mathcal{F}|q^{-1} + R_{t,B}(\mathcal{E}, \mathcal{F}),$$

where

$$|R_{t,B}(\mathcal{E}, \mathcal{F})|^2 \leq |\mathcal{E}||\mathcal{F}|q^{d-1}, \text{ if } t \neq 0.$$

As a corollary of Lemma 2.1, D. Hart and A. Iosevich [3] derived the following result.

Corollary 2.2 ([3, 4]). *For any $\mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^d$, let*

$$\mathcal{E} \cdot \mathcal{F} = \{\mathbf{u} \cdot \mathbf{v} : \mathbf{u} \in \mathcal{E}, \mathbf{v} \in \mathcal{F}\}.$$

We have $\mathbb{F}_q^ \subseteq \mathcal{E} \cdot \mathcal{F}$ when $|\mathcal{E}||\mathcal{F}| \gg q^{d+1}$.*

We also need the following corollary.

Corollary 2.3. *Let $B(\cdot, \cdot)$ be a nondegenerate bilinear form in $\mathbb{F}_q^d \times \mathbb{F}_q^d$. For any $\mathcal{E} \subset \mathbb{F}_q^d \setminus (0, \dots, 0)$, let*

$$B^*(\mathcal{E}) = \{B(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in \mathcal{E}\} \setminus \{0\}.$$

We have

$$|B^*(\mathcal{E})| \geq q \left(1 - \frac{q + q^{3/2}}{|\mathcal{E}| + q^{3/2}} \right).$$

Proof. For any $\mathbf{x} \in \mathcal{E}$, there exist at most q vectors $\mathbf{y}$ such that $\mathbf{x} \cdot \mathbf{y} = 0$. Hence,

$$(2.1) \quad \sum_{t \in \mathbb{F}_q^*} \nu_{t,B}(\mathcal{E}, \mathcal{E}) \geq |\mathcal{E}|^2 - q|\mathcal{E}|.$$

Lemma 2.1 implies that

$$(2.2) \quad \nu_{t,B}(\mathcal{E}, \mathcal{E}) \leq \frac{|\mathcal{E}|^2}{q} + q^{1/2}|\mathcal{E}|,$$

for any $t \in \mathbb{F}_q^*$. The corollary follows immediately from (2.1) and (2.2). $\square$

2.2. Cross-product set. Let $\mathcal{H}$ be the d -dimensional vector space over a finite field $\mathbb{F}_q$. Let $\{\mathbf{v}^1, \dots, \mathbf{v}^d\}$ be an orthogonal basis of $\mathcal{H}$. For any d vectors $\mathbf{u}^1, \dots, \mathbf{u}^d \in \mathcal{H}$, each vector $\mathbf{u}^i$ can be written uniquely as a linear combination of $\{\mathbf{v}^1, \dots, \mathbf{v}^d\}$, i.e.

$$\mathbf{u}^i = \sum_{j=1}^d u_j^i \mathbf{v}^j, \quad u_j^i \in \mathbb{F}_q, 1 \leq j \leq d.$$

We have

$$(2.3) \quad \mathbf{u}^1 \wedge \mathbf{u}^2 \wedge \dots \wedge \mathbf{u}^d = \det((u_j^i)_{i,j=1}^d) \mathbf{v}^1 \wedge \dots \wedge \mathbf{v}^d.$$

For any $\mathcal{E} \subseteq \mathcal{H}$, define

$$(2.4) \quad \mathcal{D}_{\mathcal{E},d}^* := \left\{ \det((u_j^i)_{i,j=1}^d) : \mathbf{u}^i = \sum_{j=1}^d u_j^i \mathbf{v}^j \in \mathcal{E}, 1 \leq i \leq d \right\} \setminus \{0\}.$$

For any $\mathbf{x} \in \mathbb{F}_q^d$ and $\mathcal{E} \subseteq \mathbb{F}_q^d$, let

$$g_{\mathcal{E}}(\mathbf{x}) = \#\{(\mathbf{u}^1, \dots, \mathbf{u}^{d-1}) \in \mathcal{E}^{d-1} : \mathbf{u}^1 \wedge \dots \wedge \mathbf{u}^{d-1} = \mathbf{x}\}.$$

Define the cross-product set of $\mathcal{E}$:

$$\mathcal{F}_{\mathcal{E}}^* = \{\mathbf{x} \in \mathbb{F}_q^d : g_{\mathcal{E}}(\mathbf{x}) \neq 0\} \setminus \{(0, \dots, 0)\}.$$

For any $\mathbf{x} \in \mathbb{F}_q^d \setminus \{(0, \dots, 0)\}$, let $\mathcal{H}^{\mathbf{x}} := \mathbf{x}^{\perp} = \{\mathbf{y} \in \mathbb{F}_q^d : \mathbf{x} \cdot \mathbf{y} = 0\}$. It is clear that $\mathbf{x} \in \mathcal{F}_{\mathcal{E}} \setminus \{(0, \dots, 0)\}$ if and only if there exist $\mathbf{u}^1, \dots, \mathbf{u}^{d-1} \in \mathcal{H}^{\mathbf{x}} \cap \mathcal{E}$ such that

$$(2.5) \quad \mathbf{u}^1 \wedge \dots \wedge \mathbf{u}^{d-1} = \mathbf{x}.$$

It follows from (2.3), (2.4), and (2.5) that

$$\mathcal{F}_{\mathcal{E}}^* \cap \{l\mathbf{x} : l \in \mathbb{F}_q^*\} = \mathcal{D}_{\mathcal{E} \cap \mathcal{H}^{\mathbf{x}}, d-1}^*.$$

Hence, we have proved the following lemma.

Lemma 2.4. *For any $\mathcal{E} \subseteq \mathbb{F}_q^d$, we have*

$$|\mathcal{F}_{\mathcal{E}}^*| = \sum_{H \in G(d-1, d)} |\mathcal{D}_{\mathcal{E} \cap H, d-1}^*|,$$

where $G(d-1, d)$ is the set of all $(d-1)$ -dimensional subspaces of $\mathbb{F}_q^d$.

3. PROOF OF THEOREM 1.2

The proof proceeds by induction. We first consider the base case, $d = 3$. We show that if $|\mathcal{E}| > 2q^2$, then the cross-product set $\mathcal{F}_{\mathcal{E}}^*$ is large. From Lemma 2.4, we have

$$(3.1) \quad |\mathcal{F}_{\mathcal{E}}^*| = \sum_{\mathcal{H} \in G(2,3)} |\mathcal{D}_{\mathcal{E} \cap \mathcal{H}, 2}^*|.$$

Since each non-zero vector lies in $(q+1)$ two-dimensional subspaces of $\mathbb{F}_q^2$,

$$\sum_{\mathcal{H} \in G(2,3)} |\mathcal{E} \cap \mathcal{H}| = (q+1)|\mathcal{E}|.$$

Let

$$G_{(2,3)}^{\mathcal{E}} = \{\mathcal{H} \in G(2,3) : |\mathcal{E} \cap \mathcal{H}| > q\};$$

we have

$$\sum_{\mathcal{H} \in G_{(2,3)}^{\mathcal{E}}} |\mathcal{E} \cap \mathcal{H}| > (q+1)|\mathcal{E}| - q|G(2,3)| = (q+1)|\mathcal{E}| - q(q^2 + q + 1) > q^3.$$

Corollary 2.3 implies that

$$|\mathcal{D}_{\mathcal{E} \cap \mathcal{H}, 2}^*| \geq q \left(1 - \frac{q + q^{3/2}}{|\mathcal{E} \cap \mathcal{H}| + q^{3/2}} \right),$$

for any $\mathcal{H} \in G(2,3)$. Since

$$f(x) = 1 - \frac{q + q^{3/2}}{x + q^{3/2}}$$

is a concave function on $[q, q^2]$,

$$\begin{aligned} \sum_{\mathcal{H} \in G_{(2,3)}^{\mathcal{E}}} |\mathcal{D}_{\mathcal{E} \cap \mathcal{H}, 2}^*| &\geq q \sum_{\mathcal{H} \in G_{(2,3)}^{\mathcal{E}}} \left(1 - \frac{q + q^{3/2}}{|\mathcal{E} \cap \mathcal{H}| + q^{3/2}} \right) \\ &\geq q \frac{\sum_{\mathcal{H} \in G_{(2,3)}^{\mathcal{E}}} |\mathcal{E} \cap \mathcal{H}|}{q^2} \left(1 - \frac{q + q^{3/2}}{q^2 + q^{3/2}} \right) \\ (3.2) \quad &> q^2 (1 - q^{-1/2}) > q^2/2. \end{aligned}$$

It follows from (3.1) and (3.2) that $|\mathcal{F}_{\mathcal{E}}^*| > q^2/2$. Hence, $|\mathcal{E}||\mathcal{F}_{\mathcal{E}}^*| > q^4$. Corollary 2.2 implies that $\mathbb{F}_q^* \subseteq \mathcal{E} \cdot \mathcal{F}_{\mathcal{E}}^* \subseteq \text{vol}(\mathcal{E})$. By choosing a matrix of identical rows, we have $0 \in \text{vol}(\mathcal{E})$. The base case $d = 3$ follows.

Suppose that the theorem holds for $d-1 \geq 3$; we show that it also holds for d . Similarly, we show that if $|\mathcal{E}| > (d-1)q^{d-1}$, then the cross-product set $\mathcal{F}_{\mathcal{E}}^*$ is large. Since each non-zero vector lies in $(q^{d-1} - 1)/(q-1)$ $(d-1)$ -dimensional subspaces of $\mathbb{F}_q^d$,

$$\sum_{\mathcal{H} \in G(d-1,d)} |\mathcal{E} \cap \mathcal{H}| = \frac{q^{d-1} - 1}{q - 1} |\mathcal{E}|.$$

Let

$$G_{(d-1,d)}^{\mathcal{E}} = \{\mathcal{H} \in G(d-1,d) : |\mathcal{E} \cap \mathcal{H}| > (d-2)q^{d-2}\}.$$

We have

$$\begin{aligned} \sum_{\mathcal{H} \in G_{(d-1,d)}^{\mathcal{E}}} |\mathcal{E} \cap \mathcal{H}| &> \frac{q^{d-1} - 1}{q - 1} |\mathcal{E}| - (d - 2)q^{d-2} |G(d - 1, d)| \\ &= \frac{(q^{d-1} - 1)|\mathcal{E}| - (d - 2)q^{d-2}(q^d - 1)}{q - 1} \\ &> q^d, \end{aligned}$$

when q is sufficiently large (in fact, $q > d$ is enough). Since $|\mathcal{E} \cap \mathcal{H}| \leq q^{d-1}$ for each $\mathcal{H} \in G_{(d-1,d)}^{\mathcal{E}}$,

$$(3.3) \quad |G_{(d-1,d)}^{\mathcal{E}}| > q^d / q^{d-1} = q.$$

By induction hypothesis, for any $\mathcal{H} \in G_{(d-1,d)}^{\mathcal{E}}$,

$$(3.4) \quad |\mathcal{D}_{\mathcal{E} \cap \mathcal{H}, d-1}^*| = q - 1.$$

Putting (3.3), (3.4) and Lemma 2.4 together, we have

$$|\mathcal{F}_{\mathcal{E}}^*| = \sum_{\mathcal{H} \in G_{(d-1,d)}^{\mathcal{E}}} |\mathcal{D}_{\mathcal{E} \cap \mathcal{H}, d-1}^*| > \sum_{\mathcal{H} \in G_{(d-1,d)}^{\mathcal{E}}} |q - 1| > q(q - 1) > q^2/2.$$

Hence, $|\mathcal{E}| |\mathcal{F}_{\mathcal{E}}^*| > q^{d+1}$. Corollary 2.2 implies that $\mathbb{F}_q^* \subseteq \mathcal{E} \cdot \mathcal{F}_{\mathcal{E}}^* \subseteq \text{vol}(\mathcal{E})$. By choosing a matrix of identical rows, we have $0 \in \text{vol}(\mathcal{E})$. This completes the proof of the theorem.

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