CLOSED DENSELY DEFINED OPERATORS COMMUTING WITH MULTIPLICATIONS IN A MULTIPLIER PAIR

DON HADWIN, ZHE LIU, AND ERIC NORDGREN

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Dedicated to the memory of Bill Arveson, an inspiration to us all

ABSTRACT. For a multiplier pair (X, Y) we study the closed densely defined operators T on X that commute with all of the multiplications by right multipliers in X . We apply our general results to special cases involving H^p , completions of $L^\infty[0, 1]$ with respect to certain norms, and the completion of a II_1 factor von Neumann algebra with respect to a unitarily invariant norm, where we show that each such T is a “left multiplication”. However, we give an example of a closed densely defined operator on the Bergman space that commutes with all multiplications by H^∞ -functions but is not a multiplication operator.

1. INTRODUCTION

In [6], [7] the first and third authors introduced and studied the notion of a *multiplier pair* (X, Y) , where X is a Banach space that is a vector subspace of a Hausdorff topological vector space Y with a separately continuous bilinear map (multiplication) $\cdot : X \times X \rightarrow Y$ such that the set $\mathcal{L}_0 = \{x \in X : x \cdot X \subseteq X\}$ of left multipliers and the set $\mathcal{R}_0 = \{x \in X : X \cdot x \subseteq X\}$ of right multipliers are dense in X , and such that there are dense subsets $\mathcal{E} \subseteq \mathcal{L}_0$, $\mathcal{F} \subseteq X$, $\mathcal{G} \subseteq \mathcal{R}_0$ such that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

whenever $a \in \mathcal{E}$, $b \in \mathcal{F}$, $c \in \mathcal{G}$. Moreover, there is an $e \in X$ such that $e \cdot x = x \cdot e = x$ for every $x \in X$. It was shown that, for each $x \in X$, the linear transformations R_x and L_x defined by

$$L_x a = x \cdot a, \quad R_x a = a \cdot x$$

are closed densely defined operators on X . Moreover, L_x is bounded on $\mathcal{R}_0$ if and only if $x \in \mathcal{L}_0$ and R_x is bounded on $\mathcal{L}_0$ if and only if $x \in \mathcal{R}_0$. Thus $\mathcal{R} = \{R_x : x \in \mathcal{R}_0\}$ and $\mathcal{L} = \{L_x : x \in \mathcal{L}_0\}$ are unital subalgebras of $B(X)$ (the set of bounded linear operators on X). It was proved in [6] that $\mathcal{L}$ and $\mathcal{R}$ are each other's commutant, which implies that $\mathcal{L}$ and $\mathcal{R}$ are closed in the weak operator topology. In the setting of multiplier pairs a general notion of a composition operator was defined.

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Suppose $B(W)$ denotes the Banach algebra of all bounded linear transformations on a Banach space W , and suppose $\mathcal{A}, \mathcal{B}$ are unital Banach subalgebras of $B(W)$ such that $\mathcal{B}' = \mathcal{A}$ and $\mathcal{A}' = \mathcal{B}$, where $\mathcal{S}'$ denotes the *commutant* of a set of operators $\mathcal{S}$, i.e., the set of operators commuting with every element of $\mathcal{S}$. Suppose T is a linear transformation whose domain $\mathcal{D}(T)$ and range $\mathcal{R}(T)$ are linear subspaces of W . We say that T *commutes* with $\mathcal{B}$ (or T is *affiliated with* $\mathcal{A}$) if, for every $S \in \mathcal{B}$, we have $S(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$ and, for every $x \in \mathcal{D}(T)$, we have $STx = TSx$. This is equivalent to saying that, for every invertible $S \in \mathcal{B}$, we have $ST = TS$. It is also equivalent to saying that, for every $S \in \mathcal{B}$,

$$(S \oplus S) \operatorname{Graph}(T) \subseteq \operatorname{Graph}(T),$$

where $\operatorname{Graph}(T) = \{(x, y) \in W \times W : y = Tx\}$ is the graph of T . It easily follows that if $\operatorname{Graph}(T)$ is closed, then the set

$$\mathcal{S} = \{S \in B(W) : S(\mathcal{D}(T)) \subseteq \mathcal{D}(T), \forall x \in \mathcal{D}(T) \ STx = TSx\}$$

is a unital algebra that is closed in the weak operator topology. In order to show that $\mathcal{B} \subseteq \mathcal{S}$, it is sufficient to show that $\mathcal{S}$ contains a set of operators $\mathcal{B}_0$ such that the unital weak operator closed algebra generated by $\mathcal{B}_0$ is $\mathcal{B}$.

In this paper we study the problem of determining for a multiplier pair (X, Y) the closed densely defined operators T on X that commute with $\mathcal{R}$ (i.e., are affiliated with $\mathcal{L}$). The symmetry of the situation makes this problem “equivalent” to that of finding the closed densely defined operators on X that commute with $\mathcal{L}$ (i.e., are affiliated with $\mathcal{R}$).

In [6] it was shown that, for every $x \in X$, L_x commutes with every operator in $\mathcal{R}$ and R_x commutes with every operator in $\mathcal{L}$. In general, we expect that the closed densely defined operators commuting with $\mathcal{R}$ should be left multiplications of some sort. We will prove general results that affirm this notion in a large number of cases, but, when X is the Bergman space on the unit disk, we construct a closed densely defined operator commuting with $\mathcal{L} = \mathcal{R} = H^\infty$ that is not a multiplication by any function.

Examples of this problem have been studied by D. Suárez [18] and S. Seubert [17] in the case of the algebra generated by the unilateral shift operator, by D. Sarason [13] in the case of a restricted shift operator [14], [15], and by H. Bercovici [1] in the case of a C_0 -contraction. A result for von Neumann algebras was proved by Nelson [11]. The problem for $L^\infty[0, 1]$ acting on $L^2[0, 1]$ was discussed in [9].

In [6] many examples of multiplier pairs were constructed. Many of them satisfy more conditions than assumed in the definition. We define a multiplier pair (X, Y) to be a *special multiplier pair* if the multiplication $\cdot : Y \times Y \rightarrow Y$ is defined and separately continuous and $(Y, +, \cdot)$ is a ring with identity e .

2. GENERAL RESULTS

We begin with algebraic results that will apply to special multiplier pairs. If $\mathcal{R}$ is a ring and $\mathcal{S} \subseteq \mathcal{R}$, we say that $\mathcal{S}$ is *left-separating* if, for every $x \in \mathcal{R}$, $x \cdot \mathcal{S} = \{0\}$ implies $x = 0$.

Theorem 1. *Suppose $1 \in Y$ is a ring, $1 \in \mathcal{R}$ is a subring of Y and $\mathcal{R} \subset X \subset Y$, where X is a right $\mathcal{R}$ -module. Suppose $G \subset X \times X$ is a graph and a right $\mathcal{R}$ -module*

containing an element (g, h) such that

- (1) g has a left-inverse g^{-1} in Y ,
- (2) for every $x \in X$, $\{u \in \mathcal{R} : \exists v \in \mathcal{R} \text{ such that } xu = gv\}$ is left-separating for Y .

Then $G \subseteq \text{Graph}(L_{hg^{-1}})$.

Proof. Suppose $(x, y) \in G$, $u, v \in \mathcal{R}$ and $xu = gv$. Then $(x, y)u - (g, h)v = (0, yu - hv) \in G$. Since $(0, 0) \in G$ and G is a graph, we know $yu - hv = 0$. Thus $yu = hv = hg^{-1}(gv) = hg^{-1}(xu)$. Hence, for every $u \in \mathcal{R}$ for which there is a $v \in \mathcal{R}$ such that $xu = gv$, we have

$$[y - L_{hg^{-1}}x]u = 0.$$

It follows from (2) that $y = L_{hg^{-1}}x$ for every $(x, y) \in G$. □

If R is an integral domain, the field of fractions $\hat{R}$ of R is the field of formal quotients $\frac{a}{b}$ with $a, b \in R$ and the natural addition and multiplication. We think of $R \subseteq \hat{R}$ by identifying $x \in R$ with $\frac{x}{1} \in \hat{R}$.

Corollary 1. Suppose F is the field of fractions of an integral domain R , and $R \subset X \subset F$ and X is an R -module. Suppose $G \subset X \times X$ is a graph and an R -module. Then there is a $\varphi \in F$ such that G is contained in the graph of $y = \varphi x$ in $F \times F$.

Corollary 2. Suppose (X, Y) is a special multiplier pair such that Y is an integral domain and let $\hat{Y}$ denote the field of quotients over Y . Suppose also that $X \subseteq \hat{\mathcal{R}}_0$. If $G \subseteq X \times X$ is a graph that is an $\mathcal{R}_0$ -module, then there is a $\varphi \in \hat{\mathcal{R}}_0$ such that $G \subseteq \text{Graph}(L_\varphi)$. Moreover, if the domain of G is dense in X and $\tau : Y \rightarrow \mathbb{C}$ is a unital algebra homomorphism whose restriction to X is continuous, then there are $f, g \in \mathcal{R}_0$ such that $\varphi = g/f$ and $\tau(f) \neq 0$.

Proof. The first part follows immediately from Theorem 1. Next, suppose the domain of G is dense and $\tau : Y \rightarrow \mathbb{C}$ is a multiplicative linear functional whose restriction to X is continuous. It follows that the domain of G cannot be contained in $\ker \tau$. Hence there is an $(f, g) \in G$ such that $\tau(f) \neq 0$. Since $g = \varphi f$, we have that $\varphi = g/f$ and $\tau(f) \neq 0$. □

Throughout this paper we use (SOT) to denote the strong operator topology.

Theorem 2. Suppose (X, Y) is a multiplier pair and T is a closed densely defined operator on X commuting with every operator in $\mathcal{R}$ and such that $T \subseteq L_y$ for some $y \in Y$. Suppose also that, for some $g \in \mathcal{D}(T)$ and for every $x \in X$, there exist $\{u_n\}, \{v_n\}$ in $\mathcal{R}_0$ such that

- (1) $xu_n = gv_n \in \mathcal{D}(T) \subseteq \mathcal{D}(L_y)$,
- (2) $R_{u_n} \rightarrow 1$ (SOT).

Then $T = L_y$.

Proof. Suppose $x \in \mathcal{D}(L_y)$, so $yx \in X$, and choose $\{u_n\}$ and $\{v_n\}$ as above. Then $xu_n \rightarrow x$ and $(yx)u_n \rightarrow yx$, but $xu_n = gv_n \in \mathcal{D}(T)$, so $yxu_n = Txu_n$. Hence $(x, yx) = \lim_{n \rightarrow \infty} (xu_n, Txu_n) \in \mathcal{G}(T)^- = \mathcal{G}(T)$. Hence $x \in \mathcal{D}(T)$. Thus $\mathcal{D}(T) = \mathcal{D}(L_y)$, so $T = L_y$. □

Theorem 3. Suppose (X, Y) is a special multiplier pair such that, for each $y \in Y$, there is a sequence $\{q_n\}$ in $\mathcal{R}_0$ such that

- (1) $R_{q_n} \rightarrow 1$ in the (SOT) on X ,
- (2) $yq_n \in \mathcal{R}_0$ for every n .

If T is a closed densely defined operator on X commuting with $\mathcal{R}$ and there is a $g \in \mathcal{D}(T)$ that is invertible in Y , then $T = L_{T(g)g^{-1}}$.

Proof. It follows from condition (1) that $\|e - eq_n\| \rightarrow 0$, which, from the separate continuity of multiplication in Y , implies $w - wq_n = w \cdot (e - eq_n) \rightarrow 0$ and $w - q_nw = (e - eq_n) \cdot w \rightarrow 0$ in Y . Hence $\{q_1, q_2, \dots\}$ is both left-separating and right-separating in Y . Suppose $x \in X$ and let $y = g^{-1}x \in Y$. Choose $\{q_n\}$ as above and note that

$$xq_n = g(yq_n),$$

so, by Theorem 1, we see that $T \subseteq L_{T(g)g^{-1}}$. If $x \in \mathcal{D}(L_{T(g)g^{-1}})$, then $xq_n = R_{yq_n}(g) \in \mathcal{D}(T)$ (since T commutes with $R_{T(g)g^{-1}}$), $\|xq_n - x\| \rightarrow 0$ and $T(xq_n) = L_{T(g)g^{-1}}(xq_n) = (L_{T(g)g^{-1}}x)q_n \rightarrow L_{T(g)g^{-1}}x$. Since the operator T is closed, $(x, L_{T(g)g^{-1}}x)$ is in the graph of T . Hence $T = L_{T(g)g^{-1}}$. $\square$

We now consider a special case in which X is a Hilbert space and $\mathcal{L}$ and $\mathcal{R}$ are von Neumann algebras.

Lemma 1. Suppose (X, Y) is a multiplier pair, X is a Hilbert space and $\mathcal{L}$ is a von Neumann algebra. Suppose T is a closed densely defined operator affiliated with $\mathcal{L}$. Then there are $u, v \in \mathcal{L}$ with $0 \leq u$ and $\ker u = 0$ such that $T \subseteq L_{u^{-1}v}$.

Proof. Since $\text{Graph}(T)$ is closed and invariant for the von Neumann algebra $\mathcal{M} = \{A \oplus A : A \in \mathcal{R}\}$, the projection P from $X \oplus X$ onto $\text{Graph}(T)$ is in the commutant of $\mathcal{M}$, which is $\mathcal{M}_2(\mathcal{R}') = \mathcal{M}_2(\mathcal{L})$. Hence, we can write $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in \mathcal{L}$. If $\begin{pmatrix} x \\ Tx \end{pmatrix} \in \text{Graph}(T)$, then

$$\begin{pmatrix} x \\ Tx \end{pmatrix} = P \begin{pmatrix} x \\ Tx \end{pmatrix} = \begin{pmatrix} Ax + BTx \\ Cx + DTx \end{pmatrix},$$

which implies

$$Tx = Cx + DTx$$

or

$$(1 - D)Tx = Cx.$$

Suppose $x \in X$, $\|x\| = 1$ and $Dx = x$.

Then

$$\begin{aligned} \left\| P \begin{pmatrix} 0 \\ x \end{pmatrix} \right\|^2 &= \left(P \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right) = \left(\begin{pmatrix} Bx \\ Dx \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right) \\ &= (Dx, x) = \left\| \begin{pmatrix} 0 \\ x \end{pmatrix} \right\|^2. \end{aligned}$$

Thus $P \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$, which means $\begin{pmatrix} 0 \\ x \end{pmatrix} \in \text{Graph}(T)$, which, in turn, implies $x = T0 = 0$. Thus $\ker(1 - D) = 0$. We have from $(1 - D)Tx = Cx$ that $Tx = (1 - D)^{-1}Cx$ for every $x \in \mathcal{D}(T)$. $\square$

We now look at a case that will apply to operator algebras with a *strictly cyclic separating vector*. Suppose X is a Banach space and $\mathcal{A}$ is a norm closed unital subalgebra of $B(X)$ and suppose $0 \neq e \in X$ satisfies

- (1) $\mathcal{A}e = X$ (i.e., e is a strictly cyclic vector for $\mathcal{A}$),
- (2) e is a separating vector for $\mathcal{A}$, i.e., for every $A \in \mathcal{A}$,

$$Ae = 0 \implies A = 0.$$

We can define a multiplication $\cdot$ on X by

$$(Ae) \cdot (Be) = (AB)e.$$

In this case if we let $X = Y$, we have (X, Y) as a multiplier pair with $\mathcal{L}_0 = \mathcal{R}_0 = X$.

Conversely, suppose (X, Y) is a multiplier pair with $\mathcal{L}_0 = \mathcal{R}_0 = X$. Then e is a strictly cyclic separating vector for $\mathcal{L} = \{L_x : x \in \mathcal{L}_0\}$, since $\mathcal{L}e = \mathcal{L}_0 = X$ and since $L_x e = 0$ implies $x e = x = 0$.

Proposition 1. *Suppose (X, Y) is a multiplier pair and $\mathcal{L}_0 = \mathcal{R}_0 = X$. Suppose T is a closed densely defined operator on X affiliated with $\mathcal{L}$. Then $T \in \mathcal{L}$.*

Proof. The map $\gamma : \mathcal{L} \rightarrow X$ defined by $\gamma(L_x) = x$ is continuous and bijective. It follows from the open mapping theorem that γ^{-1} is continuous. Since $\mathcal{D}(T)$ is dense in X , there is a $g \in \mathcal{D}(T)$ with $\|e - g\| < 1/\|\gamma^{-1}\|$, which means that $\|1 - L_g\| = \|\gamma^{-1}(e - g)\| < 1$. Hence L_g is invertible in $B(X)$ and in the commutant of $\mathcal{R}$. Hence there is an $h \in \mathcal{L}_0$ such that $\eta = hg = e$. It follows from the fact that $\mathcal{D}(T) \subseteq \mathcal{R}(\mathcal{D}(T))$ that $\mathcal{D}(T) = X$, which implies $T \in \mathcal{R}' = \mathcal{L}$. $\square$

We conclude with some results in a different direction.

Proposition 2. *Suppose (X, Y) is a multiplier pair. Then:*

- (1) *If $a \in X$, then L_a is a closed densely defined operator on X that commutes with every operator in $\mathcal{R}$.*
- (2) *If (X, Y) is a special multiplier pair and $y \in Y$, then L_y is a closed operator on X that commutes with every operator in $\mathcal{R}$.*
- (3) *If (X, Y) is a special multiplier pair, $y \in Y$ and, for each $x \in \mathcal{R}_0$, there is a sequence $\{e_n\}$ in $\mathcal{L}_0$ such that*
 - (a) $\|e_n x - x\| \rightarrow 0$,
 - (b) $y e_n \in X$ for every $n \geq 1$,*then $\mathcal{D}(L_y)$ is dense in X .*

Proof. (1) This was proved in [6].

(2) If $\{(x_n, yx_n)\}$ is a sequence in $\text{Graph}(L_y)$ and $\|(x_n, yx_n) - (x, w)\| \rightarrow 0$, then $\|x_n - x\| \rightarrow 0$ and $\|yx_n - w\| \rightarrow 0$. Thus $x_n \rightarrow x$ and $yx_n \rightarrow w$ in Y . Since (X, Y) is a special multiplier pair, multiplication is separately continuous, so $yx_n \rightarrow yx$ in Y . This implies $w = yx$. Thus L_y is closed. Since $(Y, +, \cdot)$ is a ring, L_y is in the commutant of $\mathcal{R}$.

(3) It is clear from conditions (a) and (b) that the closure of $\mathcal{D}(L_y)$ contains $\mathcal{R}_0$, so $\mathcal{D}(L_y)$ is dense in X . $\square$

3. APPLICATIONS

We now apply the results of the preceding section to some special cases of multiplier pairs.

3.1. Analytic functions. Suppose $1 \leq p < \infty$ and $X = H^p$ (on the unit disk) and that $Y = N$ is the set of meromorphic functions in the Nevanlinna class, i.e., functions of the form f/g with $f, g \in H^\infty$ and g not identically 0. Then (X, Y) is a special multiplier pair with $\mathcal{L}_0 = \mathcal{R}_0 = H^\infty$. The Smirnov class N^+ consists of all members of N having a denominator that is an outer function. Sarason [13] has observed that the closed densely defined operators that commute with the unilateral shift on H^2 are multiplications induced by members of the Smirnov class, and we prove it here more generally in Theorem 4 as a consequence of Corollary 1. Sarason also observed [14] that members of N^+ have a canonical form related to H^2 , and the following lemma shows that the analogous result related to H^p holds for $1 \leq p < \infty$ and is established by the same proof.

Lemma 2. *If $1 \leq p < \infty$, $\phi \in N$, and $\phi \neq 0$, then there exist relatively prime inner functions u and v and outer functions a and b satisfying $|a|^p + |b|^p = 1$ a.e. on the unit circle such that*

$$\phi = \frac{vb}{ua}.$$

Proof. Recall that an outer function is positive at zero and is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Suppose ϕ is a nonzero function in N and the inner-outer factorization is applied to each of the numerator and denominator of ϕ , so

$$\phi = \frac{uf_1}{vf_2},$$

where u and v are relatively prime inner functions and f_1 and f_2 are outer functions in H^∞ .

Observe that on the unit circle $\mathbb{T}$,

$$\max\{|f_1|, |f_2|\} \leq (|f_1|^p + |f_2|^p)^{1/p} \leq |f_1| + |f_2|,$$

and therefore $(|f_1|^p + |f_2|^p)^{1/p}$ is log integrable. Thus there exists an outer function ψ in H^∞ such that $|\psi| = (|f_1|^p + |f_2|^p)^{1/p}$ a.e. on $\mathbb{T}$. Put $a = f_2/\psi$ and $b = f_1/\psi$ and observe that the definition of ψ implies that $|a|^p + |b|^p = 1$ a.e. on $\mathbb{T}$. The asserted representation of ϕ follows. $\square$

Corollary 3. *If $\phi \in N$, where $\phi = (vb)/(ua)$ as in Lemma 2, then the graph $\text{Graph}(M_\phi)$ of M_ϕ is the closed subset $\{(uag) \oplus (vbg) : g \in H^p\}$ of $H^p \oplus H^p$.*

Proof. If $g \in H^p$, then $uag \in H^p$ and $M_\phi uag = vbg \in H^p$, and it follows that the set asserted to be the graph of M_ϕ is a subset of the graph. For the opposite inclusion suppose both f and ϕf belong to H^p . Then both $|f|^p$ and $|\phi|^p |f|^p$ are integrable on $\mathbb{T}$, and because

$$\frac{|f|^p}{|a|^p} = \frac{|a|^p + |b|^p}{|a|^p} |f|^p = |f|^p + |\phi|^p |f|^p$$

on $\mathbb{T}$, it follows that if $g_1 = f/a$, then $g_1 \in H^p$. (If $f \in H^p$, a is outer, and $f/a \in L^p$, then $f/a \in H^p$. See Nikolskii [12, Theorem 3.9.6].) Thus $f = ag_1$ and $u\phi f = vbg_1$. Since u and v are relatively prime and b is outer, the last equation

shows that u is a factor of g_1 , and thus $g_1 = ug$ for some $g \in H^p$. We have shown that $f = aug$ and $\phi f = vbg$, and thus the required inclusion is established. $\square$

Theorem 4. *Suppose $1 \leq p < \infty$ and $\mathcal{G} \subseteq H^p \oplus H^p$ is a graph that is invariant under $M_z \oplus M_z$. Then there is a meromorphic $\phi \in N$ such that $\mathcal{G} \subseteq \text{Graph}(M_\phi)$. If the domain of $\mathcal{G}$ is dense in H^p , then ϕ is in the Smirnov class. If, in addition, $\mathcal{G}$ is closed, then ϕ is in the Smirnov class and $\mathcal{G} = \text{Graph}(M_\phi)$.*

Proof. The first assertion follows from Corollary 1. Suppose the domain $\mathcal{D}(\mathcal{G})$ of $\mathcal{G}$ is dense. Then for every z with $|z| < 1$, there is a $g \in \mathcal{D}(\mathcal{G})$ such that $g(z) \neq 0$, so $\phi g \in H^p$ implies that ϕ has a removable singularity at z . Hence ϕ is analytic on the open unit disk. By Corollary 3, the domain of M_ϕ , which includes that of $\mathcal{G}$, is uaH^p . Thus uaH^p is dense, and it follows that $u = 1$. Thus ϕ is in N^+ .

Suppose that $\mathcal{G}$ is closed. If $H^p \oplus H^p$ is given the norm defined by $\|f \oplus g\| = (\|f\|_p^p + \|g\|_p^p)^{1/p}$, then the proof of Corollary 3 shows that the map $V : H^p \rightarrow H^p \oplus H^p$ defined by

$$Vg = uag \oplus vbg$$

is an isometry of H^p onto $\text{Graph}(M_\phi)$. Let $\mathcal{M}$ be the inverse image of $\mathcal{G}$ under V . Then $\mathcal{G}$ is a closed subspace of H^p , and for $g \in \mathcal{M}$ we have

$$VM_zg = uazg \oplus vbzg = M_z \oplus M_z Vg \in \mathcal{G}.$$

Hence $\mathcal{M}$ is invariant under M_z , and, by Duren [4, Theorem 7.4] or Helson [10, page 25], $\mathcal{M} = \omega H^p$ for some inner function ω , and thus

$$\mathcal{G} = \{\omega uag \oplus \omega vbg : g \in H^p\} = (M_\omega \oplus M_\omega) \text{Graph}(M_\phi).$$

It follows that if the domain of $\mathcal{G}$ is dense, then ω , as well as u , must be 1, and hence the asserted equality holds. $\square$

As a corollary to the proof we have the following.

Corollary 4. *If $1 \leq p < \infty$ and $\mathcal{G} \subseteq H^p \oplus H^p$ is a closed graph that is invariant under $M_z \oplus M_z$, then there is a meromorphic function ϕ in the Nevanlinna class and an inner function ω such that*

$$\mathcal{G} = (M_\omega \oplus M_\omega) \text{Graph}(M_\phi).$$

The following is a direct consequence of the preceding corollary (i.e., ω is constant).

Corollary 5. *Suppose $1 \leq p < \infty$ and $\mathcal{G} \subseteq H^p \oplus H^p$ is a closed graph that is invariant under $M_z \oplus M_z$ and the sum of its domain and range is dense in H^p . Then it is the graph of some meromorphic function in the Nevanlinna class.*

3.2. Measure theory. A symmetric norm on $L^\infty[0, 1]$, with respect to Lebesgue measure μ , is a norm α such that

- (1) $\alpha(f) = \alpha(|f|)$ for every $f \in L^\infty[0, 1]$,
- (2) $\alpha(1) = 1$,
- (3) $\alpha(f \circ \tau) = \alpha(f)$ for every $f \in L^\infty[0, 1]$ and every invertible measure-preserving $\tau : [0, 1] \rightarrow [0, 1]$.

We say that a symmetric norm α on $L^\infty[0, 1]$ is *continuous* if

$$\lim_{t \rightarrow 0^+} \alpha(\chi_{[0,t]}) = 0.$$

We define $L^\alpha[0, 1]$ to be the completion of $L^\infty[0, 1]$ with respect to the norm α . We can realize the elements of $L^\alpha[0, 1]$ as (equivalence classes of) measurable functions in $L^1[0, 1]$. Moreover, if $X = L^\alpha[0, 1]$ and Y is the set of all measurable functions on $[0, 1]$ with the topology of convergence in measure, then (X, Y) is a special multiplier pair with pointwise multiplication, and $\mathcal{R}_0 = \mathcal{L}_0 = L^\infty[0, 1]$ (see [7]). Moreover, if α is continuous, we have

$$\lim_{\mu(F) \rightarrow 0} \alpha(\chi_F h) = 0$$

for every $h \in L^\alpha[0, 1]$.

Proposition 3. *Suppose α is a continuous symmetric norm on $L^\infty[0, 1]$ and $\mathcal{G} \subseteq L^\alpha[0, 1] \oplus L^\alpha[0, 1]$ is a closed linear subspace and a graph that is invariant under $M_f \oplus M_f$ for every $f \in L^\infty[0, 1]$. Then there is a measurable function φ (with possibly infinite values) such that $\mathcal{G} \subseteq \mathcal{G}(M_\varphi)$.*

Proof. Since $\mathcal{D}(\mathcal{G})$ is an $L^\infty[0, 1]$ -module, we have $|g| \in \mathcal{D}(\mathcal{G})$ whenever $g \in \mathcal{D}(\mathcal{G})$. Moreover, if $\{g_n\}$ is a sequence in $\mathcal{D}(\mathcal{G})$ and $(|g_n|, h_n) \in \mathcal{G}$ for $n \geq 1$, then

$$g = \sum_{n=1}^{\infty} t_n |g_n| \in \mathcal{D}(\mathcal{G}), \text{ where } 0 < t_n < \frac{1}{2^n[\alpha(g_n) + \alpha(h_n) + 1]} \text{ for } n \geq 1. \text{ Since}$$

$$\{x : g(x) > 0\} = \bigcup_{n \geq 1} \{x : |g_n(x)| > 0\}, \text{ if we choose } g_n \text{ so that}$$

$$\mu(\{x : g_n(x) \neq 0\}) > \sup\{\mu(\{x : h(x) \neq 0\}) : h \in \mathcal{D}(\mathcal{G})\} - \frac{1}{n},$$

then we see that

$$\mu(\{x : g(x) \neq 0\}) = \sup\{\mu(\{x : h(x) \neq 0\}) : h \in \mathcal{D}(\mathcal{G})\}.$$

If $E = \{x : g(x) \neq 0\}$, then $\chi_E h = h$ a.e. (μ) for every $h \in \mathcal{D}(\mathcal{G})$, and, since $\mathcal{G}$ is a graph and an $L^\infty[0, 1]$ -module, we see that $\chi_E h = h$ a.e. (μ) for every h in the range of $\mathcal{G}$. We can now replace $L^\alpha[0, 1]$ with $\chi_E L^\alpha[0, 1] = X$ and let Y be the set of measurable functions that vanish on $[0, 1] \setminus E$. Now g has a multiplicative inverse, and, for each $h \in X$ and each positive integer $n \geq 1$, we have $h\chi_{E_n} = g(\chi_{E_n} h/g)$, where $E_n = \{x : g(x) \geq 1/n \text{ and } |f(x)| \leq n\}$. Since $\bigcup_{n \geq 1} E_n = E$ (a.e.), the set $\{\chi_{E_n} : n \geq 1\}$ is left separating for Y . Applying Theorem 1 to $\chi_E \mathcal{G}$, we see that

$$\mathcal{G} = \chi_E \mathcal{G} \subseteq L_{f/g},$$

where $(g, f) \in \mathcal{G}$. Moreover, the fact that $\mu(E \setminus E_n) \rightarrow 0$ and α is continuous implies

$$R_{\chi_{E_n}} \rightarrow 1$$

in the strong operator topology. It follows from Theorem 2 that $\chi_E \mathcal{G}$ is the graph of $L_{f/g}$ on $\chi_E L^\alpha[0, 1]$. If we let $\varphi = \frac{f}{g}\chi_E + \infty\chi_{([0,1] \setminus E)}$, we see that $\mathcal{G} = \mathcal{G}(L_\varphi)$. $\square$

Corollary 6. *Suppose α is a continuous symmetric norm on $L^\infty[0, 1]$ and T is a closed densely defined linear transformation on $L^\alpha[0, 1]$ that commutes with L_f for every $f \in L^\infty[0, 1]$. Then there is a measurable function $\varphi : [0, 1] \rightarrow \mathbb{C}$ such that $T = L_\varphi$.*

Theorem 5. *Suppose α is a symmetric norm on $L^\infty[0, 1]$ and T is a closed densely defined operator on $L^\alpha[0, 1]$ that commutes with M_x . Then there is a measurable function $\varphi : [0, 1] \rightarrow \mathbb{C}$ such that $T = M_\varphi$.*

Proof. If α is continuous, the desired conclusion follows from Corollary 6. If α is not continuous, then α is equivalent to $\|\cdot\|_\infty$ on $L^\infty[0, 1]$ (see [7]), so $L^\alpha[0, 1] = L^\infty[0, 1]$, and the desired conclusion follows from Proposition 1. $\square$

3.3. Finite von Neumann algebras. Suppose $\mathcal{M}$ is a II_1 factor von Neumann algebra with a faithful normal tracial state τ . A *unitarily invariant norm* on $\mathcal{M}$ is a norm β such that

- (1) $\beta(UTV) = \beta(T)$ for all $U, T, V \in \mathcal{M}$ with U, V unitary,
- (2) $\beta(1) = 1$.

Since $\mathcal{M}$ is a II_1 factor, there is a chain $\{P_t : t \in [0, 1]\}$ of projections in $\mathcal{M}$ such that $\tau(P_t) = t$ for every $t \in [0, 1]$. The map $\chi_{[0, t]} \mapsto P_t$ extends to a unital $*$ -homomorphism $\pi : L^\infty[0, 1] \rightarrow \mathcal{M}$ such that

$$\tau(\pi(f)) = \int_0^1 f(x) dx$$

for every $f \in L^\infty[0, 1]$. If we define $\alpha : L^\infty[0, 1] \rightarrow [0, \infty)$ by

$$\alpha(f) = \beta(\pi(f)),$$

we obtain a symmetric norm α . It turns out that α is independent of the chain $\{P_t : t \in [0, 1]\}$. Moreover, every element $A \in \mathcal{M}$ has a polar decomposition $A = U(A^*A)^{1/2}$ and $\beta(A) = \beta((A^*A)^{1/2})$. There is a chain $\{P_t : t \in [0, 1]\}$ whose generated von Neumann algebra contains $(A^*A)^{1/2}$, and there is a $\varphi \in L^\infty[0, 1]$ such that $\pi(\varphi) = (A^*A)^{1/2}$, and we get $\beta(A) = \alpha(\varphi)$. In this way we can recapture β from α . It is shown in [5] that every symmetric norm α on $L^\infty[0, 1]$ determines a unitarily invariant norm β on $\mathcal{M}$, and we will call this norm $\|\cdot\|_\alpha$. We let $L^\alpha(\mathcal{M}, \tau)$ denote the completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_\alpha$. If the symmetric norm α is continuous, then

$$\lim_{\tau(P) \rightarrow 0, P=P^*=P^2 \in \mathcal{M}} \|P\|_\alpha = 0.$$

If α is not continuous, then α is equivalent to $\|\cdot\|_\infty$ on $L^\infty[0, 1]$ and $\|\cdot\|_\alpha$ is equivalent to the operator norm on $\mathcal{M}$, so $L^\beta(\mathcal{M}, \tau) = \mathcal{M}$.

There is a topology of *convergence in measure* on $\mathcal{M}$ (see [11]), and $L^\alpha(\mathcal{M}, \tau)$ can be viewed as a subset of the completion $\mathcal{Y}$ of $\mathcal{M}$ with respect to this topology. It was shown in [7] that $(L^\alpha(\mathcal{M}, \tau), \mathcal{Y})$ is a multiplier pair and $\mathcal{L}_0 = \mathcal{R}_0 = \mathcal{M}$, and it follows from [11, Theorem 1] that $(L^\alpha(\mathcal{M}, \tau), \mathcal{Y})$ is a special multiplier pair.

Theorem 6. *Suppose $\mathcal{M}$ is a II_1 factor von Neumann algebra with faithful normal tracial state τ , and suppose α is a unitarily invariant norm on $\mathcal{M}$. Suppose T is a closed densely defined operator on $L^\alpha(\mathcal{M}, \tau)$ that commutes with R_S for every $S \in \mathcal{M}$. Then:*

- (1) *If α is not continuous, then $T \in \mathcal{L}$.*
- (2) *If α is continuous, then $T = L_y$ for some $y \in \mathcal{Y}$.*
- (3) *If α is continuous and $y \in \mathcal{Y}$, then L_y is a closed densely defined operator on $L^\alpha(\mathcal{M}, \tau)$ that commutes with R_S for every $S \in \mathcal{M}$.*

Proof. (1) If α is not continuous, then α is equivalent to the operator norm on $\mathcal{M}$, so $L^\alpha(\mathcal{M}, \tau) = \mathcal{M}$, which implies $\mathcal{L}_0 = \mathcal{R}_0 = \mathcal{M}$. The desired conclusion now follows from Proposition 1.

(2) Suppose $x \in \mathcal{D}(T)$. It follows from [11, Theorem 2] that there is a sequence $\{p_n\}$ of projections in $\mathcal{M}$ such that $\tau(p_n) \rightarrow 0$ and $x(1 - p_n) \in \mathcal{M}$ for each $n \geq 1$, and, since $\mathcal{D}(T)(1 - p_n) \subseteq \mathcal{D}(T)$, we have $x(1 - p_n) \in \mathcal{M} \cap \mathcal{D}(T)$ for each $n \geq 1$. However, α is continuous, so

$$\alpha(x - (1 - p_n)x) = \alpha(p_n x) \leq \alpha(p_n) \|x\| \rightarrow 0.$$

Thus the closure of $\mathcal{M} \cap \mathcal{D}(T)$ contains $\mathcal{D}(T)$, and since $\mathcal{D}(T)$ is dense in $L^\alpha(\mathcal{M}, \tau)$, it follows that $\mathcal{M} \cap \mathcal{D}(T)$ is dense in $L^\alpha(\mathcal{M}, \tau)$. For each $x \in \mathcal{M} \cap \mathcal{D}(T)$, let P_x denote the projection onto the weak operator topology closure of the range of x , and choose a sequence $\{x_n\}$ in $\mathcal{M} \cap \mathcal{D}(T)$ such that

$$\lim \tau(P_{x_n}) = \sup\{\tau(P_x) : x \in \mathcal{M} \cap \mathcal{D}(T)\}.$$

For each $n \geq 1$ we can write $x_n = (x_n x_n^*)^{\frac{1}{2}} u_n$ with u_n unitary, and so $(x_n x_n^*)^{\frac{1}{2}} = R_{u_n^*} x_n \in \mathcal{M} \cap \mathcal{D}(T)$. If we choose t_n so that

$$t_n \left[\alpha \left((x_n x_n^*)^{\frac{1}{2}} \right) + \alpha \left(T(x_n x_n^*)^{\frac{1}{2}} \right) \right] < \frac{1}{2^n},$$

then $g = \sum_{n=1}^{\infty} t_n (x_n x_n^*)^{\frac{1}{2}} \in \mathcal{M} \cap \mathcal{D}(T)$. Moreover, it follows that $P_g \geq P_{(x_n x_n^*)^{\frac{1}{2}}}$ for each n . Hence

$$\tau(P_g) = \sup\{\tau(P_x) : x \in \mathcal{M} \cap \mathcal{D}(T)\}.$$

If $x \in \mathcal{D}(T)$, then $(xx^*)^{\frac{1}{2}}, g + (xx^*)^{\frac{1}{2}} \in \mathcal{D}(T)$, which means the weak operator topology closure of the range of x (which equals the range of $(xx^*)^{\frac{1}{2}}$) is contained in P_g . Hence,

$$\mathcal{M} \cap \mathcal{D}(T) \subseteq P_g \mathcal{M} \subseteq P_g L^\alpha(\mathcal{M}, \tau).$$

Since $\mathcal{M} \cap \mathcal{D}(T)$ is dense in $L^\alpha(\mathcal{M}, \tau)$, we must have $P_g = 1$, which implies g is invertible in $\mathcal{Y}$.

Now it follows from [11, Theorem 2] that for each $y \in \mathcal{Y}$ there is a sequence $\{q_n\}$ of projections in $\mathcal{M}$ such that $\tau(q_n) \rightarrow 1$ and $yq_n \in \mathcal{M}$ for every $n \geq 1$. Since α is continuous, we know from [5] that $R_{q_n} \rightarrow 1$ (SOT) on $L^\alpha(\mathcal{M}, \tau)$. Hence, Theorem 3 implies that $T = L_{T(g)} g^{-1}$.

(3) It follows from [6] that L_y is a closed operator on $L^\alpha(\mathcal{M}, \tau)$. It follows from [11, Theorem 2] that there is a sequence $\{p_n\}$ of projections in $\mathcal{M}$ such that $\tau(p_n) \rightarrow 0$ and $y(1 - p_n) \in \mathcal{M} \subseteq L^\alpha(\mathcal{M}, \tau)$, which implies

$$\bigcup_{n=1}^{\infty} (1 - p_n) \mathcal{M} \subseteq \mathcal{D}(L_y).$$

Moreover, since α is continuous,

$$\alpha(x - (1 - p_n)x) = \alpha(p_n x) \leq \alpha(p_n) \|x\| \rightarrow 0.$$

Hence, $\mathcal{D}(L_y)$ is dense in $L^\alpha(\mathcal{M}, \tau)$. □

4. AN EXAMPLE ON THE BERGMAN SPACE

The closed densely defined operators commuting with the unilateral shift, M_z on H^2 , are multiplications by functions in the Smirnov class. Is there a similar result for the Bergman shift, multiplication by z on the Bergman space A^2 ? What follows is a counterexample.

Theorem 7. *There exists a closed densely defined operator that commutes with the Bergman shift but is not a multiplication operator.*

Proof. By a result of Seip (see [16], [3, Theorem 9, p. 186], [8]) there exist two disjoint interpolation sets Γ_1, Γ_2 for A^2 whose union Γ is a sampling set. A sampling set for A^2 is defined as a subset Γ of $\mathbb{D}$ with the property that there exist positive constants c and C such that, for each $f \in A^2$,

$$c\|f\|^2 \leq \sum_{z \in \Gamma} |f(z)|^2 (1 - |z|^2)^2 \leq C\|f\|^2.$$

Thus, with μ the discrete measure having mass $(1 - |z|^2)^2$ at each point z of Γ , the restriction map $\Phi : f \mapsto f|_{\Gamma}$ is an invertible map of A^2 onto a subspace $A^2|_{\Gamma}$ of $L^2(\mu)$. An interpolation set for A^2 is a subset Γ of $\mathbb{D}$ with the property that for each element w in $L^2(\mu)$ there exists $f \in A^2$ such that $w = \Phi(f)$, i.e., the restriction map $f \mapsto f|_{\Gamma}$ on A^2 is onto $L^2(\mu)$.

Since interpolation sets are also zero sets for A^2 , we can let f_1 be a function in A^2 having Γ_1 as its zero set. Define $\mathcal{G}_1, \mathcal{G}_2$ by

$$\mathcal{G}_1 = \{(u, uf_1) : u, uf_1 \in A^2\}$$

and

$$\mathcal{G}_2 = \{(g, g) : g \in A^2, g|_{\Gamma_2} = 0\}.$$

Then $\mathcal{G}_1 = \mathcal{G}(M_{f_1})$ and $\mathcal{G}_2 \subseteq \mathcal{G}(I)$ are graphs of closed operators that commute with the Bergman shift M_z . Moreover, $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ is also a graph. To see that $\mathcal{G}$ is a graph, we need only check that if $(0, h) \in \mathcal{G}$, then $h = 0$. Suppose $(u, uf_1) \in \mathcal{G}_1$, $(g, g) \in \mathcal{G}_2$ and $(0, h) = (u, uf_1) + (g, g)$. Then $u = -g$ vanishes on Γ_2 , so uf_1 vanishes on the sampling set $\Gamma = \Gamma_1 \cup \Gamma_2$, which implies $uf_1 = 0$. But $f_1 \neq 0$, so $0 = u = -g$, whence $h = 0$.

Finally, it will be shown that the angle between $\mathcal{G}_1$ and $\mathcal{G}_2$ is greater than zero, and thus the graph $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ is itself a closed direct sum. It is to be shown that there is a γ , $0 \leq \gamma < 1$, such that, if $F_k \in \mathcal{G}_k$ for $k = 1, 2$, then

$$|\langle F_1, F_2 \rangle| \leq \gamma \|F_1\| \|F_2\|.$$

Let $\Phi^{(2)}(f, g) = (\Phi(f), \Phi(g))$. Since Φ is an isomorphism between A^2 and its image $\mathcal{A}$ in $L^2(\mu)$, it suffices to show that the preceding inequality holds for the images of F_1 and F_2 under $\Phi^{(2)}$. Since an element of $L^2(\mu)$ in the image $\mathcal{A}$ of Φ is just the restriction of an element of A^2 , we will simply write $f = \Phi(f)$ and indicate the norm and inner product in $L^2(\mu)$ with a subscript μ . We have $F_1 = (u, uf_1)$ and $F_2 = (g, g)$ with $g, u, uf_1 \in A^2$ and $g|_{\Gamma_2} = 0$. Thus

$$\langle F_1, F_2 \rangle_{\mu} = \langle u, g \rangle_{\mu} + \langle uf_1, g \rangle_{\mu} = \langle u, g \rangle_{\mu},$$

where the second equality follows from the fact that $f_1 \mid \Gamma_1 = 0$ as well as $g \mid \Gamma_2 = 0$, so $uf_1\bar{g}$ vanishes a.e.- μ . Also,

$$\|F_1\|_\mu^2 = \|u\|_\mu^2 + \|uf_1\|_\mu^2$$

and

$$\|F_2\|_\mu^2 = 2\|g\|_\mu^2.$$

We have

$$|\langle F_1, F_2 \rangle_\mu| = |\langle u, g \rangle_\mu|^2 \leq \|u\|_\mu \|g\|_\mu \leq \|F_1\|_\mu \left(\sqrt{2}/2 \right) \|F_2\|_\mu,$$

as required with $\gamma = \sqrt{2}/2$. It follows that the subspaces are at a positive angle and the sum of their closures is closed. The operator whose graph is $\mathcal{G}$ has the required properties. $\square$

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DEPARTMENT OF MATHEMATICS, KINGSBURY HALL, UNIVERSITY OF NEW HAMPSHIRE, DURHAM,
NEW HAMPSHIRE 03824-3591

E-mail address: `don@math.unh.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 SOUTH 33RD STREET,
PHILADELPHIA, PENNSYLVANIA 19104

E-mail address: `zheliu@sas.upenn.edu`

DEPARTMENT OF MATHEMATICS, KINGSBURY HALL, UNIVERSITY OF NEW HAMPSHIRE, DURHAM,
NEW HAMPSHIRE 03824-3591

E-mail address: `ean@math.unh.edu`