ON PROLONGATIONS OF CONTACT MANIFOLDS

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ABSTRACT. We apply spectral sequences to derive both an obstruction to the existence of n -fold prolongations and a topological classification. Prolongations have been used in the literature in an attempt to prove that every Engel structure on $M \times \mathbb{S}^1$ with characteristic line field tangent to the fibers is determined by the contact structure induced on a cross section and the twisting of the Engel structure along the fibers. Our results show that this statement needs some modification: to classify the diffeomorphism type of the Engel structure, we additionally have to fix a class in the first cohomology of M .

1. INTRODUCTION

The goal of this note is to present a discussion of prolongations of contact manifolds. For an introduction to basic notions of Engel structures, we refer the reader to [7, §2.2], [1, §1.2] and [8]. Given a 3-dimensional contact manifold (M, ξ) , we define its prolongation $\mathbb{P}(\xi)$ as the $\mathbb{S}^1$ -bundle over M obtained by projectivizing the contact planes ξ (cf. §4 or see [7, §2.2] and [1, §1.2]). In [1], Adachi discusses prolongations of contact manifolds and introduces a notion of an n -fold prolongation, which he defines as a *fiberwise n -fold covering* of $\mathbb{P}(\xi)$. This notion is then employed in an attempt to prove that Engel structures $\mathcal{D}$ on 4-manifolds $M \times \mathbb{S}^1$, where M is a closed, oriented 3-manifold and $\mathcal{D}$ is an Engel structure with characteristic foliation tangent to the $\mathbb{S}^1$ -fibers, are determined by the contact structure induced on a cross section of $M \times \mathbb{S}^1$ and the *twisting* of $\mathcal{D}$ along the $\mathbb{S}^1$ -fibers (see [1, Theorem 1(2)]). His definition of the n -fold prolongation and the statement in his Theorem 1(2) in [1] suggest that n -fold prolongations of contact manifolds always exist and are unique. Additionally, the proof of Theorem 1(2) in [1] seems to rest in an essential way on a lifting argument, which however does not work in general (cf. §2). In §2 we discuss an example which shows that the data specified by Adachi in Theorem 1(2) in [1] are not sufficient to determine the isotopy class of the Engel structure $\mathcal{D}$ on $M \times \mathbb{S}^1$. In §3 we apply methods from spin geometry to provide a topological classification of n -fold prolongations.

Theorem 1.1. *A contact manifold (M, ξ) admits an n -fold prolongation if and only if the mod- n reduction $e_n(\mathbb{P}(\xi))$ of the Euler class $e(\mathbb{P}(\xi))$ vanishes. The isomorphism classes of n -fold prolongations are in one-to-one correspondence with elements of $H^1(M; \mathbb{Z}_n)$.*

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The main ingredient in the proof of this theorem is the classification of fiberwise n -fold coverings of $\mathbb{S}^1$ -bundles over a given manifold M . We provide this classification in Theorem 3.3. Its proof uses techniques coming from the characterization of spin structures. As a corollary, we are able to show that Engel structures on an $\mathbb{S}^1$ -bundle over M with characteristic foliation tangent to the fibers are classified by the set of data specified by Adachi and a class in $H^1(M; \mathbb{Z}_n)$ (see Corollary 4.1).

Finally, in §5 we investigate which contact structures admit n -fold prolongations. We prove the following general existence result.

Theorem 1.2. *Every 3-dimensional contact manifold (M, ξ) admits both 2-fold prolongations and 4-fold prolongations.*

This is derived by showing that the Euler class $e(\mathbb{P}(\xi))$ lies in the subgroup $4 \cdot H^2(M; \mathbb{Z})$ of $H^2(M; \mathbb{Z})$. Furthermore, we see that, in fact, each element $q \in 4 \cdot H^2(M; \mathbb{Z})$ is given as the Euler class $e(\mathbb{P}(\xi))$ of the prolongation of a suitable contact structure ξ . Consequently, for every $n \neq 2, 4$ there is a contact structure which does not admit an n -fold prolongation.

2. AN INTRODUCTORY EXAMPLE

In this section we provide an example which illustrates that Theorem 1(2) in [1] needs some modification. We will investigate fiberwise n -fold coverings of the $\mathbb{S}^1$ -bundle $T^3 \times \mathbb{S}^1 \rightarrow T^3$. The considerations presented in this section show that fiberwise n -fold coverings are in one-to-one correspondence with elements of $H^1(T^3; \mathbb{Z}_n)$ (cf. Theorem 3.3). We consider the 3-dimensional torus T^3 with coordinates (x, y, z) . Given some vector $\alpha \in \mathbb{Z}^3 \cap [0, n-1]^3$, we define a fiberwise n -fold covering $\phi_\alpha: T^3 \times \mathbb{S}^1 \rightarrow T^3 \times \mathbb{S}^1$ by

$$\phi_\alpha(\mathbf{p}, \theta) = (\mathbf{p}, n\theta + \langle \alpha, \mathbf{p} \rangle).$$

Here we use the identification $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The restriction of ϕ_α to a fiber of $T^3 \times \mathbb{S}^1$ corresponds to the unique connected n -fold covering of the circle. Observe that α determines a cohomology class in $H^1(T^3; \mathbb{Z}_n)$: by the universal coefficient theorem we know that the group $H^1(T^3; \mathbb{Z}_n)$ is isomorphic to the group $\text{Hom}(H_1(T^3; \mathbb{Z}); \mathbb{Z}_n)$, whose elements, by fixing the *standard* generators of $H_1(T^3; \mathbb{Z})$, are uniquely determined by a vector in $\mathbb{Z}^3 \cap [0, n-1]^3$.

For α and α' with $\alpha \neq \alpha'$, the coverings ϕ_α and $\phi_{\alpha'}$ are not equivalent. Suppose they were equivalent. Then there exists a covering isomorphism ψ such that $\phi_\alpha \circ \psi = \phi_{\alpha'}$. The morphism ψ_* on fundamental groups is given by the matrix

$$\psi_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

for suitable integers $a, b, c \in \mathbb{Z}$, and we have that $(\phi_\alpha)_* \psi_* = (\phi_{\alpha'})_*$. This implies the equality $\alpha' = \alpha + n \cdot (a, b, c)$, which means that $\alpha' = \alpha$ considering the fact that they are both vectors in $\mathbb{Z}^3 \cap [0, n-1]^3$.

Conversely, given a fiberwise n -fold covering $\phi: T^3 \times \mathbb{S}^1 \rightarrow T^3 \times \mathbb{S}^1$ we associate with it a cohomology class α in $H^1(T^3; \mathbb{Z}_n)$ as follows. As outlined above, we have to assign an element in $\mathbb{Z}_n$ to each of the standard generators of $H_1(T^3; \mathbb{Z})$. Take

such a standard generator, c say, and lift it to an embedded circle $\tilde{c} \subset T^3 \times \mathbb{S}^1$. Write p_2 for the projection of $T^3 \times \mathbb{S}^1$ to the $\mathbb{S}^1$ -factor. Then we define

$$\alpha(c) = \deg(p_2 \circ \phi \circ \tilde{c}) \bmod n.$$

Observe that this is well defined, since ϕ is a fiberwise n -fold covering. This algorithm defines a morphism $\alpha: H_1(T^3; \mathbb{Z}) \rightarrow \mathbb{Z}_n$, which corresponds to a class in $H^1(T^3; \mathbb{Z}_n)$ by the universal coefficient theorem.

2.1. An example of non-equivalent prolongations. Choose ξ to be the standard contact structure on T^3 , i.e. ξ is defined as the kernel of the contact 1-form given by $\sin(2\pi z)dx + \cos(2\pi z)dy$. The contact planes are spanned by ∂_z and

$$V_{\mathbf{p}} = \cos(2\pi z) \partial_x + \sin(2\pi z) \partial_y,$$

where $\mathbf{p} = (x, y, z)$. The prolongation $\mathbb{P}(\xi)$ can be naturally identified with the 4-dimensional torus $T^3 \times \mathbb{S}^1$ with coordinates (x, y, z, θ) , and the corresponding Engel structure $\mathcal{D}(\xi)$ is spanned by the tangent vectors ∂_θ and $\cos(\pi\theta) \partial_z + \sin(\pi\theta) V_{\mathbf{p}}$. Using the fiberwise n -fold covering ϕ_α from above, we can pull back the Engel structure $\mathcal{D}(\xi)$ so that we get a new Engel structure, which we denote by $\mathcal{D}_\alpha^n(\xi)$. At the point $(\mathbf{p}, \theta)$ the Engel plane $\mathcal{D}_\alpha^n(\xi)_{(\mathbf{p}, \theta)}$ is spanned by the tangent vectors ∂_θ and

$$\cos(\pi(n\theta + \langle \alpha, \mathbf{p} \rangle)) \partial_z + \sin(\pi(n\theta + \langle \alpha, \mathbf{p} \rangle)) V_{\mathbf{p}}.$$

The Engel manifolds $(T^3 \times \mathbb{S}^1, \mathcal{D}_\alpha^n(\xi))$ are pairwise non-equivalent if we allow only isotopies through Engel structures with characteristic foliation tangent to the $\mathbb{S}^1$ -fibers. These *special* isotopies appear in the proof of Corollary 4.1. If one considers general isotopies through Engel structures, then it is not clear which of these structures are isotopic.

3. CHARACTERIZATION OF FIBERWISE COVERINGS

Let us denote by $\varphi_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ the connected n -fold covering of the unit circle $\mathbb{S}^1$, where n is some positive integer. Suppose we are given an $\mathbb{S}^1$ -bundle $P \rightarrow M$ over a closed, oriented manifold M . We define a **fiberwise n -fold covering of P** as a pair (Q, ϕ) , where Q is an $\mathbb{S}^1$ -bundle over M and ϕ is a smooth map $Q \rightarrow P$ such that its restriction $\phi|_{Q_x}$, for every $x \in M$, is a map $Q_x \rightarrow P_x$ which corresponds to the n -fold covering map $\varphi_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Before we move our focus to the characterization of fiberwise n -fold coverings, we show that their existence is tied to the following condition on the Euler classes of the bundles.

Lemma 3.1. *Let $\pi_Q: Q \rightarrow M$ and $\pi_P: P \rightarrow M$ be two principal $\mathbb{S}^1$ -bundles such that there is a bundle map $\phi: Q \rightarrow P$ whose restriction to the fibers corresponds to the map φ_n . Then $e(P) = n \cdot e(Q)$.*

Proof. The Euler class provides an isomorphism

$$H^1(M; \mathbb{S}^1) \xrightarrow{e} H^2(M; \mathbb{Z}),$$

where $H^1(M; \mathbb{S}^1)$ naturally corresponds to equivalence classes of principle $\mathbb{S}^1$ -bundles over M . So, the statement of the lemma can be rephrased in terms of Čech cochains: Let $\mathfrak{U} = \{U_\alpha\}_\alpha$ be an open covering of M , let $\{g_{\alpha\beta}^Q\}$ be the Čech cochain representing the bundle Q and let $\{g_{\alpha\beta}^P\}$ be the Čech cochain representing P . We have to prove that the Čech cohomology classes $[n \cdot (g_{\alpha\beta}^Q)] = n \cdot [(g_{\alpha\beta}^Q)]$ and $[(g_{\alpha\beta}^P)]$ are equal.

Associated to the open covering $\mathfrak{U}$, the bundles Q and P admit bundle charts $\{t_\alpha^Q\}$ and $\{t_\alpha^P\}$. Now consider the maps $\phi_\alpha = t_\alpha^P \circ \phi \circ (t_\alpha^Q)^{-1}$. Since the restriction of ϕ to the fibers of Q corresponds to φ_n , the equality

$$\phi_\alpha(p, \theta) = (p, \mu_\alpha(p) + n \cdot \theta)$$

holds for suitable $\mu_\alpha: U_\alpha \rightarrow \mathbb{S}^1$. Because the ϕ_α come from the well-defined bundle map ϕ , we have that

$$\phi_\beta(g_{\alpha\beta}^Q(p, \theta)) = g_{\alpha\beta}^P(\phi_\alpha(p, \theta))$$

for all $p \in U_\alpha \cap U_\beta$ and $\theta \in \mathbb{S}^1$. This is equivalent to saying that

$$\mu_\beta(p) + n \cdot g_{\alpha\beta}^Q(p) - \mu_\alpha(p) = g_{\alpha\beta}^P(p)$$

for all $p \in U_\alpha \cap U_\beta$. Hence, $\{n \cdot (g_{\alpha\beta}^Q)\}$ and $\{g_{\alpha\beta}^P\}$ differ by the boundary of the Čech-1-cochain $\{\mu_\alpha\}$ and the result follows (cf. [5, Appendix A]). $\square$

From the relationship of the Euler classes presented in Lemma 3.1 we see that the mod n reduction of the Euler class is an obstruction to the existence of fiberwise n -fold coverings; i.e. the existence of fiberwise n -fold coverings imply the vanishing of the mod n reduction of the Euler class. In fact, the converse statement is true as well (see Theorem 3.3). Its proof will occupy the remainder of this section. As in the characterization of spin structures used in [5, §1] (see especially sequences (1.2) and (1.4) of [5]), we find it opportune to work with Čech cohomology.

Proposition 3.2. *An $\mathbb{S}^1$ -bundle $\mathbb{S}^1 \rightarrow P \rightarrow M$ induces the long exact sequence*

$$0 \longrightarrow H^1(M; \mathbb{Z}_n) \xrightarrow{\pi^*} H^1(P; \mathbb{Z}_n) \xrightarrow{\iota^*} H^1(\mathbb{S}^1; \mathbb{Z}_n) \xrightarrow{d} H^2(M; \mathbb{Z}_n),$$

where the map d sends the generator of $H^1(\mathbb{S}^1; \mathbb{Z}_n)$ to the mod n reduction of the Euler class $e(P)$.

To give a bit of explanation, observe that an n -fold covering of the space P corresponds to an element in $H^1(P; \mathbb{Z}_n)$ (cf. [5, Appendix A]). A fiberwise n -fold covering (Q, ϕ) is an ordinary n -fold covering, and, thus, we may think of the pair $[(Q, \phi)]$ (or simply $[Q]$) as an element in the first cohomology of P . The statement that it is fiberwise φ_n is equivalent to saying that the pullback bundle $\iota^*(Q, \phi)$ is isomorphic to φ_n . In terms of the exact sequence presented in Proposition 3.2, this amounts to the equality $\iota^*[Q] = [\varphi_n]$, where $[\varphi_n]$ is a generator of $H^1(\mathbb{S}^1; \mathbb{Z}_n)$ (cf. [5, §1] and [5, Appendix A]).

Theorem 3.3. *Given an $\mathbb{S}^1$ -bundle $P \rightarrow M$, a fiberwise n -fold covering of M exists if and only if the mod n reduction $e_n(E)$ of the Euler class $e(E)$ is zero. In this case, the isomorphism classes of fiberwise n -fold coverings of M are in one-to-one correspondence with elements of $H^1(M; \mathbb{Z}_n)$.*

Proof. By Proposition 3.2 the following sequence is exact:

$$0 \longrightarrow H^1(M; \mathbb{Z}_n) \xrightarrow{\pi^*} H^1(P; \mathbb{Z}_n) \xrightarrow{\iota^*} H^1(\mathbb{S}^1; \mathbb{Z}_n) \xrightarrow{d} H^2(M; \mathbb{Z}_n).$$

The mod n reduction of the Euler class $e(P)$ will be denoted by e_n . Suppose that a fiberwise n -fold covering Q of P exists. Then $[Q]$ is an element of $H^1(P; \mathbb{Z}_n)$ such that $\iota^*[Q] = [\varphi_n]$. Thus, by exactness of the sequence,

$$0 = d(\iota^*[Q]) = d[\varphi_n] = e_n.$$

Conversely, assuming that $e_n = 0$, we have $d[\varphi_n] = e_n = 0$. By exactness, this implies the existence of an element $q \in H^1(P; \mathbb{Z}_n)$ which is mapped to $[\varphi_n]$ under ι^* . But q corresponds to a fiberwise n -fold covering of P .

The isomorphism classes of fiberwise coverings correspond to the set $(\iota^*)^{-1}([\varphi_n])$ on which π^* , by the sequence above, induces a free and transitive $H^1(M; \mathbb{Z}_n)$ -action. Hence, we obtain a one-to-one correspondence between $H^1(M; \mathbb{Z}_n)$ and the isomorphism classes of fiberwise coverings. $\square$

It remains to prove Proposition 3.2. We just sketch the proof, since it is analogous to the proofs of the exact sequences used in the characterization of spin structures (see [5, §1]).

Sketch of Proof of Proposition 3.2. We look at the Leray-Serre spectral sequence with E_2 -page given by $E_2^{p,q} = H^p(M; H^q(\mathbb{S}^1; \mathbb{Z}_n))$ (see [6]). By applying the fact that $H^q(\mathbb{S}^1; \mathbb{Z}_n)$ is non-zero for $q = 0, 1$ only, we see that $E_\infty^{1,0} = E_2^{1,0} = H^1(M; \mathbb{Z}_n)$ and that $E_\infty^{0,1} = E_3^{0,1}$. Thus, we obtain the following exact sequence:

$$0 \longrightarrow H^1(M; \mathbb{Z}_n) \longrightarrow H^1(P; \mathbb{Z}_n) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} H^2(M; \mathbb{Z}_n).$$

In fact, it is not hard to see that $E_2^{0,1}$ equals $H^1(\mathbb{S}^1; \mathbb{Z}_n)$, and we obtain the exact sequence as proposed. With a discussion similar to the spin case (see [4] and cf. [5, §1]), it is possible to prove that $d_2^{0,1}$ sends the generator of $H^1(\mathbb{S}^1; \mathbb{Z}_n)$ to the mod- n reduction of the Euler class $e(P)$. $\square$

4. ENGEL STRUCTURES WITH TRIVIAL CHARACTERISTIC LINE FIELD

An **Engel structure** is a maximally non-integrable 2-plane distribution $\mathcal{D}$ on a 4-dimensional manifold Q , i.e. $\mathcal{D}$ is defined as a 2-plane bundle for which $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is of rank 3 and $[\mathcal{E}, \mathcal{E}]$ of rank 4. Inside the Engel structure $\mathcal{D}$ there is a line field $\mathcal{L}$ given by the condition that $[\mathcal{L}, \mathcal{E}] \subset \mathcal{E}$. This line field is called the **characteristic line field** and its induced foliation on Q the **characteristic foliation** of $\mathcal{D}$. Engel structures arise in a natural way as prolongations of contact 3-manifolds. Given a contact 3-manifold (M, ξ) , one can consider the bundle $\mathbb{P}\xi$ whose fibers are the projectivizations of the contact planes; i.e. for every $p \in M$ a point $q \in (\mathbb{P}\xi)_p$ corresponds to a line $l \subset \xi_p$ in the contact plane. Note that by construction this 4-manifold carries the structure of an $\mathbb{S}^1$ -bundle $\rho: \mathbb{P}\xi \rightarrow M$ over M . Furthermore, we obtain a natural plane distribution $\mathcal{D}\xi \subset T\mathbb{P}\xi$ given by

$$(\mathcal{D}\xi)_q = T_q\rho^{-1}(l).$$

This distribution defines an Engel structure whose characteristic line field is tangent to the $\mathbb{S}^1$ -fibers of the bundle.

Now, assume we are given an oriented $\mathbb{S}^1$ -bundle $\pi: Q \rightarrow M$ over some 3-manifold M and an Engel structure $\mathcal{D} \subset TQ$ with characteristic line field $\mathcal{L}$ tangent to the fibers. Since the induced distribution $[\mathcal{D}, \mathcal{D}] \subset TQ$ is preserved by any flow tangent to $\mathcal{L}$, we obtain a well-defined contact structure $\xi = \pi_*([\mathcal{D}, \mathcal{D}])$ on M . Furthermore, one obtains the **development map**

$$\phi_{\mathcal{D}}: (Q, \mathcal{D}) \longrightarrow (\mathbb{P}\xi, \mathcal{D}\xi)$$

by assigning to a point $q \in Q$ the element $\phi_{\mathcal{D}}(q)$ in $(\mathbb{P}(\xi))_{\pi(q)}$ which corresponds to the 1-dimensional subspace $T_q\pi(\mathcal{D}_q)$ of the contact plane $\xi_{\pi(q)}$. Note that $\phi_{\mathcal{D}}(\mathcal{D}) = \mathcal{D}\xi$ and that $\phi_{\mathcal{D}}$ defines a fiberwise n -fold covering of $\mathbb{P}(\xi)$, where $n \in \mathbb{N}$ denotes the

degree of the development map restricted to a fiber. We will refer to such $(Q, \mathcal{D})$ as an **n -fold prolongation of (M, ξ)** .

Proof of Theorem 1.1. Suppose that $e_n(\mathbb{P}(\xi))$ vanishes. Then Theorem 3.3 implies the existence of a fiberwise n -fold covering $\phi: Q \rightarrow \mathbb{P}(\xi)$. The space Q is itself a $\mathbb{S}^1$ -bundle over M . We define an Engel structure $\mathcal{D}$ on Q by setting $\mathcal{D} = (T\phi)^{-1}(\mathcal{D}(\xi))$.

Conversely, given an n -fold prolongation, the development map is a fiberwise n -fold covering of $\mathbb{P}(\xi)$, which by Lemma 3.1 or Theorem 3.3 implies the vanishing of $e_n(\mathbb{P}(\xi))$.

By the statement of Theorem 3.3 and the discussion from above, we see that, in the case of existence, n -fold prolongations are in one-to-one correspondence with elements of $H^1(M; \mathbb{Z}_n)$. $\square$

Now we have everything ready to prove a modified version of [1, Theorem 1(2)].

Corollary 4.1. *Suppose we are given an oriented Engel structure $\mathcal{D}$ on an $\mathbb{S}^1$ -bundle Q over M whose characteristic line field is tangent to the $\mathbb{S}^1$ -fibers. Denote by ξ the contact structure on the base given by $\xi = \pi_*([\mathcal{D}, \mathcal{D}])$ where π is the projection of Q onto M . Let $n \in \mathbb{N}$ denote the twisting number of $\mathcal{D}$ and let α be a suitable class in $H^1(M; \mathbb{Z}_n)$ associated to $\mathcal{D}$. Then $\mathcal{D}$ is classified up to Engle diffeomorphism by the set of data (ξ, n, α) .*

Proof. Let Q be an $\mathbb{S}^1$ -bundle over a closed, oriented 3-manifold M and let $\mathcal{D}_0$ and $\mathcal{D}_1$ be two Engel structures on Q which induce the same set of data (ξ, n, α) . According to our classification of fiberwise n -fold coverings of $\mathbb{P}(\xi)$, the n -fold prolongations $(Q, \mathcal{D}_0)$ and $(Q, \mathcal{D}_1)$ are both equivalent; i.e. there is an isomorphism ψ which makes the following diagram commutative:

$$\begin{array}{ccc} (Q, \mathcal{D}_0) & \xrightarrow{\psi} & (Q, \mathcal{D}_1) \\ & \searrow \phi_{\mathcal{D}_0} \quad \swarrow \phi_{\mathcal{D}_1} & \\ & \mathbb{P}(\xi) & \end{array}$$

Conversely, to every triple (ξ, n, α) we can construct an Engel structure with these data. $\square$

5. ON EULER CLASSES OF PROLONGATIONS

In §3 we derived a characterization of fiberwise n -fold coverings and we have seen that the mod n reduction of the Euler class determines their existence (see Theorem 3.3). In §4 we have seen that every n -fold prolongation $(Q, \mathcal{D})$ of a contact manifold (M, ξ) naturally carries the structure of a fiberwise n -fold covering of the prolongation $\mathbb{P}(\xi)$ via the development map $\phi_{\mathcal{D}}$ (see Theorem 1.1). Thus, the existence of n -fold prolongations of contact manifolds is connected to the vanishing of the mod n reduction of the Euler class $e(\mathbb{P}(\xi))$. This section is devoted to determining which contact structures admit n -fold prolongations. We start with the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose we are given an oriented contact manifold (M, ξ) . Choose a Riemannian metric on M and a trivialization of the tangent bundle TM . Then the 2-plane field ξ can be described in terms of its corresponding Gauß map $f_{\xi}: M \rightarrow S^2$, which assigns to each $x \in M$ the positive normal vector to ξ_x . In

fact, $f_{\xi}^*TS^2 = \xi$. Denote by u_0 the positive generator of $H^2(\mathbb{S}^2; \mathbb{Z})$. Combining naturality of the Euler class under pullback with the fact that $e(TS^2) = 2 \cdot u_0$, we conclude that $e(\xi) = 2 \cdot f_{\xi}^*u_0$. The unit-sphere bundle ξ_1 of ξ is a fiberwise 2-fold covering of the prolongation $\mathbb{P}(\xi)$ so that Lemma 3.1 implies $e(\mathbb{P}(\xi)) = 2 \cdot e(\xi_1)$. Because of $e(\xi) = e(\xi_1)$ we have

$$e(\mathbb{P}(\xi)) = 4 \cdot f_{\xi}^*u_0.$$

Hence, both the mod 4 reduction $e_4(\mathbb{P}(\xi))$ and the mod 2 reduction $e_2(\mathbb{P}(\xi))$ of the Euler class $e(\mathbb{P}(\xi))$ vanish. By Theorem 3.3 the result follows. $\square$

As a consequence of the last proof we see that n -fold prolongations of contact manifolds (M, ξ) often do not exist. Recall that the class $f_{\xi}^*u_0$ classifies the homotopy type of ξ over the 2-skeleton of M (cf. [3, §4.2]). Since in every homotopy class of 2-plane fields there is an overtwisted contact structure, it is easy to find examples for which $e(\mathbb{P}(\xi)) = 4 \cdot f_{\xi}^*u_0$ is non-zero when reduced modulo n .

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