PARTIALLY OVERDETERMINED PROBLEM IN SOME INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we consider the partially overdetermined problem in integral equations as follows:

$$\begin{cases} u(x) = A \int_{\Omega} \frac{1}{|x-y|^{n-\alpha}} u^p(y) dy + B, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = C, & x \in \Gamma \subseteq \partial\Omega, \end{cases}$$

where $0 < \alpha < n$, $p > \frac{n}{n-\alpha}$, A , B , C are positive constants, $\Omega \subset R^n$ ($n \geq 2$) is a bounded domain with $\partial\Omega \in C^1$, and Γ is a proper open set of $\partial\Omega$. Under some assumptions on the geometry of Γ , we prove that Ω must be a ball and u is radially symmetric and monotone decreasing with respect to the radius.

1. INTRODUCTION

In this paper, we consider the following integral equation:

$$(1.1) \quad \begin{cases} u(x) = A \int_{\Omega} \frac{1}{|x-y|^{n-\alpha}} u^p(y) dy + B, & x \in \Omega, \\ u > 0, & x \in \Omega, \end{cases}$$

where $0 < \alpha < n$, $p > \frac{n}{n-\alpha}$, A , B are positive constants, $\Omega \subset R^n$ ($n \geq 2$) is a bounded domain with $\partial\Omega \in C^1$.

In [13], under some natural integrability conditions, D. Li, G. Strohmer and L. Wang proved that if the solution u of (1.1) satisfies the boundary conditions

$$(1.2) \quad u = C \text{ on } \partial\Omega,$$

where C is a positive constant, then Ω is a ball and u is radially symmetric and monotone decreasing with respect to the radius.

Problems (1.1)–(1.2) are called overdetermined problems in integral equations. Recently, some very interesting results about overdetermined problems in integral equations were obtained in [11]–[15], [18]. The overdetermined problem in PDE is an interesting problem, while many authors used different methods to obtain a large amount of literature; see [2], [8], [9], [19], [20] and the references therein.

The aim of this paper is to study the *partially* overdetermined problem. More precisely, we want to find that if the boundary condition $u = C$ does not hold on

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global $\partial\Omega$ but on a proper set of $\partial\Omega$, i.e.,

$$(1.3) \quad u = C \text{ on } \Gamma \subseteq \partial\Omega,$$

does the same symmetry of domain and solution still hold?

From a mathematical point of view, the problem can be stated as follows. Let Γ be a nonempty, proper and connected subset of $\partial\Omega$, relatively open in $\partial\Omega$. The *partially* overdetermined problem in integral equation is

$$(1.4) \quad \begin{cases} u(x) = A \int_{\Omega} \frac{1}{|x-y|^{n-\alpha}} u^p(y) dy + B, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = C & x \in \Gamma \subseteq \partial\Omega. \end{cases}$$

In order to guarantee that the same symmetry in [13] still holds for (1.4), we need some additional assumptions on the geometry of Γ . Motivated by [9], where the *partially* overdetermined problem in PDE is studied, we can show that the *partially* overdetermined problem (1.4) is actually a *global* overdetermined problem; i.e., $u = C$ holds for any $x \in \partial\Omega$. Thus, we can apply the theorem in [13] to get the desired symmetry results.

The tool used in this paper is the method of moving planes in integral forms introduced by Chen, Li and Ou [3], which has been a powerful tool for obtaining symmetry, monotonicity and nonexistence of solutions for integral equations. It is entirely different from the traditional methods of moving planes introduced by Alexanderoff [1] and further developed by Serrin [19], Gidas, Ni and Nirenberg [10], etc. Instead of using local properties of a PDE, such as differentiability and maximum principles, the global properties of the integral equations and integral norms are established. For more details about the method of moving planes in integral forms, please see [3]–[7], [11]–[17] and the references therein.

This paper is organized as follows. In Section 2, the assumptions on Γ and the main theorem are stated. We prove the symmetry results of (1.4) and show some interesting remarks in Section 3.

2. MAIN THEOREMS AND SOME PRELIMINARIES

In order to give the additional assumptions on Γ , we need the following definition.

Definition 2.1. Assume $\partial\Omega \in C^1$. Consider a (hyper)plane $T \perp \partial\Omega$, namely such that T contains the unit outer normal $\nu(P)$ at some point P on $\partial\Omega$. Then take a connected component of $\partial\Omega \setminus T$ such that its closure γ contains P . We say that γ is a hat of $\partial\Omega$ determined by T if the bounded open domain delimited by γ and T is entirely contained in Ω .

Remark 2.2. If Ω is convex, any plane T which intersects $\partial\Omega$ orthogonally determines two hats (see Figure 1); $\gamma_1 = \widehat{A_1 M_1 A_2}$ and $\gamma_2 = \widehat{A_1 M_2 A_2}$ are exactly the two hats. But if Ω is nonconvex, then $\partial\Omega \setminus T$ may have more than two connected components, as happens for instance in Figure 2. Therein, $\gamma_1 = \widehat{A_1 M_1 A_2}$, $\gamma_2 = \widehat{A_3 M_2 A_4}$, $\gamma_3 = \widehat{A_5 M_3 A_6}$ are hats, whereas $\gamma_4 = \widehat{A_1 M_4 A_6}$ is not.

FIGURE 1. Hats of convex domain

FIGURE 2. Hats of non-convex domain

Our main result is as follows.

Theorem 2.3. Assume Ω is a convex bounded domain and $\partial\Omega \in C^1$. If

(A₁) $p > \frac{n}{n-\alpha}$ and the solution u of (1.4) satisfies $u \in L^{\frac{n(p-1)}{n-\alpha}}(\Omega) \cap C^1(\Omega)$,

(A₂) $\Gamma \subset \partial\Omega$ is nonempty, connected and relatively open in $\partial\Omega$, Γ contains a hat γ of $\partial\Omega$ determined by some plane $T \perp \partial\Omega$, and

(A₃) the maximum of u over $\partial\Omega$ is attained on Γ , i.e.,

$$(2.1) \quad \max_{\partial\Omega} u(x) = \max_{\Gamma} u(x),$$

then Ω is a ball and u is radially symmetric and monotone decreasing with respect to the radius.

Remark 2.4. In [13], the symmetry result is obtained under the following integral condition:

($\tilde{A}_1$) For $1 < \alpha < n$, $p > 0$, $u \in L^q(\Omega)$, where $q = 1$ for $p \leq 1$ and $q > \max\{\frac{n}{n-\alpha}, \frac{n}{\alpha}(p-1)\}$ for $p > 1$.

Actually, $u \in C^1(\Omega)$ can be obtained by the standard bootstrap method; see [13]. From the proof of Theorem 2.3, it is easy to find that (A₁) can be replaced by ($\tilde{A}_1$) and Theorem 2.3 still holds.

Remark 2.5. With the similar additional assumptions on $\Gamma \subset \partial\Omega$ as in Theorem 2.3, one can also consider the *partially* overdetermined problem similar to (1.4) for other integral equations and get the desired results; i.e., the *global* overdetermined problem studied in [11], [12], [14], [15] can also be discussed in the *partial* type and the same symmetry results will still hold.

The convex assumption on Ω makes sure that T cuts Ω into only two connected domains, and one of them is the bounded open domain delimited by the hat γ and T , which is a key point in our proof of Theorem 2.3. Actually, we mention that the convex assumption on Ω may be relaxed; this is explained in detail after the proof of Theorem 2.3 (see Remark 3.7).

It is natural to ask how far our assumptions are from being optimal, namely whether the same symmetry results in Theorem 2.3 still hold under weaker requirements on Γ . We do not pay our main attention to the optimal conditions, but we will show how our assumptions act in our proof in Remark 3.8 after the proof of Theorem 2.3. We hope this may be helpful to find the optimal conditions.

3. PROOF OF THEOREM 2.3

The strategy of our proof is to show that the *partially* overdetermined problem (1.4) is actually a *global* overdetermined problem; i.e., $u = C$ holds for any $x \in \partial\Omega$. The tool we used is the method of moving planes in integral forms with some modifications with respect to [3]. More precisely, the moving plane in [3] is arbitrary such that the radial symmetry of Ω and solution can be obtained at the same time, but here we move just one plane which must be carefully chosen, and only the symmetry of Ω about some plane can be deduced, which is enough for us to prove that $u = C$ holds for any $x \in \partial\Omega$.

We divide the proof into five steps.

Step 1. Choosing the proper moving plane.

We will move the plane T_λ that is parallel to the plane T in assumption (A_2) , which ensures that Γ contains a hat γ . Up to a rotation and a translation, we may assume that $T = T_0 = \{x \in \mathbb{R}^n \mid x_1 = 0\}$. Without loss of generality, we assume that the hat γ lies on the left of T_0 .

First, moving the plane T_λ from $\lambda = -\infty$ to the right, there must exist a $\lambda_0 < 0$ such that T_{λ_0} is tangent to $\partial\Omega$, i.e.,

$$\lambda_0 = \min\{\lambda \mid T_\lambda \cap \bar{\Omega} \neq \emptyset\}.$$

Then, we go on moving the plane T_λ continuously from T_{λ_0} to the right. At every stage, T_λ will cut off from Ω an open cap:

$$\Sigma_\lambda = \{x \in \Omega \mid x_1 < \lambda\}.$$

Since Ω is convex, Σ_λ is connected, and

$$(3.1) \quad \partial\Sigma_\lambda \setminus T_\lambda \subset \gamma \subset \Gamma, \quad \lambda \in (\lambda_0, 0],$$

$$(3.2) \quad u = C, \quad x \in \partial\Sigma_\lambda \setminus T_\lambda, \quad \lambda \in (\lambda_0, 0].$$

Denote by $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ the reflection of $x = (x_1, x_2, \dots, x_n)$ about the plane T_λ . Set

$$\Sigma'_\lambda = \{x \in \Omega \mid x^\lambda \in \Sigma_\lambda\}.$$

Since Ω is bounded and $\partial\Omega \in C^1$, Σ'_λ will remain in Ω at the beginning when $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$ for sufficiently small $\varepsilon > 0$. As the plane T_λ keeps moving to the right, Σ'_λ will stay in Ω until at least one of the following events occurs:

- (i) Σ'_λ is internally tangent to $\partial\Omega$ at some point $\hat{z}$ not on T_λ ,
- (ii) T_λ is orthogonal to $\partial\Omega$ at some point $\hat{z}$.

Let $\hat{\lambda}$ be the smallest value of λ for which either (i) or (ii) occurs.

It is easy to find that $\hat{\lambda} \in (\lambda_0, 0]$; by the construction and (3.2), there holds

$$(3.3) \quad u = C, \quad x \in \partial\Sigma_{\hat{\lambda}} \setminus T_{\hat{\lambda}} \subset \Gamma.$$

Step 2. A starting point to move the plane T_λ .

Define

$$(3.4) \quad \Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid \omega_\lambda(x) < 0\},$$

where

$$(3.5) \quad \omega_\lambda(x) = u(x^\lambda) - u(x), \quad x \in \Sigma_\lambda.$$

When $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$ for sufficiently small $\varepsilon > 0$, we show that Σ_λ^- must be empty, i.e.,

$$(3.6) \quad \omega_\lambda(x) = u(x^\lambda) - u(x) \geq 0, \quad x \in \Sigma_\lambda.$$

This provides a starting point to move the plane T_λ .

Step 3. Moving the plane to a limiting position.

Define

$$(3.7) \quad \bar{\lambda} = \sup\{\lambda \in (\lambda_0, \hat{\lambda}] \mid \omega_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda\}.$$

Then, we prove that $\bar{\lambda} = \hat{\lambda}$, which means that the plane T_λ can be moved continuously from T_{λ_0} to $T_{\hat{\lambda}}$ as long as (3.6) holds.

Step 4. Show that Ω is symmetric about $T_{\hat{\lambda}}$.

For $\bar{\lambda} = \hat{\lambda}$, with the help of assumption (A_2) in Theorem 2.3 and (3.3), Ω must be symmetric about $T_{\hat{\lambda}}$, i.e.,

$$(3.8) \quad \Sigma_{\hat{\lambda}} \cup (\Omega \cap T_{\hat{\lambda}}) \cup \Sigma'_{\hat{\lambda}} = \Omega,$$

which implies that

$$(3.9) \quad (\partial\Sigma_{\hat{\lambda}} \cup \partial\Sigma'_{\hat{\lambda}}) \setminus T_{\hat{\lambda}} = \partial\Omega.$$

Step 5. Prove that $u = C$ holds for any $x \in \partial\Omega$ and get the symmetry of the domain and solution.

In this step, with the help of the symmetry result in Step 4, we prove that

$$(3.10) \quad u(x) = C, \quad x \in \partial\Omega.$$

Hence, using the results of the *totally* overdetermined problem in [13], we can deduce from (3.10) that Ω is a ball and that u is radially symmetric and monotone decreasing with respect to the radius.

This completes the proof of Theorem 2.3.

It is easy to get

$$(3.11) \quad \begin{aligned} u(x) = A \int_{\Sigma_\lambda} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy &+ A \int_{\Sigma_\lambda} \frac{u^p(y^\lambda)}{|x-y^\lambda|^{n-\alpha}} dy \\ &+ A \int_{\Omega_\lambda} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy + B, \end{aligned}$$

$$(3.12) \quad \begin{aligned} u(x^\lambda) = A \int_{\Sigma_\lambda} \frac{u^p(y)}{|x^\lambda-y|^{n-\alpha}} dy &+ A \int_{\Sigma_\lambda} \frac{u^p(y^\lambda)}{|x^\lambda-y^\lambda|^{n-\alpha}} dy \\ &+ A \int_{\Omega_\lambda} \frac{u^p(y)}{|x^\lambda-y|^{n-\alpha}} dy + B. \end{aligned}$$

Notice that $|x - y| = |x^\lambda - y^\lambda|$ and $|x^\lambda - y| = |x - y^\lambda|$ for $x, y \in \Sigma_\lambda$. We get from (3.11) and (3.12) the following lemma, which is a key ingredient in our integral estimates.

Lemma 3.1. *For any $x \in \Sigma_\lambda$, $\lambda \in (\lambda_0, \hat{\lambda}]$,*

$$(3.13) \quad \begin{aligned} u(x) - u(x^\lambda) &= A \int_{\Sigma_\lambda} L(x, y, \lambda) [u^p(y) - u^p(y^\lambda)] dy \\ &\quad + A \int_{\Omega_\lambda} L(x, y, \lambda) u^p(y) dy, \end{aligned}$$

where $\Omega_\lambda = \Omega \setminus \overline{(\Sigma_\lambda \cup \Sigma'_\lambda)}$ and $L(x, y, \lambda) = \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x^\lambda-y|^{n-\alpha}}$ satisfy

$$(3.14) \quad L(x, y, \lambda) > 0, \quad x, y \in \Sigma_\lambda,$$

$$(3.15) \quad L(x, y, \lambda) < 0, \quad x \in \Sigma_\lambda, \quad y \in \Omega_\lambda.$$

Notice that the boundary condition is not used in the proofs of Steps 2 and 3. That means the proofs in [13] are still true here, so we present only the results and omit the details.

Lemma 3.2. *There exists a sufficiently small $\varepsilon > 0$ such that (3.6) holds for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon]$.*

Lemma 3.3. *For $\bar{\lambda}$ defined in (3.7), $\bar{\lambda} = \hat{\lambda}$.*

Now, we complete Step 4, which proves that Ω is symmetric about $T_{\hat{\lambda}}$.

Lemma 3.4. $\Sigma_{\hat{\lambda}} \cup (\Omega \cap T_{\hat{\lambda}}) \cup \Sigma'_{\hat{\lambda}} = \Omega$, i.e., $\partial\Omega = (\partial\Sigma_{\hat{\lambda}} \setminus T_{\hat{\lambda}}) \cup (\partial\Sigma'_{\hat{\lambda}} \setminus T_{\hat{\lambda}})$.

Proof. We prove the lemma with two cases.

Case 1. $\Sigma'_{\hat{\lambda}}$ is internally tangent to $\partial\Omega$ at some point $\hat{z}$ not on $T_{\hat{\lambda}}$.

If $\Sigma_{\hat{\lambda}} \cup (\Omega \cap T_{\hat{\lambda}}) \cup \Sigma'_{\hat{\lambda}} \neq \Omega$, then $\Omega_{\hat{\lambda}} \neq \emptyset$. Since $\bar{\lambda} = \hat{\lambda}$ in Lemma 3.3, then $u(y) \leq u(y^{\hat{\lambda}})$ for $y \in \Sigma_{\hat{\lambda}}$. Let $\hat{z}^{\hat{\lambda}}$ be the reflection of $\hat{z}$ about $T_{\hat{\lambda}}$; we deduce from (3.1) that $\hat{z}^{\hat{\lambda}} \in \partial\Sigma_{\hat{\lambda}} \setminus T_{\hat{\lambda}} \subset \Gamma$. By Lemma 3.1, (3.14) and (3.15) and the condition boundary (1.3), we have

$$(3.16) \quad u(\hat{z}^{\hat{\lambda}}) - u(\hat{z}) \leq A \int_{\Omega_{\hat{\lambda}}} \left(\frac{1}{|\hat{z}^{\hat{\lambda}} - y|^{n-\alpha}} - \frac{1}{|\hat{z} - y|^{n-\alpha}} \right) u^p(y) dy < 0.$$

For $\hat{z} \in \partial\Omega$, there must be two possibilities:

- (a) If $\hat{z} \in \Gamma$, then $\hat{z}^{\hat{\lambda}} \in \Gamma$ and $u(\hat{z}^{\hat{\lambda}}) = u(\hat{z})$, which is contradictive to (3.16).
- (b) If $\hat{z} \in \partial\Omega \setminus \Gamma$, then (3.16) implies that the maximum of u on $\partial\Omega$ is not attained on Γ , which is contradictive to assumption (A_3) in Theorem 2.3.

Case 2. $T_{\hat{\lambda}}$ is orthogononal to $\partial\Omega$ at some point $\hat{z}$.

Since Γ is open and $\hat{z} \in \Gamma$, the boundary condition (1.3) deduces that

$$\frac{\partial u(\hat{z})}{\partial x_1} = 0.$$

Following the same methods in [13] (see Lemma 3.5 in [13]), we can also deduce a contradiction. We omit the details here.

The above two cases imply that $\Omega_{\hat{\lambda}}$ is empty, which implies that $\Sigma_{\hat{\lambda}} \cup (\Omega \cap T_{\hat{\lambda}}) \cup \Sigma'_{\hat{\lambda}} = \Omega$, i.e., $\partial\Omega = (\partial\Sigma_{\hat{\lambda}} \setminus T_{\hat{\lambda}}) \cup (\partial\Sigma'_{\hat{\lambda}} \setminus T_{\hat{\lambda}})$. $\square$

Finally, in Step 5, we prove that the *partially* overdetermined problem (1.4) is actually a *global* overdetermined problem and get the symmetry results about the domain and solution.

Lemma 3.5. *For any $x \in \partial\Omega$, we have $u(x) = C$. Furthermore, Ω is a ball and u is radially symmetric and monotone decreasing with respect to the radius.*

Proof. Lemma 3.4 implies that $\Omega_{\hat{\lambda}}$ is empty. Noting

$$u^p(y) \leq u^p(y^{\hat{\lambda}}), \quad \forall y \in \Sigma_{\hat{\lambda}},$$

by Lemma 3.1, (3.14) and (3.15), we have

$$(3.17) \quad u(x) - u(x^{\hat{\lambda}}) = A \int_{\Sigma_{\hat{\lambda}}} \left(\frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x^{\hat{\lambda}} - y|^{n-\alpha}} \right) [u^p(y) - u^p(y^{\hat{\lambda}})] dy \leq 0.$$

For any $x \in \partial\Omega \setminus \Gamma$, it holds that $x^{\hat{\lambda}} \in \Gamma$ and that

$$u(x^{\hat{\lambda}}) - u(x) = C - u(x) \leq 0,$$

i.e.,

$$(3.18) \quad u(x) \geq C, \quad x \in \partial\Omega \setminus \Gamma.$$

Hence, we obtain from (2.1) and (3.18) that

$$\max_{\partial\Omega \setminus \Gamma} u(x) \leq \max_{\partial\Omega} u(x) = \max_{\Gamma} u(x) = C$$

and

$$u(x) = C, \quad x \in \partial\Omega \setminus \Gamma,$$

which implies that $u(x) = C$ holds on $\partial\Omega$.

This means that the *partially* overdetermined problem (1.4) is actually a *global* overdetermined problem (1.1)-(1.2). Due to the theorem in [13], we have that Ω is a ball and u is radially symmetric and monotone decreasing with respect to the radius. $\square$

Remark 3.6. Unlike the method of moving the plane in [13], the moving planes here must be carefully chosen and not be arbitrary. Therefore, we cannot directly get the symmetry of the solution u but only show that Ω is symmetric about $T_{\hat{\lambda}}$, the key to Lemma 3.4 in Step 4, which is enough for us to prove Lemma 3.5.

Remark 3.7. We have the convex assumption on Ω such that Σ_{λ} is connected for all $\lambda \in (\lambda_0, 0]$. More precisely, for any $\lambda \in (\lambda_0, 0]$, the boundary $\partial\Sigma_{\lambda} \setminus T_{\lambda}$ is still contained in $\gamma \subset \Gamma$ and so (3.2) holds, which is very important for the next two problems: comparing the value of $u(x)$ between $x \in \Sigma_{\hat{\lambda}}$ and $x \in \Sigma'_{\hat{\lambda}}$ to deduce a contradiction (see case 1 in Lemma 3.4); guaranteeing $\hat{z} \in \Gamma$ and so $\frac{\partial u(\hat{z})}{\partial x_1} = 0$, which is the target of the contradiction (see case 2 in Lemma 3.4). If we do not assume Ω is convex, then we may face a problem that cannot be overcome; see Figure 3 and Figure 4.

In Figure 3, $\gamma_1 = \widehat{A_1 M A_2}$, $\gamma_2 = \widehat{A_3 \hat{z} A_4}$, $\Gamma = \widehat{P_1 M P_2}$, both γ_1 and γ_2 are hats and $\gamma_1 \subset \Gamma$, the limiting position $\hat{\lambda}$ is case 1, and $\hat{z}$ is the tangent point. It is easy to find that $\partial\Sigma_{\lambda} \setminus T_{\lambda}$ is not contained in Γ , i.e., $\hat{z}$, $\hat{z}^{\hat{\lambda}} \notin \Gamma$. There is no useful information on $u(\hat{z}^{\hat{\lambda}})$ and $u(\hat{z})$; thus we cannot compare $u(\hat{z}^{\hat{\lambda}})$ and $u(\hat{z})$ to deduce a contradiction.

FIGURE 3. Case 1

FIGURE 4. Case 2

In Figure 4, $\gamma_1 = \widehat{A_1 M A_2}$, $\gamma_2 = \widehat{A_3 \hat{z} A_4}$, $\Gamma = \widehat{P_1 M P_2}$, both γ_1 and γ_2 are hats and $\gamma_1 \subset \Gamma$, the limiting position $\hat{\lambda}$ is case 2, and $\hat{z}$ is the vertical point. Notice that $\partial\Sigma_\lambda \setminus T_\lambda$ is not contained in Γ , i.e., $\hat{z} \notin \Gamma$. We do not have $\frac{\partial u(\hat{z})}{\partial x_1} = 0$ to deduce a contradiction. There is no way to deal with it.

Remark 3.8. If more requirements on Γ are supposed in (A_2) , then we can relax the convex assumption on Ω to make sure the proof is still valid. Indeed, we need only ensure that

$$(3.19) \quad \partial\Sigma_\lambda \setminus T_\lambda \subset \gamma \subset \Gamma, \quad \lambda \in [\lambda_0, \hat{\lambda}].$$

We say that a hat is a *good hat* if (3.19) holds, and a *bad hat* if not. For example, in Figure 3 and Figure 4, γ_1 is a *bad hat* and γ_2 is a *good hat*. If we assume that $\gamma_2 \subset \Gamma$, it is easy to find that the proof of Theorem 2.3 is true. Hence, if Γ contains a *good hat*, then our Theorem 2.3 still holds. However, we are not sure whether the *good hat* contained in Γ is the optimal assumption. Finally, we point out an interesting problem: Try to find the optimal conditions on Γ such that the symmetry of the domain and solution in (1.4) holds.

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