CODIMENSIONS OF POLYNOMIAL IDENTITIES OF REPRESENTATIONS OF LIE ALGEBRAS

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ABSTRACT. Consider a representation $\rho: L \rightarrow \mathfrak{gl}(V)$ where L is a Lie algebra and V is a finite dimensional vector space. We prove the analog of Amitsur's conjecture on asymptotic behavior for codimensions of polynomial identities of ρ .

1. INTRODUCTION

In the 1980's, conjectures about the asymptotic behaviour of codimensions of polynomial identities were made by S.A. Amitsur and A. Regev. Amitsur's conjecture was proved in 1999 by A. Giambruno and M.V. Zaicev [1, Theorem 6.5.2] for associative algebras and in 2002 by Zaicev [2] for finite dimensional Lie algebras. Alongside polynomial identities of algebras, polynomial identities of representations are also important [3, 4]. Therefore the question arises whether the conjectures hold for codimensions of representations.

Denote by $F\langle X \rangle$ the free associative algebra on the countable set $X = \{x_1, x_2, \dots\}$, i.e. the algebra of polynomials without a constant term in the noncommuting variables X over a field F of characteristic 0. Let $\rho: L \rightarrow \mathfrak{gl}(V)$ be a linear representation of a Lie algebra L on a vector space V over F . Let $f = f(x_1, \dots, x_n) \in F\langle X \rangle$. We say that f is a *polynomial identity* of ρ if $f(\rho(a_1), \dots, \rho(a_n))v = 0$ for all $a_1, \dots, a_n \in L$, $v \in V$. The set $\text{Id}(\rho)$ of polynomial identities of ρ is a two-sided ideal of $F\langle X \rangle$.

Example 1. Let $\rho: \mathfrak{sl}_2(F) \rightarrow \mathfrak{gl}(V)$ be any representation of the algebra $\mathfrak{sl}_2(F)$. Then

$$St_4 = \sum_{\sigma \in S_4} (\text{sign } \sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} \in \text{Id}(\rho),$$

where S_4 is the symmetric group on $\{1, 2, 3, 4\}$. Indeed, in order to verify a multilinear identity it is sufficient to substitute basis elements. Since $\dim \mathfrak{sl}_2(F) = 3$ and St_4 is alternating in 4 variables, it vanishes under all evaluations in $\rho(\mathfrak{sl}_2(F))$.

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Let P_n be the space of multilinear polynomials in the noncommuting variables $x_1, x_2, \dots, x_n$. The nonnegative integer $c_n(\rho) = \dim \frac{P_n}{P_n \cap \text{Id}(\rho)}$ is called the n th *codimension* of the representation ρ .

Another approach to polynomial identities of representations is concerned with representations of Lie algebras in associative algebras. Let $\tau: L \rightarrow A_0$ be a Lie homomorphism where L is a Lie F -algebra and A_0 is an associative F -algebra. Let $f = f(x_1, \dots, x_n) \in F\langle X \rangle$. We say that f is a *polynomial identity* of τ if $f(\tau(a_1), \dots, \tau(a_n)) = 0$ for all $a_1, \dots, a_n \in L$. Again, the set $\text{Id}(\tau)$ of polynomial identities of τ is a two-sided ideal of $F\langle X \rangle$. The nonnegative integer $c_n(\tau) = \dim \frac{P_n}{P_n \cap \text{Id}(\tau)}$ is called the n th *codimension* of the representation τ .

These approaches are equivalent. The first one is a partial case of the second one with $A_0 = \text{End}_F(V)$ and $\tau = \rho$. The second one is a partial case of the first one since $\text{Id}(\tau) = \text{Id}(\rho)$, where $\rho: L \rightarrow \mathfrak{gl}_F(A_0 + F \cdot 1)$, $\rho(a) \cdot b = \tau(a)b$, $a \in L$, $b \in A_0 + F \cdot 1$.

Throughout the paper we use the first approach.

Yu. A. Bahturin has suggested that we consider the analog of Amitsur's conjecture for representations.

Conjecture. *There exists $\lim_{n \rightarrow \infty} \sqrt[n]{c_n(\rho)} \in \mathbb{Z}_+$.*

Theorem 1. *Let $\rho: L \rightarrow \mathfrak{gl}(V)$ be a representation of a Lie algebra L on a finite dimensional vector space V over a field F of characteristic 0. Then either there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$, $d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leq c_n(\rho) \leq C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$ or the equality $c_n(\rho) = 0$ holds for all sufficiently large n .*

Corollary. *The analog of Amitsur's conjecture holds for such representations.*

Our proof follows the outline of the proof by Zaicev [2]. However, in many cases we need to apply new ideas.

Codimensions of representations do not change upon an extension of the base field. The proof is analogous to the cases of codimensions of associative [1, Theorem 4.1.9] and Lie algebras [2, Section 2]. Thus without loss of generality we may assume F to be algebraically closed.

Fix a Levi decomposition $\rho(L) = G + R$, where G is a maximal semisimple subalgebra of $\rho(L)$ and R is a solvable radical of $\rho(L)$. Denote by A the associative subalgebra of $\text{End}_F(V)$ generated by $\rho(L)$. Then A becomes a G -module under the natural left action by multiplication.

Consider left associative ideals $I_1, I_2, \dots, I_r, J_1, J_2, \dots, J_r$, $r \in \mathbb{Z}_+$, of the algebra A such that $J_k \subseteq I_k$, satisfying the conditions

- (1) I_k/J_k is an irreducible A -module or $\dim(I_k/J_k) = 1$;
- (2) for any G -submodules T_k such that $I_k = J_k \oplus T_k$, there exist elements $q_1, \dots, q_{r-1} \in A \cup \{1\}$ such that $T_1 q_1 T_2 \dots T_{r-1} q_{r-1} T_r \neq 0$.

Let M be a left A -module. Denote by $\text{Ann } M$ its annihilator in A . Let

$$d = d(\rho) = \max \left(\dim \frac{\rho(L)}{\rho(L) \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_r/J_r)} \right),$$

where the maximum is found among all $r \in \mathbb{Z}_+$ and all $I_1, \dots, I_r, J_1, \dots, J_r$ satisfying Conditions (1)–(2). We claim that $d(\rho)$ coincides with d from Theorem 1.

We need several auxiliary lemmas.

Lemma 1. $[\rho(L), R] \subseteq J(A)$, where $J(A)$ is the Jacobson radical of A .

Proof. Let $V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots \supseteq V_t = \{0\}$ be a composition chain in V . Then each V_i/V_{i+1} is an irreducible $\rho(L)$ -module. Denote the corresponding homomorphism by $\varphi_i: \rho(L) \rightarrow \mathfrak{gl}(V_i/V_{i+1})$. Then by E. Cartan's theorem [5, Proposition 1.4.11], $\varphi_i(\rho(L))$ is semisimple or the direct sum of a semisimple ideal and the center of $\mathfrak{gl}(V_i/V_{i+1})$. Thus $\varphi_i([\rho(L), \rho(L)])$ is semisimple and $\varphi_i([\rho(L), \rho(L)] \cap R) = 0$. Since $[\rho(L), R] \subseteq [\rho(L), \rho(L)] \cap R$, we have $\varphi_i([\rho(L), R]) = 0$ and $[\rho(L), R]V_i \subseteq V_{i+1}$. Therefore the associative ideal of A generated by $[\rho(L), R]$ is nilpotent since $a_1(b_{11} \dots b_{1k_1})a_2(b_{21} \dots b_{2k_2})a_3 \dots a_t$ is a zero operator on V for all $a_i \in [\rho(L), R]$ and $b_{ij} \in \rho(L)$. Thus $[\rho(L), R] \subseteq J(A)$. $\square$

Lemma 2. *There exists a subspace $S \subseteq R$ such that $R = S + \rho(L) \cap J(A)$ is the direct sum of subspaces and $[G, S] = 0$.*

Proof. There is a natural ad-action of G on R : $(\text{ad } g)a = [g, a]$, $g \in G$, $a \in R$. Since $\rho(L) \cap J(A)$ is a submodule of R and G is semisimple, there exists a complementary G -submodule S . Thus $[G, S] \subseteq S$. By virtue of Lemma 1, $[G, S] \subseteq [\rho(L), R] \subseteq \rho(L) \cap J(A)$. Therefore, $[G, S] = 0$. $\square$

Lemma 3. *Let M be an irreducible A -module. Then either M is an irreducible G -module or $\dim M = 1$. Furthermore, S acts on M by scalar operators.*

Proof. Note that M is an irreducible $\rho(L)$ -module. By Lemma 1, $[\rho(L), R] \subseteq \rho(L) \cap J(A)$. Therefore $[\rho(L), R]M \subseteq J(A)M = 0$ and, by virtue of Schur's lemma, $S \subseteq R$ acts by scalar operators on M . Thus all subspaces in M invariant under the action of G are invariant under the action of $\rho(L)$. $\square$

Lemma 4. *Let M be an irreducible A -module. Then*

$$\text{Ann}_{\rho(L)} M = \text{Ann}_G M + \text{Ann}_S M + (\rho(L) \cap J(A)).$$

Proof. If M is an irreducible A -module, then $J(A)M = 0$. Thus it is sufficient to prove that if $g + s \in \text{Ann}_{\rho(L)} M$, $g \in G$, $s \in S$, then $g \in \text{Ann}_G M$ and $s \in \text{Ann}_S M$. Denote $\varphi: \rho(L) \rightarrow \mathfrak{gl}(M)$. Note that $sm = -gm$ for all $m \in M$ since $g + s \in \text{Ann}_{\rho(L)} M$. By Lemma 3, s is a scalar operator on M . Therefore, g is a scalar operator on M too, and $\varphi(g)$ belongs to the center of the semisimple algebra $\varphi(G)$. Hence $\varphi(g) = 0$, and this finishes the proof of the lemma. $\square$

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups. The symmetric group S_n acts on the space $\frac{P_n}{P_n \cap \text{Id}(\rho)}$ by permuting the variables. Irreducible FS_n -modules are described by partitions $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and their Young diagrams D_λ . The character $\chi_n(\rho)$ of the FS_n -module $\frac{P_n}{P_n \cap \text{Id}(\rho)}$ is called the *n th cocharacter* of the representation ρ . We can rewrite it as a sum $\chi_n(\rho) = \sum_{\lambda \vdash n} m(\rho, \lambda) \chi(\lambda)$ of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ (where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\text{sign } \sigma) \sigma$) be the Young symmetrizers corresponding to a Young tableau T_λ . Then $\bar{M}(\lambda) = FS e_{T_\lambda} \cong FS e_{T_\lambda}^*$ is an irreducible FS_n -module corresponding to a partition $\lambda \vdash n$. We refer the reader to [1, 6, 7] for accounts of S_n -representations and their applications to polynomial identities.

2. UPPER BOUND

Fix a composition chain of A -submodules

$$A = B_0 \supsetneq B_1 \supsetneq B_2 \supsetneq \dots \supsetneq J(A) \supsetneq \dots \supsetneq B_{\theta-1} \supsetneq B_\theta = \{0\}$$

in the module A . Let $\text{ht } a = \max_{a \in B_k} k$ for $a \in A$.

Remark. If $d = d(\rho) = 0$, then $\rho(L) \subseteq \text{Ann}(B_{i-1}/B_i)$ for all $1 \leq i \leq \theta$ and $a_1 a_2 \dots a_n = 0$ for all $a_i \in \rho(L)$ and $n \geq \theta + 1$. Thus $c_n(\rho) = 0$ for all $n \geq \theta + 1$. Therefore we assume $d > 0$.

Let $Y = \{y_{11}, y_{12}, \dots, y_{1j_1}; y_{21}, y_{22}, \dots, y_{2j_2}; \dots; y_{m1}, y_{m2}, \dots, y_{mj_m}\}$, $Y_1, \dots, Y_q$, and $\{z_1, \dots, z_m\}$ be subsets of $\{x_1, x_2, \dots, x_n\}$ such that $Y_i \subseteq Y$, $|Y_i| = d + 1$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$, $Y \cap \{z_1, \dots, z_m\} = \emptyset$, $j_i \geq 0$. Denote

$$g_{m,q} = \text{Alt}_1 \dots \text{Alt}_q((y_{11}y_{12} \dots y_{1j_1})z_1(y_{21}y_{22} \dots y_{2j_2})z_2 \dots (y_{m1}y_{m2} \dots y_{mj_m})z_m),$$

where Alt_i is the operator of alternation on the variables of Y_i .

Let $\varphi: F\langle X \rangle \rightarrow A$ be a homomorphism induced by a substitution $\{x_1, x_2, \dots, x_n\} \rightarrow \rho(L)$. We say that φ is *proper* for $g_{m,q}$ if $\varphi(z_i) \in \rho(L) \cap J(A)$ for $1 \leq i \leq m-1$, $\varphi(z_m) \in (\rho(L) \cap J(A)) \cup G \cup S$, and $\varphi(y_{ik}) \in G \cup S$ for $1 \leq i \leq m$, $1 \leq k \leq j_i$.

Lemma 5. *Let φ be a proper homomorphism for $g_{m,q}$. Then $\varphi(g_{m,q})$ can be rewritten as a sum of $\psi(g_{m+1,q'})$, where ψ is a proper homomorphism for $g_{m+1,q'}$, $q' \geq q - (\dim A)m - 2$. (Y' , Y'_i , $z'_1, \dots, z'_{m+1}$ may be different for different terms.)*

Proof. Let $\alpha_i = \text{ht } \varphi(z_i)$. We will use induction on $\sum_{i=1}^m \alpha_i$. (The sum will grow.) Note that $\alpha_i \leq \theta \leq \dim A$. Denote $I_i = B_{\alpha_i}$, $J_i = B_{\alpha_i + 1}$.

First, consider the case when $I_1, \dots, I_m$, $J_1, \dots, J_m$ do not satisfy Conditions (1)–(2). In this case we can choose G -submodules T_i , $I_i = T_i \oplus J_i$ such that

$$T_1 h_1 T_2 h_2 \dots h_{m-1} T_m = 0$$

for all $h_i \in A \cup \{1\}$. Rewrite $\varphi(z_i) = a'_i + a''_i$, $a'_i \in T_i$, $a''_i \in J_i$. Note that $\text{ht } a''_i > \text{ht } \varphi(z_i)$. Since $g_{m,q}$ is multilinear, we can rewrite $\varphi(g_{m,q})$ as a sum of similar terms $\tilde{\varphi}(g_{m,q})$, where $\tilde{\varphi}(z_i)$ equals either a'_i or a''_i . The term where all $\tilde{\varphi}(z_i) = a'_i \in T_i$ equals 0 since $I_1, \dots, I_m$, $J_1, \dots, J_m$ do not satisfy Conditions (1)–(2). For the other terms $\tilde{\varphi}(g_{m,q})$ we have $\sum_{i=1}^m \text{ht } \tilde{\varphi}(z_i) > \sum_{i=1}^m \text{ht } \varphi(z_i)$.

Thus without loss of generality we may assume that $I_1, \dots, I_m$, $J_1, \dots, J_m$ satisfy Conditions (1)–(2). In this case, $\dim(\rho(L) \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)) \geq \dim(\rho(L)) - d$. By virtue of Lemma 4,

$$\begin{aligned} & \rho(L) \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m) \\ &= G \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m) \\ &+ S \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m) + \rho(L) \cap J(A). \end{aligned}$$

Choose a basis in G that includes a basis of $G \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$ and a basis in S that includes a basis of $S \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$. Since $g_{m,q}$ is multilinear, we may assume that only basis elements are substituted for $y_{k\ell}$. Note that $g_{m,q}$ is alternating in Y_i . Hence, if $\varphi(g_{m,q}) \neq 0$, then for every $1 \leq i \leq q$ there exists $y_{k\ell} \in Y_i$ such that either $\varphi(y_{k\ell}) \in G \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$ or $\varphi(y_{k\ell}) \in S \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$.

Consider the case when $\varphi(y_{k\ell}) \in G \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$ for some $y_{k\ell}$. We can choose G -submodules T_k such that $I_k = T_k \oplus J_k$. We may assume that $\varphi(z_k) \in T_k$ since elements of J_k have greater heights. Therefore $a\varphi(z_k) \in T_k \cap J_k$ for all $a \in G \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$. Hence $a\varphi(z_k) = 0$. Moreover, $G \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$ is a Lie ideal of G and $[G, S] = 0$. Thus $\varphi(y_{k1} \dots y_{kj_k} z_k) = 0$. Expanding the alternations, we obtain $\varphi(g_{m,q}) = 0$.

Consider the case when $\varphi(y_{k\ell}) \in S \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$ for some $y_{k\ell} \in Y_q$. Expand the alternation Alt_q in $g_{m,q}$ and rewrite $g_{m,q}$ as a sum of

$$\begin{aligned} \tilde{g}_{m,q-1} = \text{Alt}_1 \dots \text{Alt}_{q-1} &((y_{11}y_{12} \dots y_{1j_1})z_1(y_{21}y_{22} \dots y_{2j_2})z_2 \\ &\dots (y_{m1}y_{m2} \dots y_{mj_m})z_m). \end{aligned}$$

The operator Alt_q may change indices; however we keep the notation $y_{k\ell}$ for the variable with the property $\varphi(y_{k\ell}) \in S \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_m/J_m)$. Now the alternation does not affect $y_{k\ell}$. Note that

$$\begin{aligned} y_{k1}y_{k2} \dots y_{kj_k}z_k &= y_{k1}y_{k2} \dots y_{k,\ell-1}y_{k,\ell+1} \dots y_{kj_k}y_{k\ell}z_k \\ &+ \sum_{\beta=\ell+1}^{j_k} y_{k1}y_{k2} \dots y_{k,\ell-1}y_{k,\ell+1} \dots y_{k,\beta-1}[y_{k\ell}, y_{k\beta}]y_{k,\beta+1} \dots y_{kj_k}z_k. \end{aligned}$$

In the first polynomial on the right hand side we replace $y_{k\ell}z_k$ with z'_k and define $\varphi'(z'_k) = \varphi(y_{k\ell})\varphi(z_k)$, $\varphi'(x) = \varphi(x)$ for other variables x . Then $\text{ht } \varphi'(z'_k) > \text{ht } \varphi(z_k)$ and we can use the inductive assumption. If $y_{k\beta} \in Y_j$ for some j , then we expand the alternation Alt_j in this term in $\tilde{g}_{m,q-1}$. If $\varphi(y_{k\beta}) \in G$, then this term is zero. If $\varphi(y_{k\beta}) \in S$, then $\varphi([y_{k\ell}, y_{k\beta}]) \in \rho(L) \cap J(A)$. We replace $[y_{k\ell}, y_{k\beta}]$ with an additional variable z'_{m+1} and define $\psi(z'_{m+1}) = \varphi([y_{k\ell}, y_{k\beta}])$, $\psi(x) = \varphi(x)$ for other variables x . Then the polynomial has the desired form. In each inductive step we reduce q by no more than 1, and the maximal number of inductive steps equals $(\dim A)m$. This finishes the proof. $\square$

Since the Jacobson radical is a nilpotent ideal, $J(A)^p = 0$ for some $p \in \mathbb{N}$.

Lemma 6. *If $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and $\lambda_{d+1} \geq p((\dim A)p + 3)$ or $\lambda_{\dim \rho(L)+1} > 0$, then $m(\rho, \lambda) = 0$.*

Proof. It is sufficient to prove that $e_{T_\lambda}^* f \in \text{Id}(\rho)$ for every $f \in P_n$ and a Young tableau T_λ , $\lambda \vdash n$, with $\lambda_{d+1} \geq p((\dim A)p + 3)$ or $\lambda_{\dim \rho(L)+1} > 0$.

Fix some basis of $\rho(L)$ that is a union of bases of G , S , and $\rho(L) \cap J(A)$. Since polynomials are multilinear, it is sufficient to substitute only basis elements. Note that $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ and b_{T_λ} alternates the variables of each column of T_λ . Hence if we make a substitution and $e_{T_\lambda}^* f$ does not vanish, then this implies that different basis elements are substituted for the variables of each column. But if $\lambda_{\dim \rho(L)+1} > 0$, then the length of the first column is greater than $\dim \rho(L)$. Therefore, $e_{T_\lambda}^* f \in \text{Id}(\rho)$.

Consider the case $\lambda_{d+1} \geq p((\dim A)p + 3)$. Let φ be a substitution of basis elements for the variables $x_1, \dots, x_n$. Then $e_{T_\lambda}^* f$ can be rewritten as a sum of polynomials $g_{m,q}$ where $1 \leq m \leq p$, $q \geq p((\dim A)p + 2)$, and z_i , $1 \leq i \leq m-1$, are replaced with elements of $\rho(L) \cap J(A)$. (For different terms $g_{m,q}$, the numbers m and q , variables z_i , y_{ij} , and sets Y_i can be different.) Indeed, we expand symmetrization

on all variables and alternation on the variables replaced with $\rho(L) \cap J(A)$. Also, we expand the alternation on the set that includes the last variable, which we denote by z_m .

Applying Lemma 5 many times, we increase m . The ideal $J(A)$ is nilpotent and $\varphi(g_{p+1,q}) = 0$ for every q and proper homomorphism φ . Reducing q by no more than $p((\dim A)p + 2)$, we obtain $\varphi(e_{T_\lambda}^* f) = 0$. $\square$

Now we can prove

Theorem 2. *If $d > 0$, then there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n(\rho) \leq C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. In the case $d = 0$, the equality $c_n(\rho) = 0$ holds for all sufficiently large n .*

Proof. Lemma 6 and [1, Lemmas 6.2.4, 6.2.5] imply

$$(1) \quad \sum_{m(\rho, \lambda) \neq 0} \dim M(\lambda) \leq C_3 n^{r_3} d^n$$

for some constants $C_3, r_3 > 0$. Denote by $\text{Id}(A) \subseteq F\langle X \rangle$ the ideal of polynomial identities of the associative algebra A . Let $\chi_n(A) = \chi\left(\frac{P_n}{P_n \cap \text{Id}(A)}\right) = \sum_\lambda m(A, \lambda) \chi(\lambda)$. Then $\text{Id}(A) \subseteq \text{Id}(\rho)$ and $m(\rho, \lambda) \leq m(A, \lambda) \leq C_4 n^{r_4}$ for some $C_4, r_4 > 0$ by [1, Lemma 4.9.2]. Together with (1) this implies the upper bound. $\square$

3. LOWER BOUND

By the definition of d , there exist left associative ideals $I_1, I_2, \dots, I_r, J_1, J_2, \dots, J_r$, $r \in \mathbb{Z}_+$, of the algebra A , satisfying Conditions (1)–(2), $J_k \subseteq I_k$, such that

$$d = \dim \frac{\rho(L)}{\rho(L) \cap \text{Ann}(I_1/J_1) \cap \dots \cap \text{Ann}(I_r/J_r)}.$$

We consider the case $d > 0$.

Without loss of generality we may assume that

$$\rho(L) \cap \bigcap_{k=1}^r \text{Ann}(I_k/J_k) \neq \rho(L) \cap \bigcap_{\substack{k=1, \\ k \neq \ell}}^r \text{Ann}(I_k/J_k)$$

for all $1 \leq \ell \leq r$. In particular, $\rho(L)$ has nonzero action on each I_k/J_k .

Our aim is to present a partition $\lambda \vdash n$ with $m(\rho, \lambda) \neq 0$ such that $\dim M(\lambda)$ has the desired asymptotic behavior. We will glue alternating polynomials constructed by Yu.P. Razmyslov's theorem from faithful irreducible modules over reductive algebras. In order to do this, we have to choose the reductive algebras.

Lemma 7. *There exist Lie ideals $G_1, \dots, G_r$ in G and elements $a_1, \dots, a_r \in S$ (some of a_i and G_j may be zero) such that*

- (1) $G_1 + \dots + G_r = G_1 \oplus \dots \oplus G_r$;
- (2) $\langle a_1, a_2, \dots, a_r \rangle_F = \langle a_1 \rangle_F \oplus \dots \oplus \langle a_r \rangle_F$;
- (3) $\sum_{k=1}^r (\dim(G_k \oplus \langle a_k \rangle_F)) = d$;
- (4) I_k/J_k is a faithful $G_k \oplus \langle a_k \rangle_F$ -module;
- (5) $G_i I_k/J_k = a_i I_k/J_k = 0$ for $i > k$.

Proof. Consider $N_\ell = \rho(L) \cap \bigcap_{k=1}^{\ell} \text{Ann}(I_k/J_k)$, $1 \leq \ell \leq r$, $N_0 = \rho(L)$. Since G is semisimple, we can choose such ideals G_ℓ that $N_{\ell-1} \cap G = G_\ell \oplus (N_\ell \cap G)$. Note that $\dim(S/\text{Ann}_S(I_\ell/J_\ell)) \leq 1$ since S acts by scalar operators by Lemma 3. Thus we can choose elements a_ℓ such that $N_{\ell-1} \cap S = \langle a_\ell \rangle_F \oplus (N_\ell \cap G)$. Hence Properties (1), (2), (4), (5) hold. By Lemma 4, $N_k = N_k \cap G + N_k \cap S + \rho(L) \cap J(A)$. Therefore,

$$N_{\ell-1} = G_\ell + N_\ell \cap G + \langle a_\ell \rangle_F + N_\ell \cap S + \rho(L) \cap J(A) = (G_\ell + \langle a_\ell \rangle_F) + N_\ell$$

(a direct sum of subspaces) and Property (3) holds. $\square$

Lemma 8. *For every $1 \leq i \leq r$ there exists a decomposition $a_i = b_i + c_i$ such that $c_i \in A$ acts as a diagonalizable operator on A and $b_i \in J(A)$. Furthermore, elements c_i commute with each other and b_i and c_i are polynomials in $a_1, \dots, a_r$.*

Proof. The solvable Lie algebra $R + J(A)$ acts on the algebra $\tilde{A} = A + F \cdot 1$ by left multiplication. By virtue of the Lie theorem, there exists a basis in $\tilde{A}$ in which all the operators from $R + J(A)$ have upper triangular matrices. Denote the corresponding embedding $A \hookrightarrow M_m(F)$ by ψ . Here $m = \dim \tilde{A}$.

Let A_1 be the associative algebra generated by $a_1, \dots, a_r$. Since $\psi(R) \subseteq \mathfrak{t}_m(F)$, we have $\psi(A_1) \subseteq UT_m(F)$. Here $UT_m(F)$ is the associative algebra of upper triangular matrices $m \times m$. There is a decomposition $UT_m(F) = Fe_{11} + Fe_{22} + \dots + Fe_{mm} + N$ where $N = \langle e_{ij} \mid 1 \leq i < j \leq m \rangle_F$ is a nilpotent ideal. Thus there is no subalgebra in A_1 isomorphic to $M_2(F)$ and, by the Wedderburn-Malcev theorem, $A_1 = Fe_1 + \dots + Fe_t + J(A_1)$ (the direct sum of subspaces) for some idempotents $e_i \in A_1$. Denote for every a_i its component in $J(A_1)$ by b_i and its component in $Fe_1 \oplus \dots \oplus Fe_t$ by c_i . Note that e_i are commuting diagonalizable operators. Thus they have a common basis of eigenvectors in $\tilde{A}$, and c_i are commuting diagonalizable operators, too.

We claim that the space $J(A_1) + J(A)$ generates a nilpotent ideal I in A . First, $\psi(J(A_1)), \psi(J(A)) \subseteq UT_m(F)$ and consist of nilpotent elements. Thus, the corresponding matrices have zero diagonal elements and $\psi(J(A_1)), \psi(J(A)) \subseteq N$. Denote $N_k = \langle e_{ij} \mid i + k \leq j \rangle_F \subseteq N$. Then

$$N = N_1 \supsetneq N_2 \supsetneq \dots \supsetneq N_{m-1} \supsetneq N_m = \{0\}.$$

Let $\text{ht}_N a = k$ if $\psi(a) \in N_k$, $\psi(a) \notin N_{k+1}$.

Recall that $(J(A))^p = 0$. We claim that $I^{m+p} = 0$. The space I^{m+p} is a span of $h_1 j_1 h_2 j_2 \dots j_{m+p} h_{m+p+1}$ where $j_k \in J(A_1) \cup J(A)$ and $h_k \in A \cup \{1\}$. If at least p elements j_k belong to $J(A)$, then the product equals 0. Thus we may assume that at least m elements j_k belong to $J_1(A)$.

Let $j_i \in J(A_1)$, $h_i \in A \cup \{1\}$. We prove by induction on ℓ that $j_1 h_1 j_2 h_2 \dots h_{\ell-1} j_\ell$ can be expressed as a sum of $\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\alpha j'_1 j'_2 \dots j'_\beta a$ where $\tilde{j}_i \in J(A_1)$, $j'_i \in J(A)$, $a \in A \cup \{1\}$, and $\alpha + \sum_{i=1}^{\beta} \text{ht}_N j'_i \geq \ell$. Indeed, suppose that $j_1 h_1 j_2 h_2 \dots h_{\ell-2} j_{\ell-1}$ can be expressed as a sum of $\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa a$ where $\tilde{j}_i \in J(A_1)$, $j'_i \in J(A)$, $a \in \tilde{A}$, and $\gamma + \sum_{i=1}^{\varkappa} \text{ht}_N j'_i \geq \ell - 1$. Then $j_1 h_1 j_2 h_2 \dots j_{\ell-1} h_{\ell-1} j_\ell$ is a sum of

$$\begin{aligned} \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa a h_{\ell-1} j_\ell &= \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa [a h_{\ell-1}, j_\ell] \\ &\quad + \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j'_1 j'_2 \dots j'_\varkappa j_\ell (a h_{\ell-1}). \end{aligned}$$

Note that, by virtue of the Jacobi identity and Lemma 1, $[ah_{\ell-1}, j_\ell] \in J(A)$. Thus it is sufficient to consider only the second term in the right hand side. However,

$$\begin{aligned} \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma \tilde{j}'_1 \tilde{j}'_2 \dots \tilde{j}'_\varkappa j_\ell (ah_{\ell-1}) &= \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma j_\ell \tilde{j}'_1 \tilde{j}'_2 \dots \tilde{j}'_\varkappa (ah_{\ell-1}) \\ &+ \sum_{i=1}^{\varkappa} \tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_\gamma \tilde{j}'_1 \tilde{j}'_2 \dots \tilde{j}'_{i-1} [j'_i, j_\ell] \tilde{j}'_{i+1} \dots \tilde{j}'_\varkappa (ah_{\ell-1}). \end{aligned}$$

Since $[j'_i, j_\ell] \in J(A)$ and $\text{ht}_N[j'_i, j_\ell] \geq 1 + \text{ht}_N j'_i$, all the terms have the desired form. Therefore,

$$j_1 h_1 j_2 h_2 \dots j_{m-1} h_{m-1} j_m \in \psi^{-1}(N_m) = \{0\},$$

$I^{m+p} = 0$, and

$$J(A) \subseteq J(A_1) + J(A) \subseteq I \subseteq J(A).$$

In particular, $b_i \in J(A_1) \subseteq J(A)$. □

Lemma 9. *There exist complementary subspaces $I_k = \tilde{T}_k \oplus J_k$ such that*

- (1) $\tilde{T}_k$ is an irreducible $G \oplus \langle c_1, \dots, c_r \rangle_F$ -submodule;
- (2) c_i acts on each $\tilde{T}_k$ as a scalar operator;
- (3) $\tilde{T}_k$ is a faithful $G_k \oplus \langle c_k \rangle_F$ -module. Moreover, $\sum_{k=1}^r \dim(G_k \oplus \langle c_k \rangle_F) = d$.
- (4) $G_i \tilde{T}_k = c_i \tilde{T}_k = 0$ for $i > k$.

Proof. The elements c_i are diagonalizable on A and commute. Therefore, an eigenspace of any c_i is invariant under the action of other c_k . Using induction, we split $A = \bigoplus_{i=1}^\alpha W_i$, where W_i are intersections of eigenspaces of c_k and elements c_k act as scalar operators on W_i . By virtue of Lemma 1 and the Jacobi identity, $[c_i, G] = 0$. Thus W_i are G -submodules and A is a completely reducible $G \oplus \langle c_1, \dots, c_r \rangle_F$ -module. Therefore, for every J_k we can choose a complementary $G \oplus \langle c_1, \dots, c_r \rangle_F$ -submodule $\tilde{T}_k$ in I_k . Then $\tilde{T}_k \cong I_k/J_k$ is an irreducible G -module by Lemma 3. Property (1) is proven.

By Lemma 3, for every a_i there exists $\gamma \in F$ such that for every $h \in I_k$ we have $a_i h = \gamma h + j$ where $j \in J_k$. Thus $c_i h = a_i h - b_i h = \gamma h + (j - b_i h)$ where $j - b_i h \in J_k$. However, $c_i h \in \tilde{T}_k$ for $h \in \tilde{T}_k$, and $j - b_i h = 0$. Property (2) is proven.

By Lemma 7, each I_i/J_i is a faithful $G_i \oplus \langle a_i \rangle_F$ -module and either $a_i = 0$ or a_i acts on I_i/J_i by a nonzero scalar operator. Thus either $a_i = c_i = 0$ or $a_i \neq 0$ and $c_i \neq 0$ acts on $\tilde{T}_i$ by a nonzero scalar operator that belongs to the center of $\text{End}_F \tilde{T}_i$. Since all homomorphic images of G_i have zero center, each $\tilde{T}_i$ is a faithful $G_i \oplus \langle c_i \rangle_F$ -module and $\dim(G_k \oplus \langle c_k \rangle_F) = \dim(G_k \oplus \langle a_k \rangle_F)$. Property (3) is proven.

Property (4) is a consequence of Property (5) of Lemma 7. □

Lemma 10. *Let M be a finite dimensional irreducible module over a semisimple Lie algebra G and $\tilde{G}$ be a Lie ideal of G such that M is a faithful $\tilde{G}$ -module. Then there exist faithful irreducible $\tilde{G}$ -submodules $\tilde{M}_i \subseteq M$ such that $M = \tilde{M}_1 \oplus \tilde{M}_2 \oplus \dots \oplus \tilde{M}_k$.*

Proof. There exists a semisimple ideal $\tilde{G}$ such that $G = \tilde{G} \oplus \bar{G}$. By [5, Proposition 7.3.1'], $M \cong \tilde{M} \otimes \bar{M}$ for some irreducible $\tilde{G}$ -module $\tilde{M}$ and irreducible $\bar{G}$ -module $\bar{M}$. Let $m_1, \dots, m_k$ be basis elements of $\bar{M}$. Then $M = (\tilde{M} \otimes m_1) \oplus (\tilde{M} \otimes m_2) \oplus \dots \oplus (\tilde{M} \otimes m_k)$ is the direct sum of irreducible $\tilde{G}$ -modules. These submodules are faithful since M is faithful. □

By Condition (2) of the definition of $d(\rho)$, there exist elements $q_1, \dots, q_{r-1} \in A \cup \{1\}$ such that $\tilde{T}_1 q_1 \tilde{T}_2 \dots \tilde{T}_{r-1} q_{r-1} \tilde{T}_r \neq 0$. Choose $n_i \in \mathbb{Z}_+$ with the maximal $\sum_{i=0}^{r-1} n_i$ such that

$$\left(\prod_{k=1}^{n_0} j_{0k} \right) \tilde{T}_1 q_1 \left(\prod_{k=1}^{n_1} j_{1k} \right) \tilde{T}_2 \dots q_{r-1} \left(\prod_{k=1}^{n_{r-1}} j_{r-1,k} \right) \tilde{T}_r \neq 0$$

for some $j_{ik} \in J(A)$. Denote $(q_i \prod_{k=1}^{n_i} j_{ik})$ again by q_i for $1 \leq i \leq r-1$ and $\prod_{k=1}^{n_0} j_{0k}$ by q_0 . Then $q_i \in A \cup \{1\}$ and

$$q_0 \tilde{T}_1 q_1 \tilde{T}_2 \dots \tilde{T}_{r-1} q_{r-1} \tilde{T}_r \neq 0,$$

but

$$(2) \quad q_0 \tilde{T}_1 q_1 \tilde{T}_2 \dots \tilde{T}_{k-1} q_{k-1} (j \tilde{T}_k) q_k \tilde{T}_{k+1} \dots \tilde{T}_{r-1} q_{r-1} \tilde{T}_r = 0$$

for all $j \in J(A)$ and $1 \leq k \leq r$.

By Lemma 10, for every k we can choose a faithful irreducible $G_k \oplus \langle c_k \rangle_F$ -submodule $T_k \subseteq \tilde{T}_k$ such that

$$(3) \quad q_0 T_1 q_1 T_2 \dots T_{r-1} q_{r-1} T_r \neq 0.$$

Lemma 11. *Let $\varphi: \bigoplus_{i=1}^r (G_i \oplus \langle a_i \rangle) \rightarrow \bigoplus_{i=1}^r (G_i \oplus \langle c_i \rangle)$ be the isomorphism defined by formulas $\varphi(g) = g$ for all $g \in G_i$ and $\varphi(a_i) = c_i$. Let f_i be multilinear polynomials and $h_1^{(i)}, \dots, h_{n_i}^{(i)} \in \bigoplus_{k=1}^r (G_k \oplus \langle a_k \rangle_F)$, $t_i \in \tilde{T}_i$ be some elements. Then*

$$\begin{aligned} & q_0 f_1(h_1^{(1)}, \dots, h_{n_1}^{(1)}) t_1 \ q_1 f_2(h_1^{(2)}, \dots, h_{n_2}^{(2)}) t_2 \dots q_{r-1} f_r(h_1^{(r)}, \dots, h_{n_r}^{(r)}) t_r \\ &= q_0 f_1(\varphi(h_1^{(1)}), \dots, \varphi(h_{n_1}^{(1)})) t_1 \ q_1 f_2(\varphi(h_1^{(2)}), \dots, \varphi(h_{n_2}^{(2)})) t_2 \\ & \quad \dots q_{r-1} f_r(\varphi(h_1^{(r)}), \dots, \varphi(h_{n_r}^{(r)})) t_r. \end{aligned}$$

In other words, we can replace a_i with c_i and the result does not change.

Proof. We rewrite $a_i = b_i + c_i = b_i + \varphi(a_i)$ and use the multilinearity of f_i . By (2), terms with b_i vanish. $\square$

Let $mk \leq n$ where $m, k, n \in \mathbb{N}$ are some numbers. Denote by $Q_{m,k,n} \subseteq P_n$ the subspace spanned by all polynomials that are alternating in k disjoint subsets of variables $\{x_1^i, \dots, x_m^i\} \subseteq \{x_1, x_2, \dots, x_n\}$, $1 \leq i \leq k$.

We need the following theorem [3, Remark 12.1].

Theorem 3 (Yu. P. Razmyslov). *Let B be a reductive Lie algebra over an algebraically closed field of characteristic zero, and let $\dim B = m$. Let $\tau: B \hookrightarrow U$ be an embedding of B to a simple associative algebra U generated by $\tau(B)$, with a nonzero center. Then there exist k and a multilinear polynomial $f \in Q_{m,k,mk} \setminus \text{Id}(\tau)$ such that $f(\tau(p_1), \dots, \tau(p_{mk}))$ belongs to the center of U for all $p_i \in B$.*

Lemma 12. *If $d \neq 0$, then there exist numbers $k, n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist disjoint subsets $X_1, \dots, X_{k\ell} \subseteq \{x_1, \dots, x_n\}$, $\ell = \lfloor \frac{n-n_0}{kd} \rfloor$, $|X_1| = \dots = |X_{k\ell}| = d$ and a polynomial $f \in P_n \setminus \text{Id}(\rho)$ alternating on the variables of each set X_j .*

Proof. Recall that each T_i is a faithful irreducible $G_i \oplus \langle c_i \rangle_F$ -module. Denote by τ_i the embedding $G_i \oplus \langle c_i \rangle \hookrightarrow \text{End}_F(T_i)$. By virtue of the density theorem [8, Section 4.3], the space $\tau_i(G_i \oplus \langle c_i \rangle)$ generates $\text{End}_F(T_i)$ as an associative algebra. The center of $\text{End}_F(T_i)$ is a linear span of its unit id_{T_i} . By virtue of Theorem 3, for some $k_i \in \mathbb{N}$ there exists a multilinear polynomial $f_i \in Q_{d_i, k_i, k_i d_i} \setminus \text{Id}(\rho_i)$, $d_i = \dim(G_i \oplus \langle c_i \rangle)$, satisfying the following property. If we substitute any elements from $\tau_i(G_i \oplus \langle c_i \rangle)$ for the variables of f_i , then the result belongs to the center of $\text{End}_F(T_i)$. Therefore, $\tau_i(f_i(h_1^{(i)}, \dots, h_{d_i k_i}^{(i)})) = \text{id}_{T_i}$ for some $h_\beta^{(i)} \in G_i \oplus \langle c_i \rangle_F$ and $f_i(h_1^{(i)}, \dots, h_{d_i k_i}^{(i)})t_i = t_i$ for all $t_i \in T_i$. We can consider products of several copies of f_i in different variables and assume that $k_1 = k_2 = \dots = k_r = k$.

Equation (3) implies that

$$(4) \quad b := q_0 t_1 q_1 t_2 \dots q_{r-1} t_r \neq 0$$

for some $t_i \in T_i$. Fix these t_i . Note that q_i are sums of $q_{i1} \dots q_{i\theta_i}$ and t_i are sums of $t_{i1} \dots t_{i\xi_i}$ where $\theta_i \geq 0$, $\xi_i > 0$ and $q_{ik}, t_{ik} \in \rho(L)$. Denote the maximal θ_i by η_i and the maximal ξ_i by ζ_i . Let $n_0 = \sum_{i=1}^r (\eta_{i-1} + \zeta_i)$, $\ell = \left\lfloor \frac{n-n_0}{kd} \right\rfloor$, and $\alpha = \left\lceil \frac{n-\ell kd}{kd_1} \right\rceil + 1$.

Consider the polynomial $\tilde{f}^{(\alpha)} = w_0 f_1^\alpha \tilde{f}$ where

$$\tilde{f} = (f_1^\ell z_1) w_1 (f_2^\ell z_2) w_2 (f_3^\ell z_3) \dots (f_{r-1}^\ell z_{r-1}) w_{r-1} (f_r^\ell z_r).$$

Here the variables of all copies of f_i are different and $f^{(\alpha)}$ is a multilinear polynomial of degree $\ell kd + 2r + \alpha kd_1$.

Let $w_i = q_i$, $z_i = t_i$. Equation (4) and the choice of f_i imply that if we replace the variables of f_i with $h_\beta^{(i)}$, then the value of $\tilde{f}^{(\alpha)}$ equals b . Denote this substitution of elements by Δ .

Each copy of f_i , $1 \leq i \leq r$, is alternating on disjoint sets $X_1^{i,j}, \dots, X_k^{i,j}$ where $|X_1^{i,j}| = \dots = |X_k^{i,j}| = d_i$ and $1 \leq j \leq \ell$ is the number of the copy of f_i in $\tilde{f}$. Denote $X_{(j-1)k+v} = \coprod_{i=1}^r X_v^{i,j}$ for $1 \leq j \leq \ell$, $1 \leq v \leq k$. Then $|X_j| = d$. Consider the polynomial $f^{(\alpha)} = w_0 f_1^\alpha \text{Alt}_1 \dots \text{Alt}_{k\ell} \tilde{f}$ where the operator Alt_j alternates a polynomial on the variables of X_j . We claim that $f^{(\alpha)}$ does not vanish under the substitution Δ .

Note that the alternation does not change z_i . Thus Property (4) of Lemma 9 implies that all the terms of alternation in which variables from $X_v^{i,j}$ are replaced with variables from $X_v^{i',j}$, $i < i'$, vanish. Hence we may consider only those terms where $X_v^{i,j}$ are invariant. Recall that each f_i is alternating on $X_v^{i,j}$. Thus the result of the substitution Δ for the variables of $f^{(\alpha)}$ equals $(d_1! d_2! \dots d_r!)^{k\ell} b \neq 0$. Moreover, by Lemma 11, it is not important whether we replace the variables of f_i with a_i or c_i . Now we assume that all the variables of f_i are replaced with elements from $G_i \oplus \langle a_i \rangle$.

Now we rewrite q_i as sums of $q_{i1} \dots q_{i\theta_i}$ and t_i as sums of $t_{i1} \dots t_{i\xi_i}$ where $0 \leq \theta_i \leq \eta_i$, $0 < \xi_i \leq \zeta_i$ and $q_{ik}, t_{ik} \in \rho(L)$. Using the linearity of $f^{(\alpha)}$ in w_i and t_i , we obtain that $f^{(\alpha)}$ does not vanish under some substitution $w_i = q_{i1} \dots q_{i\theta_i}$ and $z_i = t_{i1} \dots t_{i\xi_i}$. Now we replace w_i in $\tilde{f}$ with $y_{i1} \dots y_{i\theta_i}$ (or remove w_i if $\theta_i = 0$) and z_i with $v_{i1} \dots v_{i\xi_i}$. Denote the polynomial obtained by $\hat{f}$. Then

$$\hat{f}^{(\alpha)} = y_{01} \dots y_{0\theta_0} f_1^\alpha \text{Alt}_1 \dots \text{Alt}_{k\ell} \hat{f}$$

is a multilinear polynomial of degree

$$\ell kd + \alpha kd_1 + \sum_{i=1}^r (\theta_{i-1} + \xi_i)$$

and $\hat{f}^{(\alpha)} \notin \text{Id}(\rho)$. Note that $\ell kd \leq \deg \hat{f} \leq n_0 + \ell kd \leq n$ and $\deg \hat{f}^{(\alpha)} > \ell kd + \alpha kd_1 > n$.

Expand the first α copies of f_1 in $\hat{f}^{(\alpha)}$. At least one of the polynomials obtained is not a polynomial identity. Denote it by $g^{(\alpha)}$. Remove $(\deg \hat{f}^{(\alpha)} - n)$ initial variables in $g^{(\alpha)}$ and denote the polynomial obtained by f . Then

$$f = u_1 u_2 \dots u_{(n - \deg \hat{f})} \text{Alt}_1 \dots \text{Alt}_{k\ell} \hat{f} \in P_n \setminus \text{Id}(\rho)$$

satisfies all the conditions of Lemma 12. Here u_i are some variables from $\{x_1, x_2, \dots, x_n\}$. $\square$

Lemma 13. *Let k, n_0, ℓ be the numbers from Lemma 12. Then for every $n \geq n_0$ there exists a partition $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$, $\lambda_i \geq k\ell - C$ for every $1 \leq i \leq d$, with $m(\rho, \lambda) \neq 0$. Here $C = p((\dim A)p + 3)((\dim \rho(L)) - d)$.*

Proof. Consider the polynomial f from Lemma 12. It is sufficient to prove that $e_{T_\lambda}^* f \notin \text{Id}(\rho)$ for some tableau T_λ of a desired shape λ . It is known that $FS_n = \bigoplus_{\lambda, T_\lambda} FS_n e_{T_\lambda}^*$, where the summation runs over the set of all standard tableaux T_λ , $\lambda \vdash n$. Thus $FS_n f = \sum_{\lambda, T_\lambda} FS_n e_{T_\lambda}^* f \not\subseteq \text{Id}(\rho)$ and $e_{T_\lambda}^* f \notin \text{Id}(\rho)$ for some $\lambda \vdash n$. We claim that λ is of a desired shape. It is sufficient to prove that $\lambda_d \geq k\ell - C$, since $\lambda_i \geq \lambda_d$ for every $1 \leq i \leq d$. Each row of T_λ includes numbers of no more than one variable from each X_i , since $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ and a_{T_λ} is symmetrizing the variables of each row. Thus $\sum_{i=1}^{d-1} \lambda_i \leq \ell k(d-1) + (n - \ell kd) = n - \ell k$. By virtue of Lemma 6, $\sum_{i=1}^d \lambda_i \geq n - C$. Therefore $\lambda_d \geq \ell k - C$. $\square$

Proof of Theorem 1. The Young diagram D_λ from Lemma 13 contains the rectangular subdiagram D_μ , $\mu = \underbrace{(k\ell - C, \dots, k\ell - C)}_d$. The branching rule for S_n implies

that if we consider a restriction of S_n -action on $M(\lambda)$ to S_{n-1} , then $M(\lambda)$ becomes the direct sum of all nonisomorphic FS_{n-1} -modules $M(\nu)$, $\nu \vdash (n-1)$, where each D_ν is obtained from D_λ by deleting one box. In particular, $\dim M(\nu) \leq \dim M(\lambda)$. Applying the rule $(n - d(k\ell - C))$ times, we obtain $\dim M(\mu) \leq \dim M(\lambda)$. By the hook formula, $\dim M(\mu) = \frac{(d(k\ell - C))!}{\prod_{i,j} h_{ij}}$, where h_{ij} is the length of the hook with edge in (i, j) . By the Stirling formula,

$$\dim M(\mu) \geq \frac{(d(k\ell - C))!}{((k\ell - C + d)!)^d} \sim \frac{\sqrt{2\pi d(k\ell - C)} \left(\frac{d(k\ell - C)}{e}\right)^{d(k\ell - C)}}{\left(\sqrt{2\pi(k\ell - C + d)} \left(\frac{k\ell - C + d}{e}\right)^{k\ell - C + d}\right)^d} \sim C_5 \ell^{r_5} d^{dk\ell}$$

as $\ell \rightarrow \infty$ for some constants $C_5 > 0$, $r_5 \in \mathbb{Q}$. Since $\ell = \left\lceil \frac{n - n_0}{kd} \right\rceil$, this gives the lower bound. The upper bound has been proved in Theorem 2. $\square$

4. EXAMPLES

Example 2. Let $\rho: L \rightarrow \mathfrak{gl}(V)$ be a finite dimensional representation of a Lie algebra L over a field F . If $\rho(a)$ is a nilpotent operator for any $a \in L$, then $c_n(\rho) = 0$ for $n \geq \dim V$. Conversely, if $c_n(\rho) = 0$ for some $n \in \mathbb{N}$, then $\rho(L)$ generates a nilpotent associative subalgebra A in $\text{End}_F(V)$.

Proof. Suppose L acts on V by nilpotent operators. One of the variants of the Engel theorem [9, Corollary from Theorem 3.3] implies that there exists a basis in V in which all the operators $\rho(a)$, $a \in L$, have strictly upper triangular matrices. The associative algebra of strictly upper triangular matrices is nilpotent and $\rho(a_1)\rho(a_2)\dots\rho(a_n) = 0$ in $\text{End}_F(V)$ for all $a_i \in L$ and $n \geq \dim V$. Thus $c_n(\rho) = 0$.

Suppose $c_n(\rho) = 0$ for some $n \in \mathbb{N}$. Then $\rho(a_1)\rho(a_2)\dots\rho(a_n) = 0$ for all $a_i \in L$. Hence $A^n = 0$. $\square$

Example 3. Let $\rho: L \rightarrow \mathfrak{gl}(V)$ be a finite dimensional irreducible representation of a Lie algebra L over an algebraically closed field F of characteristic 0. Then there exist constants $C > 0$, $r \in \mathbb{R}$ such that

$$Cn^r(\dim \rho(L))^n \leq c_n(\rho) \leq (\dim V)^2(\dim \rho(L))^n \text{ for all } n \in \mathbb{N}.$$

Proof. Consider polynomials as n -linear maps from $\rho(L)$ to $\text{End}_F V$. Then we have the natural map $P_n \rightarrow \text{Hom}_F((\rho(L))^{\otimes n}; \text{End}_F V)$ with the kernel $P_n \cap \text{Id}(\rho)$ that leads to the embedding

$$\frac{P_n}{P_n \cap \text{Id}(\rho)} \hookrightarrow \text{Hom}_F((\rho(L))^{\otimes n}; \text{End}_F V).$$

Thus

$$c_n(\rho) = \dim \left(\frac{P_n}{P_n \cap \text{Id}(\rho)} \right) \leq \dim \text{Hom}_F((\rho(L))^{\otimes n}; \text{End}_F V) = (\dim V)^2(\dim \rho(L))^n$$

and we obtain the upper bound. Hence $d(\rho) \leq \dim \rho(L)$. Since V is an irreducible L -module, V is an irreducible A -module. The density theorem [8, Section 4.3] implies

$$A = \text{End}_F(V) \cong M_k(F), \quad k = \dim V.$$

Let $I = \langle e_{11}, e_{21}, \dots, e_{k1} \rangle \subseteq A$ and $J = 0$. Then I is a minimal left ideal of A and I/J is a faithful irreducible A -module. Thus $\text{Ann}(I/J) = 0$ and

$$\dim(\rho(L)/(\rho(L) \cap \text{Ann}(I/J))) = \dim(\rho(L)).$$

Hence $d(\rho) \geq \dim(\rho(L))$. This yields the lower bound. $\square$

Example 4. Let $\rho: L \rightarrow \mathfrak{gl}(V)$ be a finite dimensional completely reducible representation of a Lie algebra L over an algebraically closed field F of characteristic 0, i.e.

$$V = \bigoplus_{i=1}^s \bigoplus_{j=1}^{k_i} V_{ij},$$

where each V_{ij} is irreducible and $V_{ij} \cong V_{kl}$ if and only if $i = k$. Denote $\rho_{ij}: L \rightarrow \mathfrak{gl}(V_{ij})$, $\rho_{ij}(a) = \rho(a)|_{V_{ij}}$, $1 \leq i \leq s$, $1 \leq j \leq k_i$, $a \in L$. Denote

$$d = \max_{\substack{1 \leq i \leq s, \\ 1 \leq j \leq k_i}} (\dim \rho_{ij}(L)).$$

Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n(\rho) \leq C_2 n^{r_2} d^n \text{ for all } n \in \mathbb{N}.$$

Proof. Let A be the associative subalgebra of $\text{End}_F(V)$ generated by $\rho(L)$. Note that

$$\Delta = \text{End}_A(V) \cong M_{k_1}(F) \oplus \dots \oplus M_{k_s}(F).$$

Indeed, denote by φ_{ij} the isomorphism $V_{ij} \rightarrow V_{i1}$ and by p_{ij} the projection $V \rightarrow V_{ij}$. Let $\psi \in \Delta$. Then by Schur's lemma $\varphi_{k\ell} p_{k\ell} \psi \varphi_{ij}^{-1}: V_{i1} \rightarrow V_{k1}$ is a scalar operator for $i = k$ and is a zero operator for $i \neq k$. Define $\alpha_{\ell j}^{(i)}(\psi) \in F$ by the formula $\varphi_{i\ell} p_{i\ell} \psi \varphi_{ij}^{-1}(a) = \alpha_{\ell j}^{(i)}(\psi)a$ for all $a \in V_{i1}$. Then $\Phi: \Delta \rightarrow M_{k_1}(F) \oplus \dots \oplus M_{k_s}(F)$, $\Phi(\psi) = ((\alpha_{\ell j}^{(1)}(\psi)), (\alpha_{\ell j}^{(2)}(\psi)), \dots, (\alpha_{\ell j}^{(s)}(\psi)))$ is an isomorphism and

$$\Phi^{-1}((\beta_{\ell j}^{(1)}), (\beta_{\ell j}^{(2)}), \dots, (\beta_{\ell j}^{(s)})) = \sum_{i=1}^s \sum_{j,\ell=1}^{k_i} \beta_{\ell j}^{(i)} \varphi_{i\ell}^{-1} \varphi_{ij} p_{ij}.$$

Here $(\beta_{\ell j}^{(i)}) \in M_{k_i}(F)$ are arbitrary matrices.

By the density theorem [8, Section 4.3], $A = \text{End}_\Delta(V)$. Thus

$$A = \text{End}_\Delta(V) \cong \text{End}_F(V_{11}) \oplus \text{End}_F(V_{21}) \oplus \dots \oplus \text{End}_F(V_{s1}),$$

where $(\psi_1, \dots, \psi_s)$ acts on V_{ij} by the operator $\varphi_{ij}^{-1} \psi_i \varphi_{ij}$. Indeed, every operator $\psi \in \text{End}_\Delta(V)$ must commute with $p_{ij} \in \Delta$. Hence V_{ij} are invariant under ψ . Moreover, ψ must commute with $\varphi_{ij}^{-1} \varphi_{i\ell} p_{i\ell}$. Therefore,

$$\begin{aligned} \psi(\varphi_{ij}^{-1} \varphi_{i\ell} p_{i\ell}) &= (\varphi_{ij}^{-1} \varphi_{i\ell} p_{i\ell}) \psi, \\ \varphi_{ij}(\psi \varphi_{ij}^{-1} \varphi_{i\ell} p_{i\ell}) \varphi_{i\ell}^{-1} &= \varphi_{ij}(\varphi_{ij}^{-1} \varphi_{i\ell} p_{i\ell} \psi) \varphi_{i\ell}^{-1} \end{aligned}$$

and $\varphi_{ij} \psi \varphi_{ij}^{-1} = \varphi_{i\ell} \psi \varphi_{i\ell}^{-1}$ for all $1 \leq i \leq s$, $1 \leq j, \ell \leq k_i$. We obtain the isomorphism

$$\begin{aligned} \Xi: A &\rightarrow \text{End}_F(V_{11}) \oplus \text{End}_F(V_{21}) \oplus \dots \oplus \text{End}_F(V_{s1}), \\ \Xi(\psi) &= (\psi|_{V_{11}}, \psi|_{V_{21}}, \dots, \psi|_{V_{s1}}). \end{aligned}$$

If I is a left ideal in A , then $I = I_1 \oplus \dots \oplus I_s$, where $\Xi(I_i)$ is a left ideal in $\text{End}_F(V_{i1})$. Consider arbitrary left ideals $I_1, \dots, I_r, J_1, \dots, J_r$ of A that satisfy Conditions (1)–(2). There exist numbers $1 \leq i_1, \dots, i_r \leq s$ such that $I_k = \tilde{I}_k \oplus \hat{I}_k$, $J_k = \tilde{J}_k \oplus \hat{I}_k$, where $\tilde{I}_k$ and $\tilde{J}_k$ are left ideals of $\Xi^{-1}(\text{End}_F(V_{i_k 1}))$ and $\hat{I}_k$ is a left ideal of $\Xi^{-1}(\bigoplus_{\ell \neq i_k} \text{End}_F(V_{\ell 1}))$. Hence $\text{Ann}(I_k/J_k) = \Xi^{-1}(\bigoplus_{\ell \neq i_k} \text{End}_F(V_{\ell 1}))$. Thus $\text{Ann}(I_k/J_k) \cap \rho(L) = \ker \rho_{i_k 1}$ and $\dim(\rho(L)/(\text{Ann}(I_k/J_k) \cap \rho(L))) = \dim \rho_{i_k 1}(L)$. Choose A -submodules T_k such that $\tilde{I}_k = T_k \oplus \tilde{J}_k$. If $i_k \neq i_\ell$ for some k and ℓ , then $T_1 q_1 T_2 \dots T_{r-1} q_{r-1} T_r = 0$ for all $q_1, \dots, q_{r-1} \in A \cup \{1\}$. Thus $i_1 = \dots = i_r$ and $d(\rho) = \max_{1 \leq i \leq s, 1 \leq j \leq k_i} \dim \rho_{ij}(L)$. By Theorem 1 we obtain the bounds. $\square$

Example 5. Let $\rho: L \rightarrow \mathfrak{gl}(V)$ be a nontrivial finite dimensional representation of a simple Lie algebra L over an algebraically closed field F of characteristic 0. Then there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} (\dim L)^n \leq c_n(\rho) \leq C_2 n^{r_2} (\dim L)^n \text{ for all } n \in \mathbb{N}.$$

Proof. By the Weyl theorem, ρ is completely reducible. Let $\rho_i: L \rightarrow \mathfrak{gl}(V_i)$ be the corresponding irreducible representations. Then Example 4 implies that

$$d(\rho) = \max_{1 \leq i \leq s} \dim \rho_i(L) = \dim L.$$

□

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