ON SOLITON SOLUTIONS TO A CLASS OF SCHRÖDINGER-KdV SYSTEMS

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ABSTRACT. In this paper, we consider a class of coupled nonlinear Schrödinger-KdV systems in the whole space via the Nehari manifold method. The existence of nontrivial solutions with both of the components nonzero is obtained.

1. INTRODUCTION

Bound state and ground state solutions of the following coupled nonlinear Schrödinger-Korteweg-de Vries (Schrödinger-KdV for short) systems

$$(1.1) \quad \begin{cases} if_t + D^2 f = \beta f g - |f|^2 f, \\ g_t + D^3 g + g D g = \frac{\beta}{2} D(|f|^2) \end{cases}$$

were considered by many authors, where $f = f(x, t)$ is a complex-valued function, $g = g(x, t)$ is real-valued, $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$ and $D = \frac{\partial}{\partial x}$. Systems of the form (1.1) appear as models for interactions between long and short waves in a variety of physical settings: for example, in the case of the interaction between long gravity waves and capillary waves on the surface of shallow water, in the case when the group velocity of the capillary wave coincides with the velocity of the long wave, and also for resonant ion-sound/Langmuir-wave interactions in plasmas, which reduces to (1.1) under the assumption that the ion-sound wave is unidirectional; see [9], [11], [14], [19].

Solutions for (1.1) of the form $(f(x, t), g(x, t)) = (\exp(i\omega t) \exp(ikx) u_1(x - ct), u_2(x - ct))$ are the so-called solitary waves. Let $c = 2k$ and putting $c^* := k^2 + \omega$, one obtains the following system with cubic nonlinearity:

$$(1.2) \quad \begin{cases} -u_1'' + c^* u_1 = u_1^3 - \beta u_1 u_2, \\ -u_2'' + c u_2 = \frac{1}{2} u_2^2 - \frac{\beta}{2} u_1^2. \end{cases}$$

Motivated by this formula, in our work we deal with the following system:

$$(1.3) \quad \begin{cases} -\Delta u + u = u^3 - \beta uv, & u \in H^1(\mathbf{R}^N), \\ -\Delta v + v = \frac{1}{2} |v| v - \frac{\beta}{2} u^2, & v \in H^1(\mathbf{R}^N), \end{cases}$$

where $N = 2, 3$.

Systems (1.1) and (1.2) were studied in [3]. By applying the concentration-compactness method to a constrained minimizing problem, the authors obtained

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the existence of smooth bound states for (1.2). Recently, in [4], the authors considered the Cauchy problem and the existence of bound states for coupled nonlinear Schrödinger-gKdV systems which are natural extensions of Schrödinger-KdV systems. In [2], the authors considered the following system:

$$\begin{cases} i(u_t + c_1 u_x) + \delta_1 u_{xx} = \alpha uv, \\ v_t + c_2 v_x + \delta_2 v_{xxx} + \gamma(v^2)_x = \beta(|u|^2)_x. \end{cases}$$

By studying the minimizers of the energy functional subject to constraints on conserved functionals associated with symmetries of the system, they proved the existence and stability of a two-parameter family of ground states associated with a two-parameter family of symmetries. In [16], the following system was studied:

$$\begin{cases} iu_t + u_{xx} = \alpha uv, \\ v_t + \gamma Dv_x + \gamma(v^2)_x = \beta(|u|^2)_x, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x). \end{cases}$$

Using the concentration compactness technique, the authors proved the existence and some property of the solitary waves for this system. In [6], the Cauchy problem for a class of Schrödinger equations, coupled with KdV equations, was considered. In [7] and [10], the authors analyzed the stability of the solitary wave for Schrödinger-KdV systems.

Let us also mention that in [1], [12] and [13], the following system was considered:

$$\begin{cases} -\Delta u + u = u^3 + \beta v^2 u, & u \in H^1(\mathbf{R}^N), \\ -\Delta v + v = v^3 + \beta u^2 v, & v \in H^1(\mathbf{R}^N). \end{cases}$$

In [1], the authors dealt with this system and proved the existence of bound and ground states provided the coupling parameter is small and large, respectively. Their method is to study the functional constrained on the Nehari manifold and evaluate the Morse index of the semi-trivial solution (of the form $(0, v)$ or $(u, 0)$ with $v \neq 0$ and $u \neq 0$). In [12], the authors proved the existence of a nontrivial solution for this system by minimax arguments on the Nehari manifold. In [13], the authors proved the existence of infinitely many nodal solutions for this system by the method developed in [17]. Our work is based on the method used in [1]. The main result of this paper reads as:

Theorem 1.1. *Assume $N = 2$ or 3 and $\gamma > 0$ is a constant defined in (3.7). If $|\beta| > \gamma$, problem (1.3) possesses a radial symmetric bound state with both components nonzero.*

2. FUNCTIONAL FRAMEWORK

We will use variational methods to find a solution of problem (1.3). We denote the Hilbert space of all radial functions in $H^1(\mathbf{R}^N)$ by $H_r^1(\mathbf{R}^N)$, endowed with the norm $\|u\|^2 = |\nabla u|_2^2 + |u|_2^2$, where $|u|_p$ is the usual norm in the Lebesgue space $L^p(\mathbf{R}^N)$. Let $\mathcal{H} = H_r^1(\mathbf{R}^N) \times H_r^1(\mathbf{R}^N)$ and $\|(u, v)\|_{\mathcal{H}}^2 = \|u\|^2 + \|v\|^2$. Define $I : \mathcal{H} \rightarrow \mathbf{R}$ as

$$I(u, v) = \frac{1}{2} \|(u, v)\|_{\mathcal{H}}^2 - \frac{1}{4} \int_{\mathbf{R}^N} u^4 dx - \frac{1}{6} \int_{\mathbf{R}^N} |v|^3 dx + \frac{\beta}{2} \int_{\mathbf{R}^N} u^2 v dx.$$

It is easy to see that I is well defined and of class $C^2(\mathcal{H}, \mathbf{R})$. Define the Nehari manifold as

$$\mathcal{M} := \{(u, v) \in \mathcal{H} \setminus (0, 0) : \langle \nabla I(u, v), (u, v) \rangle = 0\}.$$

Lemma 2.1. *For every $(u, v) \in \mathcal{H} \setminus \{(0, 0)\}$, there exists a unique $\bar{t} = \bar{t}(u, v) > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}$. The maximum of $g(t) = I(tu, tv)$ for $t \geq 0$ is achieved at $\bar{t}$.*

Proof. Define a functional $P : \mathcal{H} \rightarrow \mathbf{R}$ as

$$P(u, v) = \|(u, v)\|_{\mathcal{H}}^2 - \int_{\mathbf{R}^N} u^4 dx - \frac{1}{2} \int_{\mathbf{R}^N} |v|^3 dx + \frac{3\beta}{2} \int_{\mathbf{R}^N} u^2 v dx.$$

Then $(u, v) \in \mathcal{H} \setminus \{(0, 0)\}$ belongs to $\mathcal{M}$ if and only if $P(u, v) = 0$. Suppose that there exists $\bar{t} > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}$. Then it holds that

$$\begin{aligned} P(\bar{t}u, \bar{t}v) &= \langle I'(\bar{t}u, \bar{t}v), (\bar{t}u, \bar{t}v) \rangle \\ &= \bar{t}^2 \|(u, v)\|_{\mathcal{H}}^2 - \bar{t}^4 \int_{\mathbf{R}^N} u^4 dx - \frac{1}{2} \bar{t}^3 \int_{\mathbf{R}^N} |v|^3 dx + \frac{3\beta}{2} \bar{t}^3 \int_{\mathbf{R}^N} u^2 v dx = 0. \end{aligned}$$

Hence

$$\begin{aligned} (2.4) \quad \varrho(\bar{t}) &:= \|(u, v)\|_{\mathcal{H}}^2 - \bar{t}^2 \int_{\mathbf{R}^N} u^4 dx - \frac{1}{2} \bar{t} \int_{\mathbf{R}^N} |v|^3 dx + \frac{3\beta}{2} \bar{t} \int_{\mathbf{R}^N} u^2 v dx \\ &= - \left(\int_{\mathbf{R}^N} u^4 dx \right) \bar{t}^2 + \left(\frac{3\beta}{2} \int_{\mathbf{R}^N} u^2 v dx - \frac{1}{2} \int_{\mathbf{R}^N} |v|^3 dx \right) \bar{t} + \|(u, v)\|_{\mathcal{H}}^2 = 0. \end{aligned}$$

If $\int_{\mathbf{R}^N} u^4 dx \neq 0$, since $\varrho(0) = \|(u, v)\|_{\mathcal{H}}^2 > 0$, then we can find a unique $\bar{t} > 0$ such that (2.4) holds; that is to say, $\bar{t}(u, v) \in \mathcal{M}$. Hence, there exists a unique $\bar{t} = \bar{t}(u, v) > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}$. If $\int_{\mathbf{R}^N} u^4 dx = 0$, then $v \neq 0$. Hence, there also exists a unique $\bar{t} = \bar{t}(u, v) > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{M}$.

For an element $(\tilde{u}, \tilde{v}) \in \mathcal{M}$, it holds that

$$\langle I'(\tilde{u}, \tilde{v}), (\tilde{u}, \tilde{v}) \rangle = \|(\tilde{u}, \tilde{v})\|_{\mathcal{H}}^2 - \int_{\mathbf{R}^N} \tilde{u}^4 dx - \frac{1}{2} \int_{\mathbf{R}^N} |\tilde{v}|^3 dx + \frac{3\beta}{2} \int_{\mathbf{R}^N} \tilde{u}^2 \tilde{v} dx = 0.$$

Thus we have

$$(2.5) \quad I(\tilde{u}, \tilde{v}) = \frac{1}{6} \|(\tilde{u}, \tilde{v})\|_{\mathcal{H}}^2 + \frac{1}{12} \int_{\mathbf{R}^N} \tilde{u}^4 dx$$

and

$$\|(\tilde{u}, \tilde{v})\|_{\mathcal{H}}^2 = \int_{\mathbf{R}^N} \tilde{u}^4 dx + \frac{1}{2} \int_{\mathbf{R}^N} |\tilde{v}|^3 dx - \frac{3\beta}{2} \int_{\mathbf{R}^N} \tilde{u}^2 \tilde{v} dx \leq C_1 \|(\tilde{u}, \tilde{v})\|_{\mathcal{H}}^4 + C_2 \|(\tilde{u}, \tilde{v})\|_{\mathcal{H}}^3.$$

So $\|(\tilde{u}, \tilde{v})\|_{\mathcal{H}} \geq \rho$ for some $\rho > 0$ and there holds $I(\tilde{u}, \tilde{v}) \geq \frac{1}{6} \rho^2$.

Since

$$g(t) = \frac{t^2}{2} \|(u, v)\|_{\mathcal{H}}^2 - \frac{t^4}{4} \int_{\mathbf{R}^N} u^4 dx - \frac{t^3}{6} \int_{\mathbf{R}^N} |v|^3 dx + \frac{\beta t^3}{2} \int_{\mathbf{R}^N} u^2 v dx,$$

$g(t) \rightarrow 0$ as $t \rightarrow 0$, $g(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ and $g(\bar{t}) \geq \frac{1}{6} \rho^2$, so the maximum of $g(t)$ is achieved in the interior of $[0, +\infty]$. Assume that $g(\bar{t}) = \max g(t)$; then $g'(\bar{t}) = 0$. Hence $\langle I'(\bar{t}u, \bar{t}v), (\bar{t}u, \bar{t}v) \rangle = 0$, so $\bar{t} = \bar{t}$. $\square$

Remark 2.2. Since $\langle P'(u, v), (u, v) \rangle = -\|(u, v)\|_{\mathcal{H}}^2 - \int_{\mathbf{R}^N} u^4 dx < -\rho^2$ for $(u, v) \in \mathcal{M}$, $\mathcal{M}$ is a C^1 complete manifold.

Lemma 2.3. *(u, v) is a nontrivial critical point of I if and only if (u, v) is a constrained critical point of I on $\mathcal{M}$.*

Proof. Suppose $I'|_{\mathcal{M}}(u, v) = 0$. By the Lagrangian multiplier method, one can see $I'(u, v) - \omega P'(u, v) = 0$ in $\mathcal{H}$ with $(u, v) \in \mathcal{M}$. By the definition of $\mathcal{M}$, $\langle I'|_{\mathcal{M}}(u, v), (u, v) \rangle = -\omega \langle P'(u, v), (u, v) \rangle = 0$. From Remark 2.2, $\omega = 0$. Thus $I'(u, v) = 0$ in $\mathcal{H}$. If (u, v) is a non-trivial critical point of I , it is of course a critical point of I constrained on $\mathcal{M}$. $\square$

Lemma 2.4. *I satisfies the Palais-Smale condition on $\mathcal{M}$.*

Proof. Suppose that $\{(u_n, v_n)\} \subseteq \mathcal{M}$ is a Palais-Smale sequence. Then $I(u_n, v_n) \rightarrow c$ and $I'(u_n, v_n)|_{\mathcal{M}} \rightarrow 0$ for some $c \in \mathbf{R}$. From (2.5), we can infer that (u_n, v_n) is bounded in $\mathcal{H}$. So we can suppose that $(u_n, v_n) \rightharpoonup (u, v)$ for some $(u, v) \in \mathcal{H}$. Since $H_r^1(\mathbf{R}^N)$ can be compactly embedded into $L^p(\mathbf{R}^N)$ for $2 < p < 2^*$ (see [18]), we can infer that $u_n \rightarrow u$, $v_n \rightarrow v$ in $L^p(\mathbf{R}^N)$ for $2 < p < 2^*$, $u_n \rightarrow u$, $v_n \rightarrow v$ almost everywhere in $\mathbf{R}^N$. Furthermore, there hold

$$\begin{aligned} \int_{\mathbf{R}^N} u_n^4 dx + \frac{1}{2} \int_{\mathbf{R}^N} |v_n|^3 dx - \frac{3\beta}{2} \int_{\mathbf{R}^N} u_n^2 v_n dx \\ \rightarrow \int_{\mathbf{R}^N} u^4 dx + \frac{1}{2} \int_{\mathbf{R}^N} |v|^3 dx - \frac{3\beta}{2} \int_{\mathbf{R}^N} u^2 v dx \end{aligned}$$

and

$$\int_{\mathbf{R}^N} u^4 dx + \frac{1}{2} \int_{\mathbf{R}^N} |v|^3 dx - \frac{3\beta}{2} \int_{\mathbf{R}^N} u^2 v dx \geq \rho^2.$$

We have

$$(2.6) \quad I'(u_n, v_n)|_{\mathcal{M}} = I'(u_n, v_n) - \omega_n P'(u_n, v_n) \rightarrow 0$$

for a sequence of real numbers $\{\omega_n\}$ by the Lagrangian multiplier method. Then it holds that

$$\langle I'(u_n, v_n)|_{\mathcal{M}}, (u_n, v_n) \rangle = \langle I'(u_n, v_n), (u_n, v_n) \rangle - \omega_n \langle P'(u_n, v_n), (u_n, v_n) \rangle \rightarrow 0.$$

Since $\langle I'(u_n, v_n), (u_n, v_n) \rangle = 0$, we have $\omega_n \langle P'(u_n, v_n), (u_n, v_n) \rangle \rightarrow 0$. From Remark 2.2, it implies that $\omega_n \rightarrow 0$. Denote $I(u, v) := \frac{1}{2} \|(u, v)\|_{\mathcal{H}}^2 - F(u, v) - G(u, v) + \beta T(u, v)$, where $F(u, v) = \frac{1}{4} \int_{\mathbf{R}^N} u^4 dx$, $G(u, v) = \frac{1}{6} \int_{\mathbf{R}^N} |v|^3 dx$ and $T(u, v) = \frac{1}{2} \int_{\mathbf{R}^N} u^2 v dx$. Then we have

$$I'(u_n, v_n) = (u_n, v_n) - F'(u_n, v_n) - G'(u_n, v_n) + \beta T'(u_n, v_n)$$

and

$$P'(u_n, v_n) = 2(u_n, v_n) - 4F'(u_n, v_n) - 3G'(u_n, v_n) + 3\beta T'(u_n, v_n).$$

Hence from (2.6) there holds

$$\begin{aligned} (1 - 2\omega_n)(u_n, v_n) &= (1 - 4\omega_n)F'(u_n, v_n) + (1 - 3\omega_n)G'(u_n, v_n) \\ &\quad + (3\omega_n - 1)\beta T'(u_n, v_n) + o(1). \end{aligned}$$

Now we prove that F' , G' and T' are compact operators. For any $h \in H_r^1(\mathbf{R}^N)$, we have

$$\begin{aligned} |\langle T'_u(u_n, v_n) - T'_u(u, v), h \rangle| &= \left| \int_{\mathbf{R}^N} (u_n v_n - uv) h dx \right| \leq \int_{\mathbf{R}^N} |u_n v_n - uv| |h| dx \\ &\leq \int_{\mathbf{R}^N} |u_n v_n - u_n v| |h| dx + \int_{\mathbf{R}^N} |u_n v - uv| |h| dx \\ &= \int_{\mathbf{R}^N} |v_n - v| |u_n| |h| dx + \int_{\mathbf{R}^N} |u_n - u| |v| |h| dx \\ &\leq |v_n - v|_3 |u_n|_3 |h|_3 + |u_n - u|_3 |v|_3 |h|_3 \rightarrow 0, \\ \langle T'_v(u_n, v_n) - T'_v(u, v), h \rangle &= \frac{1}{2} \int_{\mathbf{R}^N} (u_n^2 - u^2) h dx \leq \frac{1}{2} \int_{\mathbf{R}^N} |u_n^2 - u^2| |h| dx \\ &\leq \frac{1}{2} \left(\int_{\mathbf{R}^N} |u_n^2 - u^2|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^N} |h|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Since $u_n^4 + u^4 - |u_n^4 - u^4| \geq 0$, by Fatou's Lemma, we have

$$\int_{\mathbf{R}^N} \liminf_{n \rightarrow +\infty} (u_n^4 + u^4 - |u_n^4 - u^4|) dx \leq \liminf_{n \rightarrow +\infty} \int_{\mathbf{R}^N} (u_n^4 + u^4 - |u_n^4 - u^4|) dx,$$

and so $\lim_{n \rightarrow +\infty} \int_{\mathbf{R}^N} |u_n^4 - u^4| dx \rightarrow 0$. Thus $\int_{\mathbf{R}^N} |u_n^2 - u^2|^2 dx \leq \int_{\mathbf{R}^N} |u_n^4 - u^4| dx \rightarrow 0$ and $\langle T'_v(u_n, v_n) - T'_v(u, v), h \rangle \rightarrow 0$. So we have proved the compactness of T' . Similarly, we have

$$\begin{aligned} \langle F'_u(u_n, v_n) - F'_u(u, v), h \rangle &= \int_{\mathbf{R}^N} (u_n^3 - u^3) h dx \leq \int_{\mathbf{R}^N} |u_n^3 - u^3| |h| dx \\ &\leq \left(\int_{\mathbf{R}^N} |u_n^3 - u^3|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} |h|_4 \leq \left(\int_{\mathbf{R}^N} |u_n^4 - u^4| dx \right)^{\frac{3}{4}} |h|_4 \rightarrow 0, \\ \langle G'_v(u_n, v_n) - G'_v(u, v), h \rangle &= \frac{1}{2} \int_{\mathbf{R}^N} (|v_n|v_n - |v|v) h dx \\ &\leq \frac{1}{2} \int_{\mathbf{R}^N} |(|v_n|v_n - |v|v)| |h| dx \\ &\leq \frac{1}{2} \left(\int_{\mathbf{R}^N} (|v_n|v_n - |v|v)^2 dx \right)^{\frac{1}{2}} |h|_2 \leq \left(\int_{\mathbf{R}^N} |v_n^4 - v^4| dx \right)^{\frac{1}{2}} |h|_2 \rightarrow 0. \end{aligned}$$

Here, we have used the fundamental inequality $a^{\frac{q}{p}} - b^{\frac{q}{p}} \geq (a^q - b^q)^{\frac{1}{p}}$ for $a > b > 0$ and $p > 0, q > 0$. Thus F' and G' are also compact operators.

Hence $(u_n, v_n) \rightarrow F'(u, v) + G'(u, v) - \beta T'(u, v)$, so $(u, v) = F'(u, v) + G'(u, v) - \beta T'(u, v)$. Thus $\|(u, v)\|_{\mathcal{H}} = 4F(u, v) + 3G(u, v) - 3\beta T(u, v)$, which means $(u, v) \in \mathcal{M}$. $\square$

3. PROOF OF THE MAIN THEOREM

In this section we show the existence of nontrivial solutions of (1.3) which are different from $\mathbf{u}_0$ and $\mathbf{u}_1$. Here $\mathbf{u}_0 = (0, U)$, $\mathbf{u}_1 = (0, -U)$ and U is the unique positive radial solution of $-\Delta u + u = \frac{1}{2}u^2$; see [8]. Set

$$(3.7) \quad \gamma = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|^2}{\int_{\mathbf{R}^N} U \varphi^2 dx}.$$

From the Sobolev embedding theorem, γ is a positive real number.

Theorem 3.1. (i) If $\beta < -\gamma$, then $\mathbf{u}_0$ is a saddle point of I on $\mathcal{M}$. In particular, $\inf_{\mathcal{M}} I < I(\mathbf{u}_0)$.

(ii) If $\beta > \gamma$, $\mathbf{u}_1$ is a saddle point of I on $\mathcal{M}$. In particular, $\inf_{\mathcal{M}} I < I(\mathbf{u}_1)$.

Let $D^2 I_{\mathcal{M}}(\mathbf{u}_0)$ and $D^2 I_{\mathcal{M}}(\mathbf{u}_1)$ denote the second differential of I constrained on $\mathcal{M}$. Since $I'(\mathbf{u}_0) = 0$ and $I'(\mathbf{u}_1) = 0$, one has that

$$\begin{aligned} D^2 I_{\mathcal{M}}(\mathbf{u}_0)[\mathbf{h}]^2 &= I''(\mathbf{u}_0)[\mathbf{h}]^2, \quad \forall \mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0} \mathcal{M}, \\ D^2 I_{\mathcal{M}}(\mathbf{u}_1)[\mathbf{h}]^2 &= I''(\mathbf{u}_1)[\mathbf{h}]^2 \quad \forall \mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1} \mathcal{M}. \end{aligned}$$

Similarly, if $\mathcal{N}$ denotes the Nehari manifold relative to the functional $I_0 : H_r^1(\mathbf{R}^N) \rightarrow \mathbf{R}$ defined by $I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{6} \int_{\mathbf{R}^N} |u|^3 dx$, i.e.,

$$\mathcal{N} = \{u \in H_r^1(\mathbf{R}^N) \setminus \{0\} : \langle I'_0(u), u \rangle = 0\} = \{u \in H_r^1 \setminus \{0\}, \|u\|^2 = \frac{1}{2} \int_{\mathbf{R}^N} |u|^3 dx\},$$

from the fact that $I'_0(U) = 0$ and $I'_0(-U) = 0$, there hold

$$\begin{aligned} D^2(I_0)_\mathcal{N}(U)[h]^2 &= I''_0(U)[h]^2 \quad \forall \mathbf{h} = (h_1, h_2) \in T_U\mathcal{N}, \\ D^2(I_0)_\mathcal{N}(-U)[h]^2 &= I''_0(-U)[h]^2 \quad \forall \mathbf{h} = (h_1, h_2) \in T_{-U}\mathcal{N}. \end{aligned}$$

Note that $\pm U$ is the only minimizer of I_0 on $\mathcal{N}$, and furthermore I_0 is nondegenerate at U in the space $H^1_r(\mathbf{R}^N)$ since $\text{Ker } I''_0(U) = \text{Ker}(-\Delta + 1 + \frac{1}{2}U^2) = \text{span}\{\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}\}$ (see [15]). Here $\frac{\partial U}{\partial x_i}$ is odd about the variable x_i and every function in the space $H^1_r(\mathbf{R}^N)$ is even in x_i , so $\text{span}\{\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}\} \cap H^1_r(\mathbf{R}^N) = \{0\}$. One has that $\exists c_0 > 0$ such that

$$(3.8) \quad I''_0(U)[h]^2 > 0, \quad \forall 0 \neq h \in T_U\mathcal{N},$$

$$(3.9) \quad I''_0(-U)[h]^2 > 0, \quad \forall 0 \neq h \in T_{-U}\mathcal{N}.$$

Lemma 3.2. $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M} \Leftrightarrow h_2 \in T_U\mathcal{N}$ and $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M} \Leftrightarrow h_2 \in T_{-U}\mathcal{N}$.

Proof. $h_2 \in T_U\mathcal{N}$ if and only if $\langle U, h_2 \rangle = \frac{3}{4} \int_{\mathbf{R}^N} U^2 h_2 dx$ and $h_2 \in T_{-U}\mathcal{N}$ if and only if $\langle -U, h_2 \rangle = -\frac{3}{4} \int_{\mathbf{R}^N} U^2 h_2 dx$, while $\mathbf{h} \in T_{(u,v)}\mathcal{M}$ if and only if

$$\langle u, h_1 \rangle + \langle v, h_2 \rangle = 2 \int_{\mathbf{R}^N} u^3 h_1 dx + \frac{3}{4} \int_{\mathbf{R}^N} |v| v h_2 dx - \frac{3}{2} \beta \int_{\mathbf{R}^N} u v h_1 dx - \frac{3}{4} \beta \int_{\mathbf{R}^N} u^2 h_2 dx.$$

Hence $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_0}\mathcal{M}$ if and only if $\langle U, h_2 \rangle = \frac{3}{4} \int_{\mathbf{R}^N} U^2 h_2 dx$ and $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M}$ if and only if $\langle -U, h_2 \rangle = -\frac{3}{4} \int_{\mathbf{R}^N} U^2 h_2 dx$. $\square$

Proof of Theorem 3.1. If $(u, v) \in \mathcal{H}$ and $\mathbf{h} = (h_1, h_2) \in \mathcal{H}$, then one has

$$\begin{aligned} (3.10) \quad I''(u, v)[\mathbf{h}]^2 &= \|h_1\|^2 + \|h_2\|^2 - 3 \int_{\mathbf{R}^N} u^2 h_1^2 dx - \int_{\mathbf{R}^N} v h_2^2 dx \\ &\quad + \beta \int_{\mathbf{R}^N} v h_1^2 dx + \beta \int_{\mathbf{R}^N} u h_1 h_2 dx. \end{aligned}$$

In particular, if $(u, v) = \mathbf{u}_0 = (0, U)$, we get

$$I''(\mathbf{u}_0)[\mathbf{h}]^2 = \|h_1\|^2 + \|h_2\|^2 - \int_{\mathbf{R}^N} U h_2^2 dx + \beta \int_{\mathbf{R}^N} U h_1^2 dx.$$

Take $\mathbf{h} = (h_1, 0) \in T_{\mathbf{u}_0}\mathcal{M}$; then we have

$$I''(\mathbf{u}_0)[(h_1, 0)]^2 = \|h_1\|^2 + \beta \int_{\mathbf{R}^N} U h_1^2 dx.$$

By the definition of γ , we can infer that when $\beta < -\gamma$, there exists $\widetilde{h}_1 \in H^1_r(\mathbf{R}^N) \setminus \{0\}$ such that

$$\gamma \leq \frac{\|\widetilde{h}_1\|^2}{\int_{\mathbf{R}^N} U \widetilde{h}_1^2 dx} < -\beta.$$

So

$$(3.11) \quad I''(\mathbf{u}_0)[(\widetilde{h}_1, 0)]^2 = \|\widetilde{h}_1\|^2 + \beta \int_{\mathbf{R}^N} U \widetilde{h}_1^2 dx < 0.$$

From (3.8) and (3.11), we can infer that $\mathbf{u}_0$ is a saddle point.

If $(u, v) = \mathbf{u}_1 = (0, -U)$, we get

$$I''(\mathbf{u}_1)[\mathbf{h}]^2 = \|h_1\|^2 + \|h_2\|^2 + \int_{\mathbf{R}^N} U h_2^2 dx - \beta \int_{\mathbf{R}^N} U h_1^2 dx.$$

Take $\mathbf{h} = (h_1, 0) \in T_{\mathbf{u}_1}\mathcal{M}$. We have

$$I''(\mathbf{u}_1)[(h_1, 0)]^2 = \|h_1\|^2 - \beta \int_{\mathbf{R}^N} U h_1^2 dx.$$

By the definition of γ , we can infer that when $\beta > \gamma$, there exists $\widetilde{h}_2 \in H_r^1(\mathbf{R}^N) \setminus \{0\}$ such that

$$\gamma \leq \frac{\|\widetilde{h}_2\|^2}{\int_{\mathbf{R}^N} U \widetilde{h}_2^2 dx} < \beta.$$

So

$$(3.12) \quad I''(\mathbf{u}_1)[(\widetilde{h}_2, 0)]^2 = \|\widetilde{h}_2\|^2 - \beta \int_{\mathbf{R}^N} U \widetilde{h}_2^2 dx < 0.$$

From (3.9) and (3.12), $\mathbf{u}_1$ is a saddle point. $\square$

The following lemma is a direct consequence of Theorem 3.1.

Lemma 3.3. (i) If $\beta > \gamma$, then I has a global minimum $(\tilde{u}, \tilde{v})$ on $\mathcal{M}$ and $I(\tilde{u}, \tilde{v}) < I(\mathbf{u}_1) = \inf_{\mathcal{N}} I_0$.
(ii) If $\beta < -\gamma$, then I has a global minimum $(\tilde{u}, \tilde{v})$ on $\mathcal{M}$ and $I(\tilde{u}, \tilde{v}) < I(\mathbf{u}_0) = \inf_{\mathcal{N}} I_0$.

Proof. By the Ekeland variational principle (see [5]) and Lemma 2.4, we can infer that $\inf_{\mathcal{M}} I$ is achieved at some $(\tilde{u}, \tilde{v}) \neq 0$. Moreover, if $\beta < -\gamma$, Theorem 3.1(i) implies that $I(\tilde{u}, \tilde{v}) < I(\mathbf{u}_0)$. If $\beta > \gamma$, Theorem 3.1(ii) implies that $I(\tilde{u}, \tilde{v}) < I(\mathbf{u}_1)$. From the fact that $I(\mathbf{u}_0) = I_0(U) = \inf_{\mathcal{N}} I_0 = I_0(-U) = I(\mathbf{u}_1)$, the proof is completed. $\square$

Proof of Theorem 1.1. By Lemma 3.3, we can get a minimizer $(\tilde{u}, \tilde{v})$ of the problem $\inf_{\mathcal{M}} I$ which is a solution of (1.3). We need to prove $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$. Otherwise, if $\tilde{v} = 0 \in H_r^1(\mathbf{R}^n)$, from equation (1.3), we see also that $\tilde{u} = 0$. This is impossible since $(0, 0) \notin \mathcal{M}$. If $\tilde{u} = 0$, $\tilde{v} \in \mathcal{N}$, then $I_0(\tilde{v}) = I(\tilde{u}, \tilde{v}) \leq I(0, u) = I_0(u)$ for any $u \in \mathcal{N}$, so $\tilde{v}$ achieves the minimum of I_0 on $\mathcal{N}$. Thus we have

$$I(\tilde{u}, \tilde{v}) = I_0(\tilde{v}) = \inf_{\mathcal{N}} I_0.$$

This contradicts Lemma 3.3. $\square$

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