INVERSE DEGREE OF AN AFFINE SPACE TRIANGULAR AUTOMORPHISM

SHU KAWAGUCHI

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ABSTRACT. For any $\mathbb{Q}$ -algebra R and any triangular automorphism $f : R^N \rightarrow R^N$ with Jacobian one on the affine space, we show that $\deg(f^{-1})$ is bounded from above by a constant depending only on N and $\deg(f)$. This is seen as a generalization of a result by Furter on the affine plane. Our proof uses (a version of) Furter's estimate on nilpotency indices and Abhyankar–Gurjar's formal inversion formula. It follows that when the Jacobian of a triangular automorphism $f : R^N \rightarrow R^N$ is not necessarily equal to one, $\deg(f^{-1})$ is bounded from above by a constant depending only on N , $\deg(f)$ and $\deg(1/\text{Jac}(f))$.

1. INTRODUCTION

For a $\mathbb{Q}$ -algebra R , let $R[X_1, \dots, X_N]$ denote the polynomial ring with coefficients in R in N variables. A map $f = (f_1, \dots, f_N) : R^N \rightarrow R^N$ is a *polynomial map* over R if each f_i belongs to $R[X_1, \dots, X_N]$. We denote the Jacobian matrix of f by $J(f) = \left(\frac{\partial f_i}{\partial X_j} \right)_{1 \leq i, j \leq N} \in \text{Mat}_{N,N}(R[X_1, \dots, X_N])$, the Jacobian determinant of f by $\text{Jac}(f) = \det J(f) \in R[X_1, \dots, X_N]$, and the degree of f by $\deg(f) = \max\{\deg(f_1), \dots, \deg(f_N)\} \in \mathbb{Z}_{\geq 0}$. A polynomial map $f : R^N \rightarrow R^N$ over R is an *automorphism* (i.e. a polynomial automorphism) if it has a polynomial inverse $g : R^N \rightarrow R^N$ over R , i.e., $g \circ f = f \circ g = \text{Id}_{R^N}$. If this is the case, g is uniquely determined by f and is denoted by f^{-1} .

Let $f : R^N \rightarrow R^N$ be an automorphism over R with $\text{Jac}(f) = 1$. An interesting problem is to bound $\deg(f^{-1})$ from above. If R is a *reduced* $\mathbb{Q}$ -algebra, then a formula of Gabber (see [3, I.1] and [4]) says that $\deg(f^{-1}) \leq \deg(f)^{N-1}$ (here the condition $\text{Jac}(f) = 1$ is not necessary). However, if one allows R to be non-reduced, then Bass [2] has shown that the problem of bounding $\deg(f^{-1})$ from above by a constant depending only on N and $\deg(f)$ (and independent of non-reduced $\mathbb{Q}$ -algebras R) is, in fact, equivalent to the Jacobian conjecture over $\mathbb{C}$.

In dimension $N = 2$, instead of treating all polynomial automorphisms, Furter [8] has considered triangular automorphisms. Recall that $f = (f_1, \dots, f_N) : R^N \rightarrow R^N$ is said to be *triangular* if $\frac{\partial f_i}{\partial X_j} = 0$ for any $1 \leq i < j \leq N$. The triangular automorphisms constitute an important subgroup of the automorphism group of R^N . Furter has proved that if R is any $\mathbb{Q}$ -algebra and $f : R^2 \rightarrow R^2$ is any *triangular* automorphism over R with $\text{Jac}(f) = 1$, then $\deg(f^{-1}) \leq 4\deg(f)^4$. (For other

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directions of results related to inverse degrees of affine space automorphisms, see for example [4], [5], [6, Chap. 2.3 and 7.3] and the references in the Notes therein, [7], [9].)

The purpose of this short note is to consider inverse degrees of triangular automorphisms with Jacobian one in arbitrary dimension.

Theorem 1.1. *Let $N \geq 1$ and $d \geq 1$. Then there exists a constant $t(N, d)$ which depends only on N and d satisfying the following property: If R is any $\mathbb{Q}$ -algebra, and $f = (f_1, f_2, \dots, f_N) : R^N \rightarrow R^N$ is any triangular automorphism over R of degree at most d with $\text{Jac}(f) = 1$, then $\deg(f^{-1}) \leq t(N, d)$.*

For an explicit form of $t(N, d)$, see the end of the proof of Theorem 1.1. In $N = 2$, our proof still gives $\deg(f^{-1}) \leq 4 \deg(f)^4$ but a slightly weaker estimate than Furter [8], where it is shown that $\deg(f^{-1}) \leq (2d - 1)((d - 1)^2(2d - 3) + 1) \leq 4d^4$. Our estimate gives $\deg(f^{-1}) \leq (2d + 2)((d - 1)^2(2d - 3) + 1) - d - 2 \leq 4d^4$ if $d \geq 2$ (see Remark 2.7).

Key ingredients of our proof are (a version in N variables of) Furter's estimate on nilpotency indices (Proposition 2.4) and Abhyankar–Gurjar's formal inversion formula (Theorem 2.2 and Corollary 2.3).

Suppose $f : R^N \rightarrow R^N$ is an automorphism over R . Then it follows from $1 = \text{Jac}(f^{-1} \circ f) = \text{Jac}(f^{-1})(f) \cdot \text{Jac}(f)$ that $\text{Jac}(f)$ is a unit in $R[X_1, \dots, X_N]$. Thus $1/\text{Jac}(f) \in R[X_1, \dots, X_N]$, and we can consider $\deg(1/\text{Jac}(f))$. Theorem 1.1 tells us that in the case $\text{Jac}(f)$ is not necessarily equal to one, $\deg(f^{-1})$ is then bounded from above by a constant depending only on N , $\deg(f)$ and $\deg(1/\text{Jac}(f))$.

Corollary 1.2. *Let $N \geq 1$, $d \geq 1$ and $e \geq 0$. Then there exists a constant $t(N, d, e)$ which depends only on N , d and e satisfying the following property: If R is any $\mathbb{Q}$ -algebra, and $f = (f_1, f_2, \dots, f_N) : R^N \rightarrow R^N$ is any triangular automorphism over R of degree at most d with $\deg(1/\text{Jac}(f)) \leq e$, then $\deg(f^{-1}) \leq t(N, d, e)$.*

Finally, we remark that some condition on $\deg(1/\text{Jac}(f))$ (such as $\text{Jac}(f) = 1$ in Theorem 1.1, or $\deg(1/\text{Jac}(f)) \leq e$ in general) is necessary. Indeed, for any $d \geq 2, d' \geq 2$, by [8, Lemma 1], if one drops the condition $\text{Jac}(f) = 1$, then there exists a triangular automorphism $f : R^N \rightarrow R^N$ over R of degree d such that $\deg(f^{-1}) = d'$.

2. PROOF

Let R be a $\mathbb{Q}$ -algebra. Let $f = (f_1(X_1), f_2(X_1, X_2), \dots, f_N(X_1, X_2, \dots, X_N)) : R^N \rightarrow R^N$ be a triangular automorphism, where $f_i = f_i(X_1, \dots, X_i) \in R[X_1, \dots, X_i]$. We assume that $\text{Jac}(f) = 1$. Let $g = (g_1, \dots, g_N) : R^N \rightarrow R^N$ denote the inverse of f .

Lemma 2.1. *The automorphism g is triangular with $\text{Jac}(g) = 1$.*

Proof. Since $1 = \text{Jac}(f \circ g) = \text{Jac}(f)(g) \cdot \text{Jac}(g)$, we have $\text{Jac}(g) = 1$. We will show that g is a triangular automorphism.

Since $\text{Jac}(f) = 1$ and f is triangular, we have

$$\frac{\partial f_1}{\partial X_1}(X_1) \cdot \frac{\partial f_2}{\partial X_2}(X_1, X_2) \cdots \frac{\partial f_N}{\partial X_N}(X_1, X_2, \dots, X_N) = 1.$$

It follows that for each i , $\frac{\partial f_i}{\partial X_i}(X_1, \dots, X_i)$ is a unit in $R[X_1, \dots, X_i]$.

We write $g_i = g_i(Y_1, \dots, Y_N)$ for $i = 1, \dots, N$. Since $g \circ f = \text{Id}_{R^N}$, we have

$$(2.1) \quad g_i(f_1(X_1), \dots, f_N(X_1, \dots, X_N)) = X_i.$$

Taking $\frac{\partial}{\partial X_N}$ in (2.1), we obtain $\frac{\partial g_i}{\partial Y_N}(f_1, \dots, f_N) \cdot \frac{\partial f_N}{\partial X_N} = 0$. Since $\frac{\partial f_N}{\partial X_N}$ is a unit and f is an automorphism of $R[X_1, \dots, X_N]$, we have $\frac{\partial g_i}{\partial Y_N}(Y_1, \dots, Y_N) = 0$. Next, taking $\frac{\partial}{\partial X_{N-1}}$ in (2.1), we obtain

$$\begin{aligned} 0 &= \frac{\partial g_i}{\partial Y_{N-1}}(f_1, \dots, f_N) \cdot \frac{\partial f_{N-1}}{\partial X_{N-1}} + \frac{\partial g_i}{\partial Y_N}(f_1, \dots, f_N) \cdot \frac{\partial f_N}{\partial X_{N-1}} \\ &= \frac{\partial g_i}{\partial Y_{N-1}}(f_1, \dots, f_N) \cdot \frac{\partial f_{N-1}}{\partial X_{N-1}}. \end{aligned}$$

Since $\frac{\partial f_{N-1}}{\partial X_{N-1}}$ is a unit, we similarly have $\frac{\partial g_i}{\partial Y_{N-1}}(Y_1, \dots, Y_N) = 0$. We can proceed with the same procedure for any $j \geq i+1$, and we see that $\frac{\partial g_i}{\partial Y_j}(Y_1, \dots, Y_N) = 0$ for any $N \geq j > i \geq 1$. Hence g is triangular. $\square$

We are going to give an upper bound for $\deg(g)$. First we recall Abhyankar–Gurjar’s formal inversion formula (see [1] and [3, III.2]).

Theorem 2.2 (Abhyankar–Gurjar’s formal inversion formula). *Let $f : R^N \rightarrow R^N$ be a polynomial map over R with $f(0) = 0$ and $J(f)(0) = I_N$. Put $f = \text{Id} - h$, where $h = (h_1, \dots, h_N)$. Let $g = (g_1, \dots, g_N)$ be the formal power series inverse of f . Then as an equality in $R[[X_1, \dots, X_N]]$, we have*

$$g_i = \sum_{p=(p_1, \dots, p_N) \in \mathbb{Z}_{\geq 0}^N} \frac{1}{p_1! \cdots p_N!} \left(\frac{\partial}{\partial X_1} \right)^{p_1} \cdots \left(\frac{\partial}{\partial X_N} \right)^{p_N} (X_i \cdot \text{Jac}(f) \cdot h_1^{p_1} \cdots h_N^{p_N}).$$

As a corollary, if $X_i - f_i$ is nilpotent for each i , we have the following estimate on the degree of g_i (as a polynomial in $R[X_1, \dots, X_N]$). This actually gives a generalization of [8, Theorem 1] of $N = 1$.

Corollary 2.3. *Let $f : R^N \rightarrow R^N$ be a polynomial map over R of degree at most d with $f(0) = 0$ and $J(f)(0) = I_N$. Let g be the inverse of f . We assume further that there exist $\lambda_1, \dots, \lambda_N \geq 1$ with $(X_1 - f_1)^{\lambda_1} = \cdots = (X_N - f_N)^{\lambda_N} = 0$. Then*

$$\deg(g) \leq (d-1) \sum_{k=1}^N (\lambda_k - 1) + 1.$$

Proof. We set $h_i = X_i - f_i$. By Theorem 2.2, we get

$$g_i = \sum_{p=(p_1, \dots, p_N) \in \mathbb{Z}_{\geq 0}^N} \frac{1}{p_1! \cdots p_N!} \left(\frac{\partial}{\partial X_1} \right)^{p_1} \cdots \left(\frac{\partial}{\partial X_N} \right)^{p_N} (X_i \cdot h_1^{p_1} \cdots h_N^{p_N}).$$

The degree of each term of the right-hand side is at most $(\sum_{k=1}^N p_k (\deg(h_k) - 1)) + 1$, and it is zero if $p_k \geq \lambda_k$ for some k . Hence the right-hand side is a finite sum, and the above equality holds in $R[X_1, \dots, X_N]$. By degree counting, we get the assertion. $\square$

We also recall Furter’s estimate on nilpotent indices. For later use, we give a version in N variables.

Proposition 2.4. *Let $N \geq 1$, $d \geq 1$ and $e \geq 1$. Let*

$$P(X_1, \dots, X_N) \in (R[X_1, \dots, X_N])^\times$$

such that $\deg(P) \leq d$ and $\deg(P^{-1}) \leq e$. We write $P = p_0 + p_1 X_N + \dots + p_d X_N^d$, where $p_i = p_i(X_1, \dots, X_{N-1}) \in R[X_1, \dots, X_{N-1}]$. Then

$$(p_0 - p_0(0), p_1, \dots, p_d)^{N \min\{d, e\}(d+e-1)+1} = 0$$

as an ideal in $R[X_1, \dots, X_{N-1}]$.

Proof. The case $N = 1$ is due to Furter [8, Prop. 1]. For the reader's convenience, we recall the proof of [8, Prop. 1], following the referee's suggestion. Since $P(X_1) = p_0 + p_1 X_1 + \dots + p_d X_1^d$ is a unit of $R[X_1]$, p_0 is a unit of R . Replacing P by $p_0^{-1}P$, we may assume that $p_0 = 1$ to show the assertion. We set the monic polynomial $Q(X_1) := X_1^d P(1/X_1) = X_1^d + p_1 X_1^{d-1} + \dots + p_d \in R[X_1]$ and set $R_1 := R[T]/(Q(T))$, where T is indeterminate. Then R_1 is a $\mathbb{Q}$ -algebra containing R . We denote by ε_1 the image of T in R_1 . Since $Q(\varepsilon_1) = 0$ in R_1 , there exists a monic polynomial $Q_1(X_1) \in R_1[X_1]$ with $\deg(Q_1) = \deg(Q) - 1$ such that $Q(X_1) = (X_1 - \varepsilon_1)Q_1(X_1)$ in $R_1[X_1]$. By induction on $\deg(Q)$, we see that there exist a $\mathbb{Q}$ -algebra R' containing R and $\varepsilon_1, \dots, \varepsilon_d \in R'$ such that $Q(X_1) = \prod_{i=1}^d (X_1 - \varepsilon_i)$ in $R'[X_1]$. Then $P(X_1) = X_1^d Q(1/X_1) = \prod_{i=1}^d (1 - \varepsilon_i X_1)$ in $R'[X_1]$. For each j , we have

$$(1 - \varepsilon_j X_1)^{-1} = (P^{-1})(X_1) \left(\prod_{1 \leq i \leq d, i \neq j} (1 - \varepsilon_i X_1) \right).$$

Since the right-hand side is a polynomial of degree at most $d + e - 1$, we have $\varepsilon_j^{d+e} = 0$ for all $j = 1, \dots, d$. Let $\mathcal{E}$ be the ideal of R' generated by $\varepsilon_1, \dots, \varepsilon_d$. Then $\mathcal{E}^{d(d+e-1)+1} = 0$. Since $(p_1, \dots, p_d)R' \subseteq \mathcal{E}$, we have $(p_1, \dots, p_d)^{d(d+e-1)+1} = 0$ in R . We write $P^{-1}(X_1) = 1 + q_1 X_1 + \dots + q_e X_1^e$. Interchanging P and P^{-1} , we have $(q_1, \dots, q_e)^{e(d+e-1)+1} = 0$ in R . Since $(p_1, \dots, p_d) = (q_1, \dots, q_e)$, we have $(p_1, \dots, p_d)^{\min\{d, e\}(d+e-1)+1} = 0$, which gives the assertion in $N = 1$.

Suppose now that $N \geq 2$. Since $\deg_{X_N}(P) \leq d$ and $\deg_{X_N}(P^{-1}) \leq e$, the case $N = 1$ applied to $R[X_1, \dots, X_N] = (R[X_1, \dots, X_{N-1}])[X_N]$, regarded as the one variable polynomial ring over $R[X_1, \dots, X_{N-1}]$ with indeterminate X_N , gives

$$(p_1, \dots, p_d)^{\min\{d, e\}(d+e-1)+1} = 0.$$

Since $p_0(X_1, \dots, X_{N-1}) = P(X_1, \dots, X_{N-1}, 0)$, we have $p_0(X_1, \dots, X_{N-1}) \in (R[X_1, \dots, X_{N-1}])^\times$ and $\deg(p_0) \leq d$ and $\deg(p_0^{-1}) \leq e$. By induction on N , we get $(p_0 - p_0(0))^{(N-1) \min\{d, e\}(d+e-1)+1} = 0$. Since $N \min\{d, e\}(d + e - 1) + 1 = (\min\{d, e\}(d + e - 1) + 1) + ((N - 1) \min\{d, e\}(d + e - 1) + 1) - 1$, we have

$$(p_0 - p_0(0), p_1, \dots, p_d)^{N \min\{d, e\}(d+e-1)+1} = 0.$$

This gives the assertion. $\square$

Proposition 2.5. *Let $N \geq 2$. Let $f = (f_1, f_2, \dots, f_N) : R^N \rightarrow R^N$ be a triangular automorphism over R of degree at most d with $f(0) = 0$, $J(f)(0) = I_N$ and $\text{Jac}(f) = 1$. Let g be the inverse of f . We assume further that $f_i(X_1, \dots, X_{i-1}, 0) = 0$ for all $1 \leq i \leq N$. Then*

$$\deg(g) \leq \frac{(N+2)(N-1)}{2} (d-1)^N ((d-1)^{N-1} + d-2) + 1.$$

Proof. As in the proof of Lemma 2.1, we have

$$(2.2) \quad \frac{\partial f_1}{\partial X_1}(X_1) \cdot \frac{\partial f_2}{\partial X_2}(X_1, X_2) \cdots \frac{\partial f_N}{\partial X_N}(X_1, X_2, \dots, X_N) = 1.$$

Putting $X_{k+1} = \cdots = X_N = 0$, we get

$$\frac{\partial f_1}{\partial X_1}(X_1) \cdots \frac{\partial f_k}{\partial X_k}(X_1, \dots, X_k) \cdots \frac{\partial f_N}{\partial X_N}(X_1, \dots, X_k, 0, \dots, 0) = 1.$$

It follows that $\deg(\frac{\partial f_k}{\partial X_k}(X_1, \dots, X_k)) \leq (d-1)$ and $\deg(\frac{\partial f_k}{\partial X_k}(X_1, \dots, X_k)^{-1}) \leq (d-1)^{N-1}$.

Let $k \leq N-1$. We write

$$\frac{\partial f_k}{\partial X_k}(X_1, \dots, X_k) = p_0 + p_1 X_k + \cdots + p_{d-1} X_k^{d-1},$$

where $p_k = p_k(X_1, \dots, X_{k-1}) \in R[X_1, \dots, X_{k-1}]$. By Proposition 2.4, we have

$$(p_0 - p_0(0), p_1, \dots, p_{d-1})^{k(d-1)^{N-1}((d-1)^{N-1}+d-2)+1} = 0.$$

Since $J(f)(0) = I_N$, we get $p_0(0) = 1$. Further, since $f_k(X_1, \dots, X_{k-1}, 0) = 0$, we have

$$f_k(X_1, \dots, X_k) = X_k + (p_0 - p_0(0))X_k + \frac{1}{2}p_1 X_k^2 + \cdots + \frac{1}{d}p_{d-1} X_k^d.$$

It follows that $(X_k - f_k)^{k(d-1)^{N-1}((d-1)^{N-1}+d-2)+1} = 0$.

Let $k = N$. Equation (2.2) tells us that $\frac{\partial f_N}{\partial X_N}$ belongs to $R[X_1, \dots, X_{N-1}]$. We write

$$\frac{\partial f_N}{\partial X_N} = q_0 + q_1 X_{N-1} + \cdots + q_{d-1} X_{N-1}^{d-1},$$

where $q_k = q_k(X_1, \dots, X_{N-2}) \in R[X_1, \dots, X_{N-2}]$. By Proposition 2.4, we have

$$(q_0 - q_0(0), q_1, \dots, q_{d-1})^{(N-1)(d-1)^{N-1}((d-1)^{N-1}+d-2)+1} = 0.$$

Since $J(f)(0) = I_N$, we get $q_0(0) = 1$. Further, since $f_N(X_1, \dots, X_{N-1}, 0) = 0$, we have $f_N(X_1, \dots, X_N) = X_N + (\frac{\partial f_N}{\partial X_N} - q_0(0))X_N$. It follows that

$$(X_N - f_N)^{(N-1)(d-1)^{N-1}((d-1)^{N-1}+d-2)+1} = 0.$$

By Corollary 2.3, we get

$$\begin{aligned} \deg(g) &\leq \left(\sum_{k=1}^{N-1} k + (N-1) \right) (d-1)^N ((d-1)^{N-1} + d-2) + 1 \\ &= \frac{(N+2)(N-1)}{2} (d-1)^N ((d-1)^{N-1} + d-2) + 1. \end{aligned}$$

This completes the proof. $\square$

Lemma 2.6. For $1 \leq i \leq N-1$, let $s_i(X_1, \dots, X_i) \in R[X_1, \dots, X_i]$ be polynomials of degree at most d . We consider a triangular automorphism

$$t = (X_1, X_2 + s_1(X_1), \dots, X_n + s_{n-1}(X_1, \dots, X_{n-1})) : R^N \rightarrow R^N.$$

Then $\deg(t^{-1}) \leq d^{N-1}$.

Proof. Let $u = (u_1, \dots, u_N)$ be the inverse of t . We show that $\deg(u_i) \leq d^{i-1}$. Since $u_1(X_1) = X_1$, we have $\deg(u_1) \leq 1 = d^{1-1}$. Since $u_i(X_1, \dots, X_i) = X_i - s_i(u_1, \dots, u_{i-1})$, we have by induction $\deg(u_i) \leq d \cdot d^{i-2} = d^{i-1}$. This gives the assertion. $\square$

Proof of Theorem 1.1. Since the case $N = 1$ is obvious, we consider the case $N \geq 2$.

First note that we may assume $f(0) = 0$ and $J(f)(0) = I_N$. Indeed, we put $J(f)(0) = (a_{ij})$. Since $J(f)(0)$ is a triangular matrix and $\text{Jac}(f) = 1$, we get $\prod_i a_{ii} = 1$. Thus each a_{ii} is a unit of R . It follows that there exists a linear triangular automorphism $L : R^N \rightarrow R^N$ such that $\tilde{f} := L(f - f(0))$ is a triangular automorphism with $\tilde{f}(0) = 0$, $J(\tilde{f})(0) = I_N$ and $\text{Jac}(\tilde{f}) = 1$. Since $\deg(\tilde{f}^{-1}) = \deg(f^{-1})$, it suffices to prove Theorem 1.1 for $\tilde{f}$. Replacing $\tilde{f}$ by f , this shows that we may assume $f(0) = 0$ and $J(f)(0) = I_N$. In this case, the inverse g of f also satisfies $g(0) = 0$ and $J(g)(0) = I_N$.

We set $\varphi_1(X_1) = f_1(X_1)$ and, for $i \geq 2$,

$$\varphi_i(X_1, \dots, X_i) = f_i(X_1, \dots, X_i) - f_i(X_1, \dots, X_{i-1}, 0),$$

and consider the polynomial map $\varphi = (\varphi_1, \dots, \varphi_N) : R^N \rightarrow R^N$. Then φ is a triangular map of degree at most d with $\varphi(0) = 0$, $J(\varphi)(0) = I_N$ and $\text{Jac}(\varphi) = 1$. We note that φ is an automorphism. Indeed, suppose that $Y_1 = \varphi_1(X_1), \dots, Y_N = \varphi(X_1, \dots, X_N)$. Since f is a triangular automorphism, Y_1 is written as a polynomial in X_1 . Since $Y_i + f_i(X_1, \dots, X_{i-1}, 0) = f_i(X_1, \dots, X_{i-1}, X_i)$, we have $X_i = g_i(Y_1, \dots, Y_i + f_i(X_1, \dots, X_{i-1}, 0))$. Then, inductively, X_i is written as a polynomial in $Y_1, \dots, Y_i$, which shows that φ is an automorphism.

Let $\psi = (\psi_1, \dots, \psi_N)$ denote the inverse of φ . We set $t_1(X_1) = X_1$ and, for $i \geq 2$,

$$t_i(X_1, \dots, X_i) = X_i + f_i(\psi_1(X_1), \dots, \psi_{i-1}(X_1, \dots, X_{i-1}), 0),$$

and consider the polynomial map $t = (t_1, \dots, t_N) : R^N \rightarrow R^N$. Then t is a triangular automorphism, and we have

$$\begin{aligned} t_i(\varphi_1, \dots, \varphi_i) &= \varphi_i + f_i(\psi_1(\varphi_1), \dots, \psi_{i-1}(\varphi_1, \dots, \varphi_{i-1}), 0) \\ &= \varphi_i + f_i(X_1, \dots, X_{i-1}, 0) = f_i(X_1, \dots, X_i). \end{aligned}$$

Thus $f = t \circ \varphi$.

The inverse f is given by $\psi \circ t^{-1}$. Since φ satisfies the assumption of Proposition 2.5, we have

$$\deg(\psi) \leq \frac{(N+2)(N-1)}{2} (d-1)^N ((d-1)^{N-1} + d-2) + 1.$$

Thus $\deg(t) \leq d \left(\frac{(N+2)(N-1)}{2} (d-1)^N ((d-1)^{N-1} + d-2) + 1 \right)$. Then Lemma 2.6 tells us that $\deg(t^{-1}) \leq d^{N-1} \left(\frac{(N+2)(N-1)}{2} (d-1)^N ((d-1)^{N-1} + d-2) + 1 \right)^{N-1}$.

Finally we have

$$\begin{aligned} \deg(f^{-1}) &\leq \deg(\psi) \cdot \deg(t^{-1}) \\ &\leq d^{N-1} \left(\frac{(N+2)(N-1)}{2} (d-1)^N ((d-1)^{N-1} + d-2) + 1 \right)^N. \end{aligned}$$

This completes the proof of Theorem 1.1. $\square$

Remark 2.7. Let $d \geq 2$. In $N = 2$, $\text{Jac}(\psi) = 1$ gives $\frac{\partial \psi_1}{\partial X_1} \cdot \frac{\partial \psi_2}{\partial X_2} = 1$. Thus $\frac{\partial \psi_2}{\partial X_2}$ does not depend on X_2 , so that $\deg_{X_2}(\psi_2) = 1$. Then we find that

$$\begin{aligned} \deg(g_1) &\leq \deg(\psi_1(X_1)) \leq 2(d-1)^2(2d-3), \\ \deg(g_2) &= \deg(\psi_2(X_1, X_2 - f_2(\psi_1(X_1), 0))) \\ &\leq \deg(X_2 - f_2(\psi_1(X_1), 0)) + (\deg(\psi_2) - 1) \\ &= d(2(d-1)^2(2d-3) + 1) + 2(d-1)^2(2d-3) \\ &= (2d+2)((d-1)^2(2d-3) + 1) - d - 2. \end{aligned}$$

It follows that we still have $\deg(f^{-1}) \leq (2d+2)((d-1)^2(2d-3) + 1) - d - 2 \leq 4d^4$.

Proof of Corollary 1.2. Let $f = (f_1, f_2, \dots, f_N) : R^N \rightarrow R^N$ be a triangular automorphism over R of degree at most d with $\deg(1/\text{Jac}(f)) \leq e$. Let $g = (g_1, \dots, g_N)$ denote the inverse of f .

From f , we construct a triangular automorphism $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_N, \tilde{f}_{N+1}) : R^{N+1} \rightarrow R^{N+1}$ as follows. For $i = 1, \dots, N$, we set $\tilde{f}_i = f_i$. For $i = N+1$, we set

$$\tilde{f}_{N+1} = X_{N+1} \cdot (1/\text{Jac}(f)).$$

Then $\tilde{f}$ is a triangular map. If we set $\tilde{g}_i = g_i$ for $i = 1, \dots, N$ and $\tilde{g}_{N+1} = X_{N+1} \cdot \text{Jac}(f)(g_1, \dots, g_N)$, then $\tilde{f}^{-1}$ is given by $(\tilde{g}_1, \dots, \tilde{g}_N, \tilde{g}_{N+1})$. Thus $\tilde{f}$ is a triangular automorphism. Further, we have $\text{Jac}(\tilde{f}) = \text{Jac}(f) \cdot (1/\text{Jac}(f)) = 1$. Since $\tilde{f} : R^{N+1} \rightarrow R^{N+1}$ is a triangular automorphism over R of degree at most $\max\{d, e+1\}$ with $\text{Jac}(\tilde{f}) = 1$, Theorem 1.1 shows that

$$\deg(\tilde{f}^{-1}) \leq t(N+1, \max\{d, e+1\}).$$

In particular,

$$\begin{aligned} \deg(f^{-1}) &= \max\{g_1, \dots, g_N\} = \max\{\tilde{g}_1, \dots, \tilde{g}_N\} \\ &\leq \max\{\tilde{g}_1, \dots, \tilde{g}_N, \tilde{g}_{N+1}\} = \deg(\tilde{f}^{-1}) \leq t(N+1, \max\{d, e+1\}). \end{aligned}$$

If we put $t(N, d, e) := t(N+1, \max\{d, e+1\})$, then we obtain the assertion. $\square$

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN

E-mail address: kawaguch@math.sci.osaka-u.ac.jp

Current address: Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: kawaguch@math.kyoto-u.ac.jp