A WEAK VECTOR-VALUED BANACH-STONE THEOREM

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ABSTRACT. For a locally compact Hausdorff space X and a Banach space E , we denote by $C_0(X, E)$ the space of E -valued continuous functions on X which vanish at infinity, endowed with the supremum norm. In the spirit of the classical Banach-Stone theorem, we prove that if $C_0(X, E)$ is isomorphic to $C_0(Y, E)$, where E has non-trivial cotype and such that E is separable or E^* has the Radon-Nikodým property, then either X and Y are finite or X and Y have the same cardinality. In other words, we obtain a vector-valued extension of a 1978 B. Cengiz result, the scalar case $E = \mathbb{R}$ or $E = \mathbb{C}$.

1. INTRODUCTION

We follow the standard notation and terminology for measure theory and Banach space theory that can be found in [9], [11] and [16]. If E is the real or complex scalar field, as usual, the space $C_0(X, E)$ will be denoted by $C_0(X)$. In the case where X is compact, these spaces will be designated by $C(X, E)$ and $C(X)$, respectively. If there is a linear isomorphism T from the Banach space E onto the Banach space F , we will write, in short, $E \sim F$. Moreover, to emphasize that $\|T^{-1}\| \|T\| < \lambda$ for some $1 < \lambda < +\infty$, we will also write $E \overset{\lambda}{\sim} F$. The cardinality of a set A will be denoted by $|A|$.

The classical Banach-Stone theorem states that if $C_0(X)$ and $C_0(Y)$ are isometrically isomorphic, then X and Y are homeomorphic (in short, $X \approx Y$); see [2] and [21]. This theorem has been extended in different ways; see for instance [15]. We stress that in [1] and [6] this theorem was independently strengthened as follows.

Theorem 1.1. *Let X and Y be locally compact Hausdorff spaces. Then*

$$C_0(X) \overset{2}{\sim} C_0(Y) \implies X \approx Y.$$

In [4], Cambern showed that 2 is the best constant for the above theorem; see also [8]. Moreover, in [7], without the bound-two hypothesis, Cengiz proved the following *weak version of Theorem 1.1*.

Theorem 1.2. *Let X and Y be locally compact Hausdorff spaces. Then*

$$C_0(X) \sim C_0(Y) \implies |X| = |Y|.$$

The Banach-Stone theorem has also been generalized for the real vector-valued continuous functions. In this setting, the furthest-reaching result is the following one, due to Behrends and Cambern [3].

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Theorem 1.3. *Suppose that E is a uniformly non-square space. Then there exists $1 < \lambda \leq 2$ such that for every locally compact Hausdorff space X and Y we have*

$$C_0(X, E) \overset{\lambda}{\sim} C_0(Y, E) \implies X \approx Y.$$

Recall that a Banach space E is said to be uniformly non-square if there is a $\delta > 0$ such that there do not exist x and y in the unit ball of E for which

$$\left\| \frac{1}{2}(x + y) \right\| > 1 - \delta \text{ and } \left\| \frac{1}{2}(x - y) \right\| > 1 - \delta.$$

Theorems 1.2 and 1.3 naturally lead to the following question.

Problem 1.4. Suppose that E is a uniformly non-square Banach space and X and Y are locally compact Hausdorff spaces such that

$$C_0(X, E) \sim C_0(Y, E).$$

What can be said about the cardinalities of X and Y ?

The principal purpose of the present paper is to answer completely the above question. In an attempt to solve Problem 1.4, we have obtained a more general result concerning the isomorphisms between $C_0(X, E)$ spaces. Indeed, Enflo [12] proved that if E is a uniformly non-square Banach space, then E can be renormed to an equivalent uniformly convex Banach space. In particular, E has non-trivial cotype [10, Theorem 14.1, p. 283]; that is, it has cotype q for some $2 \leq q < \infty$.

Recall that a Banach space $E \neq \{0\}$ has cotype $2 \leq q < \infty$ if there is a constant $\kappa > 0$ such that no matter how we select finitely many vectors $v_1, v_2, \dots, v_n$ from E ,

$$(1.1) \quad \left(\sum_{i=1}^n \|v_i\|^q \right)^{\frac{1}{q}} \leq \kappa \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t)v_i \right\|^2 dt \right)^{\frac{1}{2}},$$

where $r_i : [0, 1] \rightarrow \mathbb{R}$ denote the *Rademacher functions*, defined by setting

$$r_i(t) = \text{sign}(\sin 2^i \pi t).$$

Thus, inspired by [7] we prove the following vector-valued extension of Theorem 1.2.

Theorem 1.5. *Let E be a Banach space having non-trivial cotype. If E^* has the Radon-Nikodým property or E is separable, then for every locally compact Hausdorff spaces X and Y we have*

$$C_0(X, E) \sim C_0(Y, E) \implies |X| |Y| < \aleph_0 \quad \text{or} \quad |X| = |Y|.$$

Observe that Theorem 1.5 is also a *weak version* of Theorem 1.3 and provides an immediate solution to Problem 1.4. Indeed, it suffices to recall that every uniformly convex space E is reflexive [16, p. 31] and therefore E^* has the Radon-Nikodým property [9, Corollary 13, p. 76].

Notice also that Theorem 1.5 covers the cases $E = l_1$ and $E = l_p(\Gamma)$, where Γ is an arbitrary set and $1 < p < \infty$. We do not know whether the statement of this theorem is also true for the remaining case $E = l_1(\Gamma)$, where Γ is an uncountable set. Finally, notice that the classical Milutin Theorem [18, Theorem 21.5.10] shows that we cannot remove the non-trivial cotype hypothesis in Theorem 1.5 in the case

where E is separable. Indeed, for every infinite countable compact Hausdorff K we have

$$C(K, C([0, 1])) \sim C(K \times [0, 1]) \sim C([0, 1] \times [0, 1]) \sim C([0, 1], C([0, 1])).$$

Moreover, since for every uncountable set Γ , $l_1(\Gamma)$ has the Radon-Nikodým property [9, Corollary 8, p. 83] and

$$C_0(\mathbb{N}, C_0(\Gamma)) \sim C_0(\Gamma) \sim C_0(\Gamma, C_0(\Gamma))$$

holds, it follows that the non-trivial cotype hypothesis in Theorem 1.5 cannot also be removed in the case where E^* has the Radon-Nikodým property.

The remainder of this paper is organized as follows. In Section 2, we state some results on Banach spaces having non-trivial cotype which are necessary to prove Theorem 1.5 (Proposition 2.1 and Lemma 2.2). In Section 3, we recall a Radon-Nikodým type vector measure theorem (Theorem 3.1) and also prove a simple result (Proposition 3.3) which allows us to provide a unified proof of Theorem 1.5 in its distinct cases: E^* has the Radon-Nikodým property or E is separable.

In order to prove Theorem 1.5, we will work with some convenient lower-semicontinuous functions associated to bounded linear operators from $C_0(X, E)$ to $C_0(Y, E)$. So, in Section 4, we introduce these functions (Lemma 4.1) and recall Cengiz's result concerning certain lower-semicontinuous functions and real measures (Proposition 4.2). Finally, in Section 5, we prove Theorem 1.5.

2. ON $C_0(X, E)$ SPACES WITH E HAVING NON-TRIVIAL COTYPE

In this section we state some auxiliary results on Banach spaces having non-trivial cotype. For an E -valued measure μ , $|\mu|$ denotes the variation of μ and the symbol $rcabv(X, E)$ comprises the Banach space of all regular, countably additive, Borel, bounded variation measures, endowed with the variation norm. As usual, when E is a scalar field, this space is simply denoted by $rcabv(X)$. Throughout we will use the Singer Representation Theorem. That is, there exists an isometric isomorphism between $C_0(X, E)^*$ and $rcabv(X, E^*)$ such that every linear functional φ and the corresponding measure μ are related by

$$\langle \varphi, f \rangle = \int f d\mu, \quad f \in C_0(X, E),$$

where the integral is termed the immediate integral of Dinculeanu [11, p. 11]; see [5, p. 2], [17] and also [14] when X is a compact space.

Proposition 2.1. *Let E be a Banach space having non-trivial cotype and X and Y be locally compact Hausdorff spaces. Suppose that T is an isomorphism of $C_0(X, E)$ into $C_0(Y, E)$ and set*

$$X_y = \{x \in X : \|T^*(\varphi\delta_y)(\{x\})\| > 0 \text{ for some } \varphi \in S_{E^*}\},$$

where δ_y stands for the unit point mass at y and S_{E^*} denotes the unit sphere of E^* . Then

- (a) X_y is countable for every $y \in Y$.
- (b) If Y is infinite and for every $x \in X$ there exists a $y \in Y$ such that $x \in X_y$, then $|X| \leq |Y|$.

Proof. (a) We assume that for some $y \in Y$, X_y is uncountable and show that this assumption leads to a contradiction.

Hence there exist some $0 < \epsilon < 1$ such that

$$\{x \in X : \|T^*(\varphi\delta_y)(\{x\})\| > \epsilon \text{ for some } \varphi \in S_{E^*}\}$$

is infinite.

Since E has cotype q , for some $2 \leq q < \infty$, by (1.1) there exists a constant $Q > 0$ in such a way that no matter how we select finitely many vectors $v_1, v_2, \dots, v_m \in E$, if $0 < \delta \leq \|v_i\|$ for each $1 \leq i \leq m$, there is an appropriate choice of scalars $r_i = \pm 1$ such that

$$(2.1) \quad \left\| \sum_{i=1}^m r_i v_i \right\| \geq \delta Q \sqrt[q]{m}.$$

Pick $n \in \mathbb{N}$ satisfying $\epsilon Q \sqrt[q]{n} > 2\|T\|$. Also fix distinct points $x_1, \dots, x_n \in X$ and $\varphi_1, \dots, \varphi_n \in S_{E^*}$ such that

$$\|T^*(\varphi_i\delta_y)(\{x_i\})\| > \epsilon, \quad 1 \leq i \leq n.$$

Thus, there are $v_1, \dots, v_n$ in the unit ball of E in such a way that

$$(2.2) \quad \langle T^*(\varphi_i\delta_y)(\{x_i\}), v_i \rangle > \epsilon, \quad 1 \leq i \leq n.$$

Since $T^*(\varphi_i\delta_y)$ is regular for each $1 \leq i \leq n$, we can take $U_1, \dots, U_n$ mutually disjoint open neighborhoods of $x_1, \dots, x_n$, respectively, verifying

$$|T^*(\varphi_i\delta_y)|(U_i \setminus \{x_i\}) \leq \frac{\epsilon}{2}.$$

By the Urysohn Lemma [13, Theorem 1.5.11, p. 41], we can find $h_i \in C_0(X)$ with $0 \leq h_i \leq 1$, $h_i(x_i) = 1$ and $h_i(x) = 0$ if $x \in X \setminus U_i$. Define $f_i \in C_0(X, E)$ by $f_i = v_i \cdot h_i$. By (2.2) we have

$$\begin{aligned} \|(Tf_i)(y)\| &\geq |\langle \varphi_i, (Tf_i)(y) \rangle| = \left| \int f_i dT^*(\varphi_i\delta_y) \right| \\ &\geq |\langle T^*(\varphi_i\delta_y)(\{x_i\}), v_i \rangle| - \left| \int f_i dT^*(\varphi_i\delta_y) - \langle T^*(\varphi_i\delta_y)(\{x_i\}), v_i \rangle \right| \\ &> \epsilon - |T^*(\varphi_i\delta_y)|(U_i \setminus \{x_i\}) \geq \frac{\epsilon}{2}. \end{aligned}$$

According to (2.1) there is an appropriate choice of scalars $r_i = \pm 1$ such that

$$(2.3) \quad \left\| \sum_{i=1}^n r_i (Tf_i)(y) \right\| \geq \frac{\epsilon Q \sqrt[q]{n}}{2}.$$

On the other hand, since $U_i \cap U_j = \emptyset$ if $i \neq j$ and $\|f_i\| \leq 1$ for each $1 \leq i \leq n$, we have

$$\left\| \sum_{i=1}^n r_i f_i \right\| \leq 1.$$

Therefore, by (2.3) and the choice of ϵ we conclude that

$$\|T\| \geq \left\| T \left(\sum_{i=1}^n r_i f_i \right) \right\| \geq \left\| T \left(\sum_{i=1}^n r_i f_i \right) (y) \right\| > \|T\|,$$

which is the desired contradiction.

(b) It follows by our hypothesis that

$$X = \bigcup_{y \in Y} X_y.$$

By Proposition 2.1 (a), X_y is countable for all $y \in Y$. Hence $|X| \leq |Y|$. $\square$

Note that Theorem 1.5 in the case where X and Y are finite is an immediate consequence of the following result.

Lemma 2.2. *Let E be a Banach space having non-trivial cotype and X be a finite Hausdorff topological space. Then for every locally compact Hausdorff space Y we have*

$$C_0(X, E) \sim C_0(Y, E) \implies |Y| < \aleph_0.$$

Proof. There exists some $n \in \mathbb{N}$ such that

$$(2.4) \quad E^n \sim C_0(X, E) \sim C_0(Y, E).$$

Then Y must be finite. Otherwise, it is well known that $C_0(Y, E)$ contains a copy of c_0 . Therefore, by (2.4) and [19, Theorem 1], E contains a copy of c_0 , which is absurd, because E has non-trivial cotype. $\square$

3. ON WEAKLY*-MEASURABLE FUNCTIONS

This section is devoted to presenting certain weakly*-measurable functions which will be used later. First of all, recall that, given a Banach space E and a locally compact Hausdorff space Y , a function $\gamma : Y \rightarrow E^*$ is said to be simple whenever its image is finite and $f^{-1}(v)$ is a Borel set for every $v \in E^*$. Moreover, γ is measurable if there exists a sequence of simple functions converging pointwise to γ . Finally, γ is weakly*-measurable if for every $v \in E$ the numerical function $y \mapsto \langle \gamma(y), v \rangle$ is measurable [11, Definition 5, p. 4, and Definition 16, p. 8].

We will need the following Radon-Nikodým type vector measure theorem [11, Theorem 34, p. 37].

Theorem 3.1. *Let E be a separable Banach space, Y a locally compact Hausdorff space and $\mu \in rcbv(Y, E^*)$. Then there exists a weakly*-measurable function $\gamma : Y \rightarrow E^*$ such that*

- (a) $\|\gamma(y)\| = 1$ for every $y \in Y$.
- (b) The application $y \mapsto \langle \gamma(y), f(y) \rangle$ is measurable and

$$\int f d\mu = \int \langle \gamma(y), f(y) \rangle d|\mu|(y)$$

for every function $f : Y \rightarrow E$ which is a uniform limit of a sequence of E -valued simple functions.

Remark 3.2. Notice that if E^* has the Radon-Nikodým property [9, Definition 3, p. 61], then it follows that we can choose γ measurable, satisfying the same statements of Theorem 3.1.

Proposition 3.3. *Let E be a Banach space, Y a locally compact Hausdorff space, $\mu \in rcbv(Y)$ and $\gamma : Y \rightarrow E^*$ a function such that $\|\gamma(y)\| = 1$ for every $y \in Y$. Suppose that*

- (a) E is separable and γ is weakly*-measurable or
- (b) γ is measurable.

Then for every $\epsilon > 0$ there is a compact $K \subset Y$ such that $|\mu|(Y \setminus K) < \epsilon$ and the restriction of γ to K is continuous with respect to the weak*-topology of E^* .

Proof. (a) Consider S_{E^*} endowed with its weak*-topology. Let $\{v_n : n \in \mathbb{N}\}$ be a dense sequence in S_{E^*} and set

$$d(\psi, \varphi) = \sum_{n=1}^{+\infty} \frac{1}{2^n} |\langle \psi, v_n \rangle - \langle \varphi, v_n \rangle|, \quad \psi, \varphi \in S_{E^*}.$$

It follows that d is a metric inducing the topology of S_{E^*} .

Pick $\epsilon > 0$. For each $n \in \mathbb{N}$, since the range $\gamma(Y) \subseteq S_{E^*}$, we can cover $\gamma(Y)$ by a finite collection of open balls $\{U_1, \dots, U_{r_n}\}$ of radius $\frac{1}{2n}$.

For each $1 \leq i \leq r_n$, suppose that $U_i = \{\psi \in S_{E^*} : d(\psi, \varphi_i) < \frac{1}{2n}\}$ and define $\eta_i : Y \rightarrow \mathbb{R}$ by $\eta_i(y) = d(\gamma(y), \varphi_i)$. Since γ is weakly*-measurable, η_i is measurable because it is a pointwise limit of measurable functions. By

$$\eta_i^{-1}([0, \frac{1}{2n}]) = \gamma^{-1}(U_i),$$

we infer that $\gamma^{-1}(U_i)$ is a Borel set for every $1 \leq i \leq r_n$.

Next, for each $1 \leq i \leq r_n$ we define

$$A_i = U_i \setminus \bigcup_{j < i} U_j \quad \text{and} \quad B_i = \gamma^{-1}(A_i).$$

Therefore B_i is a Borel set. By the regularity of $|\mu|$, we can find a compact $J_i \subset B_i$ such that

$$|\mu|(B_i \setminus J_i) \leq \frac{\epsilon}{r_n 2^n}.$$

For each $1 \leq i \leq r_n$, let $y_i \in J_i$ and define

$$\gamma_n = \sum_{i=1}^{r_n} \gamma(y_i) \chi_{J_i}.$$

Since $J_1, \dots, J_{r_n}$ are mutually disjoint, for every $y \in J_1 \cup \dots \cup J_{r_n}$,

$$d(\gamma(y), \gamma_n(y)) < \frac{1}{n}.$$

Set $K_n = J_1 \cup \dots \cup J_{r_n}$ for every $n \in \mathbb{N}$. Note that

$$|\mu|(Y \setminus K_n) \leq \frac{\epsilon}{2^n},$$

and the restriction of γ_n to K_n is continuous.

Define $K = \bigcap_{n \in \mathbb{N}} K_n$. Observe that $|\mu|(Y \setminus K) \leq \epsilon$ and the sequence $(\gamma_n)_{n \in \mathbb{N}}$, restricted to K , converges uniformly to γ on the metric d . Hence, the restriction of γ to K is continuous with respect to the weak*-topology of E^* .

(b) In this case the function γ is a pointwise limit of a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of simple functions. Given $\epsilon > 0$, the Egoroff Theorem [11, Theorem 42, p. 18] produces a measurable set $A \subset Y$ such that $|\mu|(A) \leq \frac{\epsilon}{2}$, and the sequence $(\gamma_n)_{n \in \mathbb{N}}$, restricted to $Y \setminus A$, converges uniformly to γ in norm. Working with functions γ_n similarly as in the previous case, we may find a compact $K \subset Y \setminus A$ with the required properties. $\square$

4. ON LOWER-SEMICONTINUOUS FUNCTIONS

Next we introduce some lower-semicontinuous functions [20, Definition 6.3.1, p. 97] which will be used in the sequel.

Lemma 4.1. *Let E be a Banach space and X and Y be locally compact Hausdorff spaces. Suppose that T is a bounded linear operator from $C_0(X, E)$ to $C_0(Y, E)$ and $\gamma : Y \rightarrow E^*$ is bounded and continuous in the weak*-topology of E^* . Then for every open set $U \subset X$ the application*

$$y \longmapsto |T^*(\gamma(y)\delta_y)|(U)$$

is lower-semicontinuous.

Proof. For each $f \in C_0(X, E)$ the application

$$y \longmapsto \left| \int f dT^*(\gamma(y)\delta_y) \right| = |\langle \gamma(y), (Tf)(y) \rangle|$$

is continuous, since it is a composition of continuous functions. Then, by considering $\mathcal{F} = \{f \in C_0(X, E) : \|f\| \leq 1 \text{ and } f(x) = 0 \text{ for every } x \in X \setminus U\}$,

$$y \longmapsto |T^*(\gamma(y)\delta_y)|(U) = \sup_{\mathcal{F}} \left| \int f dT^*(\gamma(y)\delta_y) \right|$$

is lower-semicontinuous. $\square$

We end this short section by summarizing in the next proposition a result about lower-semicontinuous functions and $rcbv(X)$ spaces which was established in the proof of the main theorem of [7].

Proposition 4.2. *Let X and Y be locally compact Hausdorff spaces. Fix $x \in X$ and assume that $\{\varphi_y : y \in Y\}$ is a family of positive measures in $rcbv(X)$ such that $\varphi_y(\{x\}) = 0$ for every $y \in Y$. Suppose that $\{U_i : i \in I\}$ is the collection of all open neighborhoods of x endowed with the partial ordering of reverse inclusion.*

Pick $\epsilon > 0$. For every $i \in I$ consider the following sets:

$$C_i = \overline{\{y \in Y : \varphi_y(\overline{U_i}) \leq \epsilon\}}.$$

If for every open set $U \subset Y$ the application $y \mapsto \varphi_y(U)$ is lower-semicontinuous, then for every positive measure $\mu \in rcbv(Y)$ we have

$$\mu(Y) = \sup_{i \in I} \mu(C_i).$$

5. PROOF OF THEOREM 1.5

We are now ready to prove the main theorem of this work.

Proof. First of all notice that thanks to Lemma 2.2 we can assume that X and Y are infinite. We will prove that $|X| \leq |Y|$. Thus by symmetry $|Y| \leq |X|$, and we are done.

We will apply Proposition 2.1 (b) to prove $|X| \leq |Y|$. So, it is enough to show that for every $x \in X$ there exists a $y \in Y$ such that $x \in X_y$. By contradiction, suppose that there exists a $x \in X$ such that, for each $y \in Y$ and for each $\varphi \in S_E^*$,

$$(5.1) \quad |T^*(\varphi\delta_y)|(\{x\}) = \|T^*(\varphi\delta_y)(\{x\})\| = 0.$$

Fix $\Phi \in S_E^*$ and consider $\mu_x \in rcbv(Y, E^*)$ defined by

$$(5.2) \quad \mu_x = T^{-1*}(\Phi\delta_x).$$

By Theorem 3.1 and Remark 3.2 there exists a weakly*-measurable function $\gamma : Y \rightarrow E^*$ verifying the statements of this theorem which is measurable in the case where E^* has the Radon-Nikodým property.

Pick $\epsilon > 0$ such that

$$(5.3) \quad \epsilon < \min \left\{ |\mu_x|(Y), \frac{1}{2(\|T\| + |\mu_x|(Y))} \right\}.$$

According to Proposition 3.3, there is a compact $K \subseteq Y$ such that

$$(5.4) \quad |\mu_x|(Y \setminus K) \leq \epsilon$$

and the restriction of γ to K is continuous with respect to the weak*-topology of E^* .

For each $y \in K$, write

$$(5.5) \quad \Gamma_y = |T^*(\gamma(y)\delta_y)|.$$

Thus, it follows from (5.1) that $\Gamma_y(\{x\}) = 0$, for each $y \in K$.

Now, let $\{U_i : i \in I\}$ be the collection of all open neighborhoods of x endowed with the partial ordering of reverse inclusion. For each $i \in I$, define the following measurable set:

$$(5.6) \quad C_i = \overline{\{y \in K : \Gamma_y(\overline{U_i}) \leq \epsilon\}}.$$

Moreover, by Lemma 4.1, notice that for every open set $U \subset X$ the application $y \mapsto \Gamma_y(U)$ is lower-semicontinuous. Therefore by Proposition 4.2 we have

$$(5.7) \quad |\mu_x|(K) = \sup_{i \in I} |\mu_x|(C_i).$$

Let $v \in E$ be such that $\|v\| = 1$ and $\frac{1}{2} < \langle \Phi, v \rangle$. By the Urysohn Lemma, for each $i \in I$ there is $h_i \in C_0(X)$ with $0 \leq h_i \leq 1$, $h_i(x) = 1$ and $h_i(z) = 0$ if $z \in X \setminus U_i$. Now define $f_i \in C_0(X, E)$ by $f_i = v.h_i$. Hence by (5.2),

$$(5.8) \quad \frac{1}{2} < \langle \Phi, f_i(x) \rangle = \int f_i d(\Phi\delta_x) = \int T f_i d(T^{-1*}(\Phi\delta_x)) = \int (T f_i) d\mu_x.$$

Since every $f \in C_0(Y, E)$ is a uniform limit of a sequence of E -valued simple functions, it follows from Theorem 3.1 (b) that

$$(5.9) \quad \int (T f_i) d\mu_x = \int \langle \gamma(y), (T f_i)(y) \rangle d|\mu_x|(y),$$

for each $i \in I$. We will split this integral into three parts. In order to do this, first observe that, in view of (5.4) and Theorem 3.1 (a),

$$(5.10) \quad \left| \int_{Y \setminus K} \langle \gamma(y), (T f_i)(y) \rangle d|\mu_x|(y) \right| \leq \|T\| |\mu_x|(Y \setminus K) \leq \|T\| \epsilon.$$

On the other hand, for every $y \in K$ and for each $i \in I$, by (5.5) we infer that

$$|\langle \gamma(y), (T f_i)(y) \rangle| = \left| \int f_i dT^*(\gamma(y)\delta_y) \right| \leq \Gamma_y(\overline{U_i}).$$

But, by using the fact that the restriction of γ to K is continuous with respect to the weak*-topology of E^* , if $y \in C_i$, then according to (5.6) we deduce that $|\langle \gamma(y), (T f_i)(y) \rangle| \leq \epsilon$. Then we have

$$(5.11) \quad \left| \int_{C_i} \langle \gamma(y), (T f_i)(y) \rangle d|\mu_x|(y) \right| \leq \epsilon |\mu_x|(Y).$$

So, putting (5.10) and (5.11) together, by Theorem 3.1 (a) we obtain

$$\begin{aligned} \int \langle \gamma(y), (Tf_i)(y) \rangle d|\mu_x|(y) &\leq \epsilon \|T\| + \left| \int_{K \setminus C_i} \langle \gamma(y), (Tf_i)(y) \rangle d|\mu_x|(y) \right| + \epsilon |\mu_x|(Y) \\ &\leq \epsilon (\|T\| + |\mu_x|(Y)) + \|T\| (|\mu_x|(K) - |\mu_x|(C_i)). \end{aligned}$$

Furthermore, keeping in mind (5.7), (5.8), (5.9) and the choice of ϵ in (5.3), we conclude that

$$\frac{1}{2} \leq \epsilon (\|T\| + |\mu_x|(Y)) < \frac{1}{2},$$

a contradiction which completes the proof of Theorem 1.5. $\square$

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